# PROOF OF THE HOLONOMIC APPROXIMATION THEOREM

/ talk by Yaniv Ganor

**Theorem** (Eliashberg-Mishachev): Let $Q \subset V$ be a codim$\geq 1$ stratified subset. Let $\sigma : V \to J^r(V,W)$ be a section.

$\forall \varepsilon \quad \exists \varphi_t : Q \longrightarrow V$ s.t. $\varphi_0 =$ inclusion and $\varphi_t \underset{\varepsilon C^0}{\approx}$ incl,

and $\exists f : \mathcal{O}p(\varphi_1(Q)) \to W$ s.t. $\left| J^r(f) - \sigma|_{\mathcal{O}p(\varphi_1(Q))} \right| < \varepsilon$.

Furthermore:

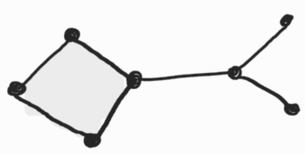
*Relative Version*: If $\sigma$ is holonomic on $\mathcal{O}p(A)$ for $A \subseteq Q$ closed, then can take $f$ s.t. $J^r(f) = \sigma$ on $\mathcal{O}p(A)$ and $\varphi_t =$ incl. on $A$.

*Parametric version*: If we have a family $\sigma^\tau$ of sections, $\tau \in \mathbb{D}^k$ s.t. they are holonomic near $\partial \mathbb{D}^k$. Then we obtain $f^\tau$, $\phi_t^\tau$ as above so that $\sigma^\tau = J^r(f^\tau)$ $\forall \tau \in \partial \mathbb{D}^k$

## STRATEGY OF PROOF:

$\boxed{1}$ Go from $Q$ to simpler pieces, namely cubes $[0,1]^k$.

Since we prove relative version, enough to prove for $I = [0,1]^k$ relative to $\partial I$. Indeed:

$Q =$ 

first approx near vertices, then near 1-cells $[0,1]$, then the 2-cells and so on.

$\boxed{2}$ define some "partial holonomicity", ie. hol in certain directions but not in others. Then "add directions" inductively.

## IN MORE DETAILS:

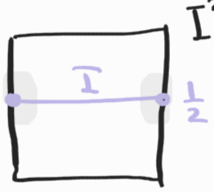
$\boxed{1}$ Holonomic approximation over a cube. $I := [0,1]$.

<u>Thm:</u> Say $I^k \subseteq \mathbb{R}^n$, $k<n$. Given $F : \mathcal{O}p(I^k) \to J^r(\mathbb{R}^n, \mathbb{R}^q)$ s.t. $F$ is hol. near $\partial I^k$. Then $\forall\, \varepsilon\ \exists\ \varepsilon$-small isotopy $\varphi_t : I^k \to \mathbb{R}^n$ fixed near $\partial I^k$ and $\partial I^k$ and $f : \mathcal{O}p(\varphi_1(I^k)) \to \mathbb{R}^q$

s.t. (1) $J^r(f) = F$ on $\mathcal{O}p(\partial I^k) = \mathcal{O}p(\varphi_1(\partial I^k))$

(2) $J^r(f)$ is $\varepsilon$-close to $F$ on $\mathcal{O}p(\varphi_1(I^k))$.

<u>Special case: $k=1$, $n=2$.</u> Think of $I \hookrightarrow I^2$ as $[0,1] \times \{\frac{1}{2}\}$:




$I^2$ in the role of $\mathbb{R}^2$.

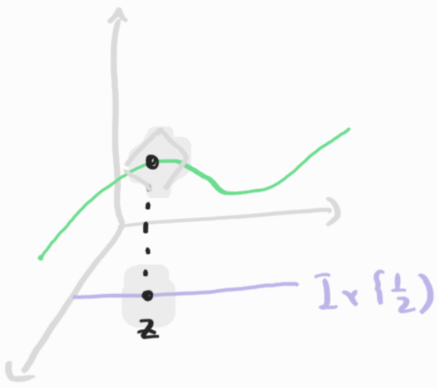
$\xrightarrow[\sigma = F]{} J^1(I^2, \mathbb{R})$

suppose we are given a section $\sigma : I^2 \to J^1(I^2, \mathbb{R})$ which is holonomic on $\mathcal{O}p(\partial I \times \{\frac{1}{2}\})$.

