APPLICATIONS OF H-PRINCIPLE IN SYMPLECTIC TOPOLOGY

By Yaniv

Recall that symp. form defines a volume form $\omega^{\wedge n}$, and hence it defines an orientation. In particular, M orientable.

Notation:

* $\mathcal{J} = \mathcal{J}(TV)$ = space of all a.c.s. on V, ie. space of complex structure on TV, $J: TV \to TV$ coverind $id: V \to V$, $J^2 = -\mathbb{1}$
* $S_{symp} = S(TV)$ = space of almost symplectic structures on V, ie linear sympleetic structures on TV (given by non degenerate 2 forms.
* $\mathcal{S}_{symp}$ = Symplectic structures on V, ie. non-deg ω s.t $d\omega = 0$
* $\mathcal{S}^a_{symp}$ = space of symp structures in class $a \in H^2(V)$.

We want to study: $\mathcal{S}_{symp} \hookrightarrow S_{symp}$, $\mathcal{S}^a_{symp} \hookrightarrow S_{symp}$ from a homotopy point of view.

Gromov's Theorem: For any $2n$-dim orientable open manifold V, the inclusion $\mathcal{S}^a_{symp} \hookrightarrow S_{symp}$ is a homotopy equiv. In particular, any 2-form $\beta \in S_{symp}$ is homotopic in S_{symp} to a symp form $\omega \in a \in H^2(V)$. Moreover, if $\omega_0, \omega_1 \in \mathcal{S}^a_{symp}$ are homotopic in S_{symp}, they're homotopic in $\mathcal{S}^a_{symp}$.

Remark: Moser trick does not apply for such homotopies as V is open and we don't know that the homotopy is constant outside of a compact.

Comparison with closed manifolds:

M^{2n} closed, $a \in H^2(M)$. Necessary condition: $0 \neq a^{\cup n} \in H^{2n}(M)$.

Q. Assuming $a^{\cup n} \neq 0$, is there a symplectic form ω in class a?

- $n=1$: YES.
- $n=2$: NO. Taubes: $(\mathbb{CP}^2)^{\# 3}$ doesn't admit a symp form.

  Uses Seiberg Witten, very specialized to dim 4.
- $n \geq 3$: OPEN.

Q. What about the paramentric version? ie. if ω_0, ω_1 symp. homotopic via "formal symp", are they homotopic via symplectic?

- $n=1$: YES
- $n=2$: OPEN
- $n \geq 3$: NO. Ruan 94'.

So, Gromov's theorem is a flexibility of open manifolds.

PROOF OF GROMOV'S THEOREM (following Eliashberg & Mishachev):

Note that any diffeomorphism $h: V \to V$ lifts to $\Lambda^p V$ = bundle of p-forms on V, as $D^p h$ = p-th exterior powers of $Dh: TV \to TV$.

We can talk about $\mathrm{Diff}(V)$-invariant sets $R \subset \Lambda^p V$.

Given such a set R and a class $a \in H^p(V)$, denote by $\mathcal{C}lo_a(R)$ the subset of $\mathrm{Sec}(R) = \{\text{sections of } \Lambda^p V \text{ with values in } R\}$ consisting of closed p-forms in class a.

Theorem (Gromov): Let $R \subset \Lambda^p V$ be an open diff(V)-invariant subset then $\mathcal{C}lo_a(R) \hookrightarrow \mathrm{Sec}(R)$ is a homotopy equivalence.

From this ⇑ thm it is clear how to conclude the proof of the main thm. So it is left to prove the latter thm.

Approximation of differential forms by exact / closed forms:

Recall that any p-form on V can be seen as a section $\begin{matrix}\Lambda^p V\\ \downarrow\\ V\end{matrix}$ and $\Lambda^p V := \Lambda^p T^* V$.

Exact p-forms and holonomic sections of

$$\begin{matrix}(\Lambda^{p-1} V)^{(1)}\\ \downarrow\\ V\end{matrix}$$

= 1st jet bundle of the bundle $\Lambda^{p-1} V$.

The fibers are germs of sections up to equivalence of 1st order appoximations

are closely related:

exterior differentiation $d : \mathrm{Sec}\, \Lambda^{p-1} V \longrightarrow \mathrm{Sec}\, \Lambda^p V$ factors as

$\mathrm{Sec}\, \Lambda^{p-1} V \xrightarrow{J^1} \mathrm{Sec}(\Lambda^{p-1} V)^{(1)} \xrightarrow{\tilde{D}} \mathrm{Sec}\, \Lambda^p V$, where $\tilde{D}$ is induced by a bundle map D called "the symbol of d"

$D : (\Lambda^{p-1} V)^{(1)} \longrightarrow (\Lambda^p V)$.

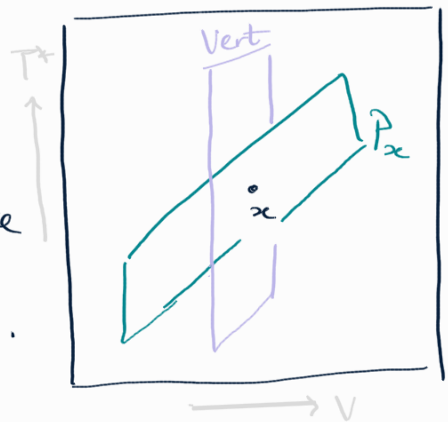
Example: $p=2$.

