THE H-PRINCIPLE IN TOPOLOGY & SYMPLECTIC GEOMETRY - LECTURE I

Consider the ODE $\left(\frac{dy}{dx}\right)^2 + e^y = 0$, $y: \mathbb{R} \to \mathbb{R}$. Does it have solutions? NO! since the algebraic equation

$u^2 + e^y = 0$ for $u, y: \mathbb{R} \to \mathbb{R}$ has no solutions.

We "decoupled the derivative from y".

What about the reverse direction? i.e. does the solvability of the algebraic equation tell us something about the solvability of the ODE?

- In general no: $\left\{\begin{matrix} y' + y^2 \\ y' \neq 0 \end{matrix}\right\}$ - no solutions

$\left\{\begin{matrix} u + y^2 = 0 \\ u \neq 0 \end{matrix}\right\}$ - plenty of sols.

Gromov: In many underdetermined PDEs (or PDRs) the answer is YES!

Theorem (Hirsch-Smale): Given V^n, W^q manifolds with $n < q$, if $\exists$ a bundle map $F: TV \to TW$ which is fiberwise injective then $\exists\, g: V \to W$ s.t. $Dg: TV \to TW$ is fibewise injective (i.e. g is an immersion).

In fact, $\{\text{Immersions}\} \xhookrightarrow[\text{w.h.e}]{} \{\text{fiberwise injective bundle maps}\}$

(ie. iso on T_n $\forall n$).

(*) An immersion $g: V \to W$ is a map s.t. $\operatorname{rank}(Dg) = n$. This can be expressed as nonvarishing of (at least one) determinant $\Rightarrow$ a diff relation.

Each g gives a bundle map $(g, Dg): TV \to TW$.

For a general bundle map (f, F) need not equal Df.

So, this theorem is exactly the "decoupling" mentiond above.

JET SPACES & JET BUNDLES

Let V, W be manifolds. The space of jets over a point $(v, w) \in V \times W$ is :

$$\left\{ \begin{matrix} \text{local maps } \mathcal{O}p\{v\} \to W \\ f(v) = w \end{matrix} \right\} \Big/ \sim \qquad \begin{matrix} f_1 \sim f_2 \text{ if their } r\text{-th} \\ \text{Taylor polynomials agree.} \end{matrix}$$

(in some chart $\Rightarrow$ in all charts).

$J^r_{v,w}(V \times W) \cong q$-tuples of polynomials of degree $\leq r$ in n variables.

These assemble to a fiber bundle over $V \times W$, $J^r(V, W)$ - Jet bundle. Transition functions between different charts - det. by chain rule.

Example. • $J^1(V,\mathbb{R}) = \underset{\text{differential}}{T^*V} \times \underset{\text{value}}{\mathbb{R}}$.

• In general, $J^1(V,W) = \mathcal{H}om(TV, TW)$.

• $J^2(\mathbb{R}^n, \mathbb{R}) = \underset{\text{src}}{\mathbb{R}^n} \times \underset{\text{tgt}}{\mathbb{R}} \times \underset{\text{1-jet}}{\mathbb{R}^n} \times \underset{\text{2 jets}}{\mathbb{R}}$.

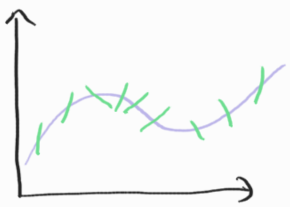
$$\begin{tikzcd} J^1(V,W) \arrow[d] \\ V \arrow[u, bend right, "J^1(f)"'] \end{tikzcd}$$

⊛ We can view $J^1(V,W)$ as a bundle over V:

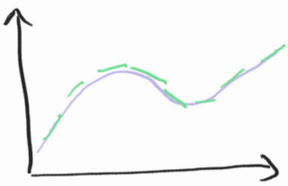
any $f: V \to W$ gives a section $J^1(f) = (f, Df)$

Def. A ==differential relation== of order r is a subset $R \subseteq J^r(V,W)$

Ho to picture a section of $J^1(\mathbb{R},\mathbb{R})$?



some section



(f, Df)

Def. • A ==formal solution== of R is any section σ with image in R.