Want to show $\exists$ arbitrarily small isotopy $\varphi_t : I \to I^2$ s.t. $\varphi_0 = \mathrm{incl}$ and a hol. section $f : \mathcal{O}p(\varphi_1(I)) \to J^1(I^2; \mathbb{R})$ s.t.

$$\begin{cases} |f - \sigma|_{C^0} < \varepsilon \\ f|_{\mathcal{O}p(\partial I \times \{\frac{1}{2}\})} = \sigma|_{\mathcal{O}p(\partial I \times \{\frac{1}{2}\})} \end{cases}$$

<u>Proof of special case:</u> At each point $z \in I \times \{\frac{1}{2}\}$, we can find $F_z : \mathcal{O}p(z) \to J^1(I^2; \mathbb{R})$ s.t. $F_z$ holonomic & $F_z(z) = \sigma(z)$



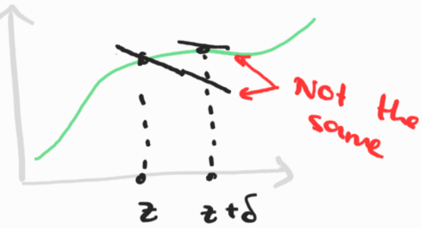

$F_z$ = linear approximation.

Note "$F_z$ continuous in $z$"

$F_\bullet : (I \times \{\tfrac{1}{2}\}) \times I^2 \to J^1(I^2, \mathbb{R})$ defined in $Op(\Delta)$ graph of inclusion

**Problem:** they don't line up!

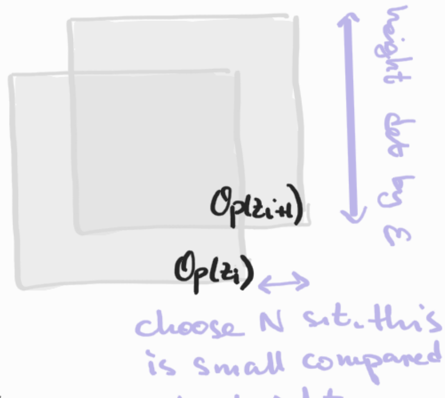
$$F_z(z+\delta) \neq F_{z+\delta}(z+\delta)$$

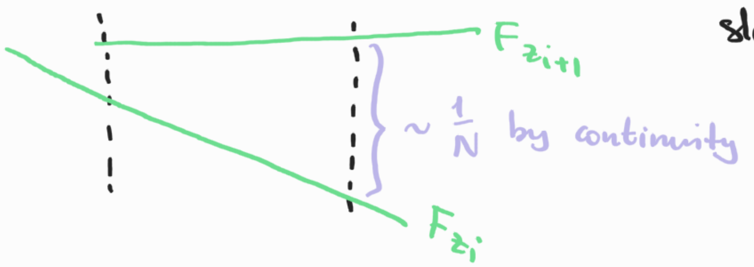



Need a way to glue them together. Take $N \gg 0$ (to be specified later), pic $z_i = \frac{i}{N}$, recall each $F_z$ is defined in $Op(\{z\} \times \{\frac{1}{2}\})$ and take them all to be small enough s.t.

$$|F_z - \sigma|_{Op}| < \frac{\varepsilon}{10}.$$

Look at $Op(z_i)$ and $Op(z_{i+1})$. How do $F_{z_i}$ and $F_{z_{i+1}}$ look like?




slope difference $\sim \frac{1}{N}$ by continuity wrt $J^1$ sections.

Restrict our attention to red part:

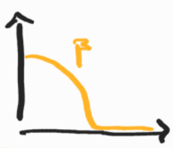

Slit = discontinuity of $f$.