$n = \dim V$

Fibers of $(\Lambda^1 V)^{(1)}$ is identified with $n\times n$ matrices:

Points in $(T^*V)^{(1)}$ are planes in TT^*V which are $\pitchfork$ to the vertical planes.

Fixing an arbitray plane P_x, all other such planes are in 1-1 correspondence with $\mathrm{Hom}(\underset{n\text{ dim}}{P_x}, \underset{n\text{dim}}{\mathrm{Vert}}) \cong n\times n$ matrices.



Having identified fibers as matrices, the fibers of $\Lambda^2 V$ are identified with skew symetric matrices. In this language we can write $D:(\Lambda^{p-1}V)^{(1)} \to \Lambda^p V$ as

$$D(A) = A - A^T.$$

The operator $D:(\Lambda^{p-1}V)^{(1)} \longrightarrow \Lambda^p V$ is an affine fibration $\Rightarrow$ any section ω of $\Lambda^p V \xrightarrow{} V$ has a lift $F_\omega: V \to (\Lambda^{p-1}V)^{(1)}$. Think of this lift as a "formal primitive". Moreover, these lifts are homotopicaly unique.

Lemma: (Approximation by exact forms): Let $K \subset V$ be a subcomplex of codim ≥ 1 and a p-form ω. Then $\exists$ C^0-small diffeotopy $h^t: V\to V$ s.t. ω can be C^0-approximated near $\tilde{K} := h^1(K)$ by an exact p-form $\tilde{\omega} = d\tilde{\alpha}$.

Moreover, given a $(p-1)$-form α on V one can choose $\tilde{\alpha}$ to be C^0-close to α near K.

Lemma (Approximation by closed form): K, V as before. ω = a p-form, $a \in H^p(V)$ cohomology class. Then $\exists$ arbitrarily small h^τ s.t. ω can be C^0-approx. near $\tilde{K}$ by a closed form $\tilde{\omega}$ of class a.

CONTACT STRUCTURES ON 3-MANIFOLDS

M^{2n+1} smooth mfld, a contact structure on M is a hyperplane field $\xi^{2n} \subset TM$ which is maximally non-integrable: There does not exist submanifolds of dim $\geq n+1$ tangent to ξ, even up to 2-nd order.

Gray Stability: ξ_t family of contact structures, const. outside of a compact $\Rightarrow \exists$ family of diffeos φ_t s.t. $D\varphi_t(\xi_0) = \xi_1$.

⊛ Locally all ctct structures look like $(\mathbb{R}^{2n+1}, \xi_{std})$

For 3-manifolds: Any 2-plane distribution is either integrable or contact at any point.

Def. A formal contact structure is a pair (β, σ) with $\beta \in \Lambda^1 V$ and $\sigma \in \Lambda^2 V$ s.t. $\beta \wedge \sigma^{\wedge n} \neq 0$. (decoupled $d\beta = \sigma$)

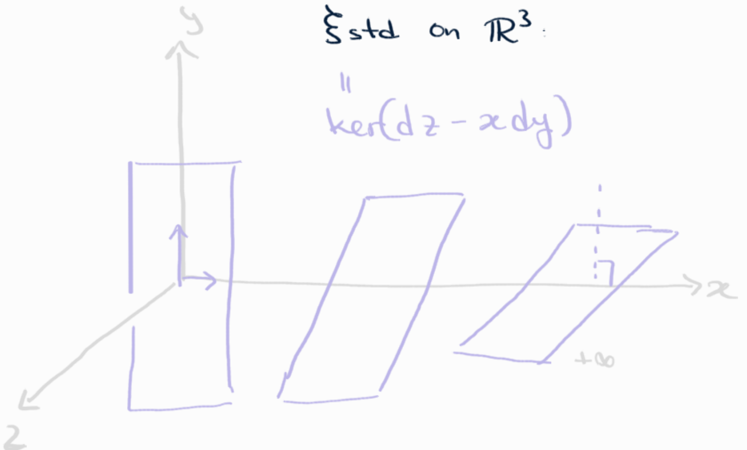
- On 3 manifolds fixing an orientation, considering σ s.t. $\beta \wedge \sigma > 0$. One has that $\forall \beta$ the space of σ contractible $\Rightarrow$ enough to consider only the β-part.
- On 3 manifolds a formal contact structure is just a 2-plane field.

Theorem (Lutz-Martinet): On 3-mflds every 2 plane dist. is homotopic to a contact structure:

$$\pi_0 \left\{ \begin{matrix} \text{contact} \\ \text{structures} \end{matrix} \right\} \longrightarrow \pi_0 \left\{ \begin{matrix} \text{2 planes} \\ \text{distr.} \end{matrix} \right\} \quad \text{is surjective.}$$

Theorem (Bennequin): There exist two contact structures on S^3, ξ_{std} and ξ_{OT}, which are not contactomorphic but are homotopic as 2-plane distributions.

ξ_{std} on $\mathbb{R}^3$: $= \ker(dz - x\,dy)$



when we go from $-\infty$ to $+\infty$ in the x direction, the planes make a half rotation.

Overtwisted: One can embed a disc D s.t. along ∂D ξ is tangent to D.

in the pic make full rotation, ie rotates more than std.

Eg. $\alpha_{OT} = \cos(r)\,dz + r\sin(r)\,d\theta$

Theorem (Eliashberg): Any homotopy class of 2-plane fields contains a unique overtwisted contact structure.

+ Parametric version.

Borman - Eliashberg - Murphy: generalized ↑ to all dimensions.

Rk. OT contact forms do not admit symplectic fillings (Gromov-Eliashberg).