• A ==holonomic section== σ is a section of the form $\sigma = J^1(f)$ for some $f: V \to W$.

• A ==genuine solution== is a holonomic formal solution.

Note $\mathrm{Sol}(R) \hookrightarrow \mathrm{Sol}^f(R)$.

Def. (1) We say R satisfies an ==h-principle== if any formal solution is homotopic (through formal solutions) to a genuine sol.

(2) a parametric h-principle holds if $i : \mathrm{Sols}(R) \hookrightarrow \mathrm{Sols}^f(R)$ is a weak homotopy equivalence. In particular, if two genuine sols are homotopic via formal sols then they are homotopic via genuine sols.

Theorem (Gromov): Let V be an open manifold, let $R \subset J^r(V,W)$ be a differential relation s.t.:

- R is open
- R is $\mathrm{Diff}(V)$-invariant

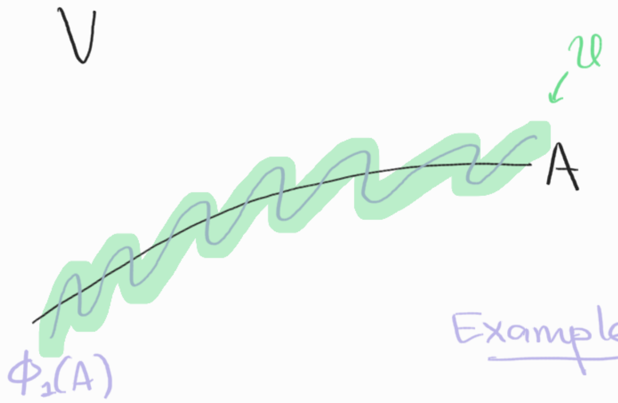
then a parametric h-principle holds.

Corollary: Hirsch-Smale thm. Indeed, immersion $\Rightarrow \det \neq 0$ can be expressed as an open, diff-inv differential invariant. Get Hirsch-Smale for open V.

For closed V: "micro extension". $\dim V < \dim W$ due to normal bundle & std nbd thms. Immersion of $V \Leftrightarrow$ Immersion of $V \times (-\varepsilon, \varepsilon)$ which is open.

How to prove Gromov's theorem?

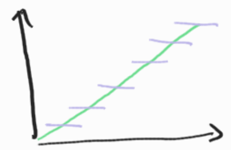
Theorem (holonomic approximation. Eliashberg-Mishachev): Let $F : V \to J^r(V,W)$ be a section and let $A \subseteq V$ be a submanifold (or stratified complex) of codim ≥ 1. Then $\exists$ a C^0-small isotopy $\Phi_t : V \to V$, a nbd $\mathcal{U} \supseteq \Phi_1(A)$ and a holonomic section $f : \mathcal{U} \to W$ s.t. $J^r(f) \underset{C^0}{\approx} F|_{\mathcal{U}}$.



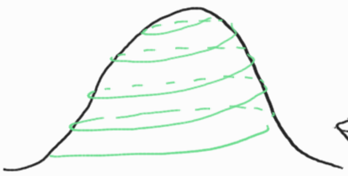
in a nbd of a perturbation $\Phi_1(A)$ of A can approx F by $J^r(f)$.

Example. "Climbing a mountain":

Take $J^1(\mathbb{R}^2, \mathbb{R})$, $F = (x, 0, 0)$. $f(x,y) = x$.
val ↗ ∂_x ↗ ∂_y ↑



can't approximate zero derivatives by ≈ 1 derivatives $(J^1(f) = (x, 1, 0))$

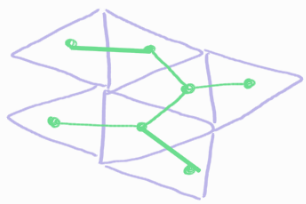


But! if we go around the mountain, then derivatives are slow.

Proof of holonomic approximation $\Rightarrow$ Gromov's thm:

Lemma: If M is an open mfld then $\exists$ set $A \subseteq M$ of codim ≥ 1 s.t. $\exists$ an isotopy Ψ_t of M, relative to A, where $\Psi_0 = \mathbb{1}$ and im $\Psi_1 = \mathcal{O}p(A)$.