Define $f$ as:

In green part $f = F_{z_{i+1}}$

In purple: $f = F_{z_i}$

In orange: interpolate




$$f = \beta(w) F_{z_i} + (1-\beta(w)) F_{z_{i+1}}$$

Indeed $f$ is holonomic, $|J^1(f) - \sigma|_{C^0} < \varepsilon$. Say $\sigma = (g, G)$

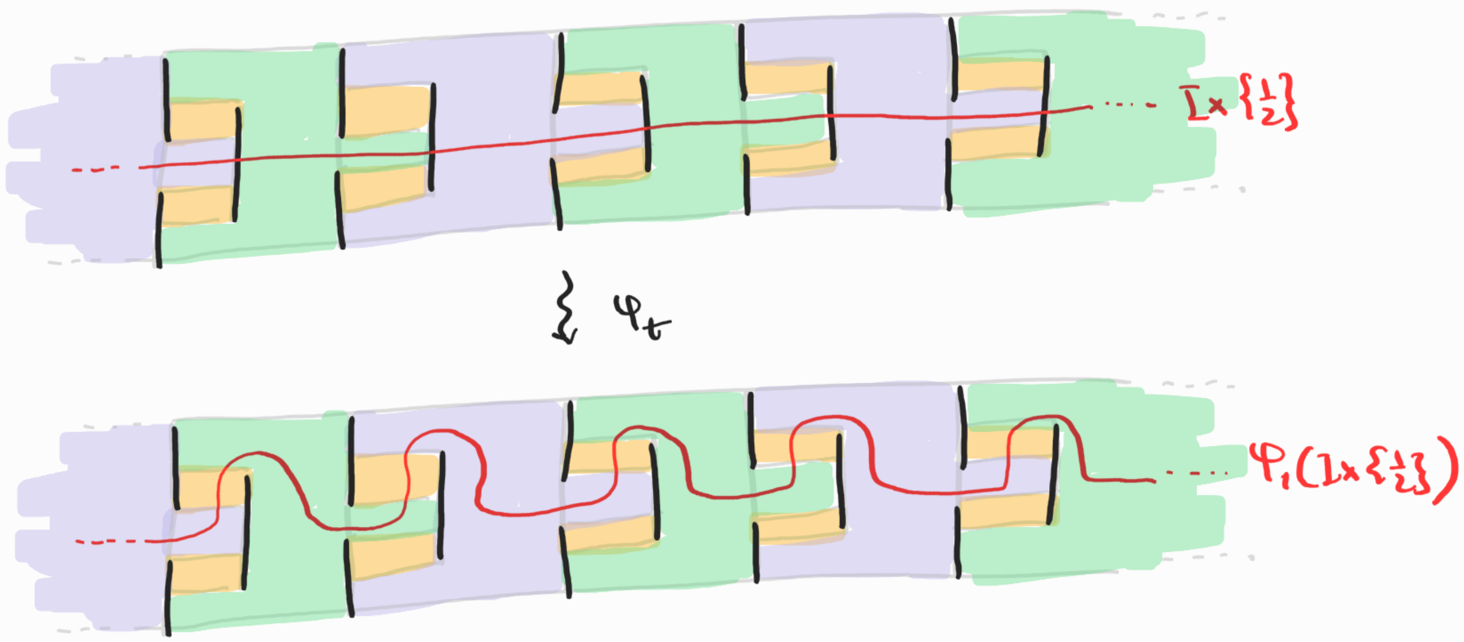
$f \approx g$ because $g \approx F_{z_i}$, $g \approx F_{z_{i+1}}$ and we interpolate

$df \approx G$ because $G \approx dF_{z_i}$, $G \approx dF_{z_{i+1}}$, $\beta'$ doesn't dep

on $N$, $|F_{z_i} - F_{z_{i+1}}| \xrightarrow{N \to \infty} 0$ so by continuity one checks

that $df \approx G$.

How to glue these together?