Idea of pf: (1) Triangulate



(2) consider the dual graph

(3) Cover this dual graph by a union of "special trees". A special tree is a one sided infinite ray with some finite tree attached:

Special trees have closed nbds which are diffeo to $\mathbb{D}^n \setminus \{\substack{\text{bdry}\\ \text{pt}}\}$. They retract to $S^{n-1} \setminus \{pt\}$.

$\Longrightarrow$ Get a retraction to some codim ≥ 1 complex.

Back to Gromov's thm: Let $F : V \to J^r(V,W)$ with image in $\mathcal{R}$.
Let A be the codim 1 skeleton on which V retracts (from the Lemma). By holonomic approx $\exists\, \mathcal{U} \supseteq \phi_1(A)$ with ϕ an isotopy and a holonomic section $J^r(f)$ s.t. $J^r(f) \underset{C^0}{\approx} F|_{\mathcal{U}}$. Since $\mathcal{R}$ is open, $J^r(f)$ also has image in $\mathcal{R}$. Take diffeo $\psi_1 : V \to \mathcal{U}$. The desired section is $\tilde{f} = f \circ \psi_1 : V \to W$. It is still in $\mathcal{R}$ since $\mathcal{R}$ is $\mathrm{Diff}(V)$-invariant.

⊗ This gives h-principle, but how to get the parametric h-principle promised in the thm? $\exists$ a parametric version of a holonomic approx where one approximates families of sections parametrized by some set (be it a path, a disc, a sphere, etc).

LET'S CLASSIFY IMMERSIONS!

We have seen $\pi_0(\text{Immersions}) = \pi_0\left(\begin{smallmatrix}\text{formal} \\ \text{immersions}\end{smallmatrix}\right)$.

- Start with $S^1 \looparrowright \mathbb{R}^2$. What is a formal immersion? Fiber injective

$$\begin{matrix} TS^1 & \to & T\mathbb{R}^2 \\ \wr\wr & & \wr\wr \\ S^1 \times \mathbb{R} & & \mathbb{R}^2 \times \mathbb{R}^2 \end{matrix}$$

$\Rightarrow$ A formal immersion is a pair (f,F) of $f : S^1 \to \mathbb{R}^2$, $F : S^1 \to \underbrace{\left\{\begin{smallmatrix}\text{injective linear} \\ \mathbb{R} \to \mathbb{R}^2\end{smallmatrix}\right\}}_{= \mathbb{R}^2 \setminus \{0\}}$

So $\pi_0(\text{formal } S^1 \looparrowright \mathbb{R}^2) = [S^1 ; \mathbb{R}^2 \setminus 0] = [S^1 ; S^1] = \pi_1(S^1) = \mathbb{Z}$.

homotopy classes of maps $S^1 \to \mathbb{R}^2 \setminus \{0\}$.

- What about $S^2 \looparrowright \mathbb{R}^3$? Want to apply same reasoning as above but TS^2 is not parallelizable.

<u>Work around</u>: immerse $S^2 \times (-\varepsilon, \varepsilon)$. This is a 3-mfld $\subseteq \mathbb{R}^3$ open set $\Rightarrow$ parallelizable.

As before, a formal immersion (f, F), again f lives in a contractible space $\to$ care about $F : S^2 \times (-\varepsilon, \varepsilon) \longrightarrow \left\{ \begin{array}{c} \text{injective linear} \\ \mathbb{R}^3 \to \mathbb{R}^3 \end{array} \right\} \cong GL(3) \sim O_3(\mathbb{R})$

Hence, $\pi_0 \begin{pmatrix} \text{formal immersions} \\ S^2 \looparrowright \mathbb{R}^3 \end{pmatrix} = [S^2, \mathbb{RP}^3] = \pi_2(\mathbb{RP}^3) \underset{\text{univ. cover}}{=} \pi_2(S^3) = 0$

We conclude that $\pi_0(S^2 \looparrowright \mathbb{R}^3) = 0 \Rightarrow$ all immersions of $S^2 \looparrowright \mathbb{R}^3$ are homotopic!