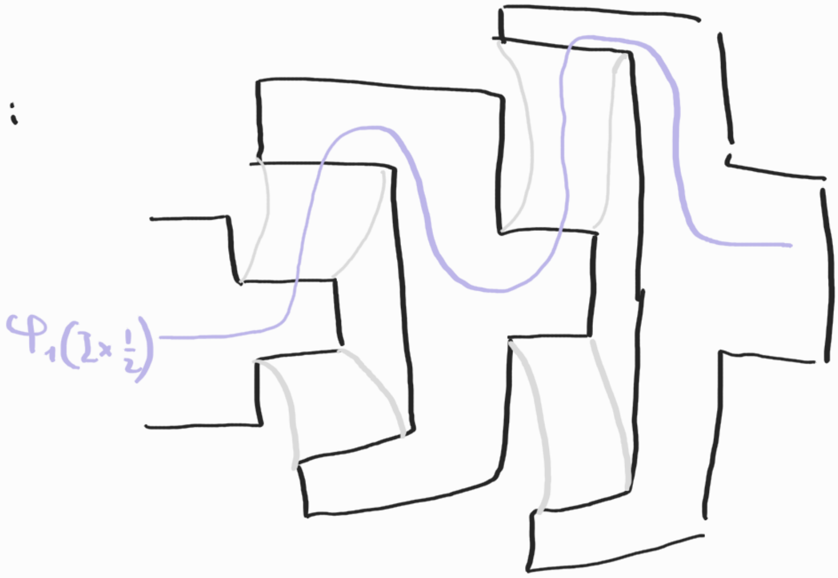

By wiggling $I \times \{\frac{1}{2}\}$ one avoids the slits of discont.

get $f$ defined on $\mathcal{O}p(\varphi_1(I \times \{\frac{1}{2}\}))$, and cont. there.

Relative version: Pick $\left.\begin{array}{l} F_{z_0} = \sigma|_{\mathcal{O}p(z_0)} \\ F_{z_{2N}} = \sigma|_{\mathcal{O}p(z_N)} \end{array}\right\}$ by assumption $\sigma$ is holonomic there

instead of picking linear approximation.

Montain climbing:




$\boxed{2}$ General case (Sketch):

Def. Suppose $X \to V$ fiber bundle and $\pi : V \to B$ also fiber bundle we say a section $F : V \to J^r(X)$

[ $J^r(X)$ = jets of sections $V \to X$, $J^r(V,W) = J^r\left(\begin{smallmatrix} V \times W \\ \downarrow \\ V \end{smallmatrix}\right)$ (ie $X = V \times W$) ]

is fiberwise holonomic (wrt $\pi$) if $\exists \tilde{F} : Op(\tilde{V}) \to J^r(X)$ with $\tilde{V} = \{(v, \pi(v)) : v \in V\} \subset V \times B$ s.t.

$\tilde{F}_b(v) = F(v)$ for $v \in \pi^{-1}(b)$

$\tilde{F}_b(\cdot)$ is holonomic for fixed $b$.

Note $\tilde{F}_b(v) \neq F(v)$ for $v$ with $\pi(v) \neq b$.

That is $F|_{\pi^{-1}(b)}$ extends to a hol section in $Op(\pi^{-1}(b))$.

Example. $\mathbb{R}^2 \to \mathbb{R}$ proj. on 1-st coord.

Section of 1st jet bundle

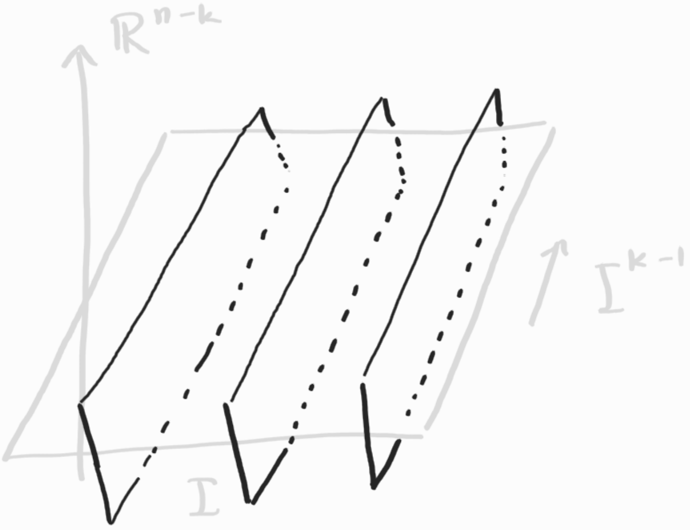

fiber direction projects to the board.

A fiberwise holonomic not holonomic fiber:


Idea for step 2: $I^k \subset \mathbb{R}^n$, consider $I^k = I^{k-\ell} \times I^\ell$

$$\begin{tikzcd} & I^{k-\ell} \times I^\ell \arrow[dl, "\pi'"'] \arrow[d, "\pi"] \\ I^{\ell-1} & I^\ell \end{tikzcd}$$

If we start with $\sigma$ which is fiber-hol wrt $\pi$, upgrade to hol wrt $\pi'$:




side view: