Talk 2 - Proof of the holonomic approximation thm

Thm [Eliashberg - Mishachev]:

Let $Q \subset V$ be a codim ≥ 1 stratified subset

Let $\sigma : V \to J^r(V, W)$ be a section

$\forall \varepsilon$

Then $\exists\, \varphi_t : Q \to V$ s.t. $\varphi_0 = \text{incl}$

and $\varphi_t \underset{\varepsilon}{\approx} C^0$ incl

and $f : \mathcal{O}p(\varphi_1(Q)) \to W$ s.t

$$\left| J^r(f) - \sigma \big|_{\mathcal{O}p(\varphi_1(Q))} \right| < \varepsilon$$

Rk: $\mathcal{O}p(Q)$ = some arbitrarily small nbhd which might keep shrinking every time it appears

Furthermore: If σ is holonomic on $\mathcal{O}p(A)$ for $A \subseteq Q$ closed, then can take f s.t $J^r(f) = \sigma$ on $\mathcal{O}p(A)$ and φ_t = inclusion on A

relative

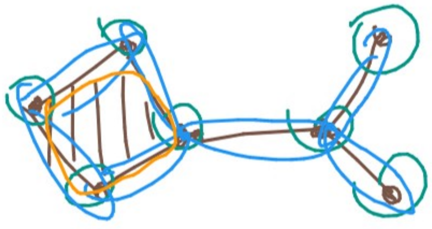
(parametric) If we have a family σ^τ of sections, $\tau \in \mathbb{D}^k$ s.t they are holonomic near $\partial\mathbb{D}^k$. Then we obtain $f^\tau, \boxed{\varphi_\varepsilon^\tau}$ as above so that $\sigma^\tau = J^r(f^\tau)$ $\forall \tau \in \partial\mathbb{D}^k$

Strategy of proof:

Step 1 go from Q to simpler pieces, namely cubes $[0,1]^k$

Since we prove a relative version

Q:



hol approx near vertices

then near the 1-cells $[0,1]$

then the 2-cell $[0,1]^2$

(role of σ)

→ enough to prove hol approx for $I=[0,1]^k$ rel ∂I

Step 2 (we will come back to)

define some "partial holonomicity"

hol. in certain directions but not in others

→ Inductive argument - adds another direction of holonomicity every time

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Step 1: Hol. Approx. Over a Cube $I=[0,1]$

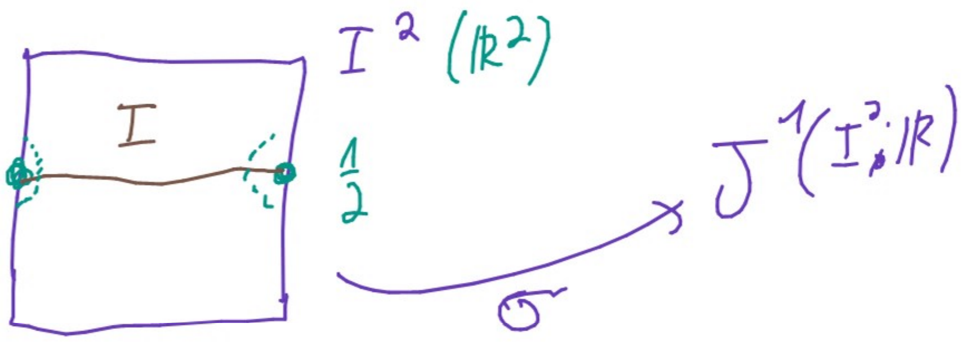
Thm: Say $I^k \subseteq \mathbb{R}^n$, $k<n$, Given

$F: \mathcal{O}p(I^k) \to J^r(\mathbb{R}^n, \mathbb{R}^q)$ s.t. F is hol. near ∂I^k. then $\forall \varepsilon$, $\exists$ ε-small-isotopy $\varphi_t: I^k \to \mathbb{R}^n$ fixed near ∂I^k and $f: \mathcal{O}p(\varphi_1(I^k)) \to \mathbb{R}^q$ s.t. (1) $J^r(f) = F$ on $\mathcal{O}p(\overset{\varphi_1(\partial I^k)}{\overset{\|}{\partial I^k}})$

(2) $J^r(f)$ is ε-close to σ on $\mathcal{O}p(\varphi_1(I^k))$

Special case $k=1$ $n=2$.

Think of $I \hookrightarrow I^2$ as $[0,1] \times \{\frac{1}{2}\}$



Suppose we are given a section σ

$\sigma : I^2 \to J^1(I^2;\mathbb{R})$ which is holonomic

on $\mathcal{O}p(\partial I \times \{\frac{1}{2}\})$ want to show

$\exists$ arbitrarily small isotopy $\varphi_t : I \to I^2$

s.t. $\varphi_0 = \text{incl}$ and a hol sect

$f : \mathcal{O}p(\varphi_1(I)) \to J^1(I^2;\mathbb{R})$

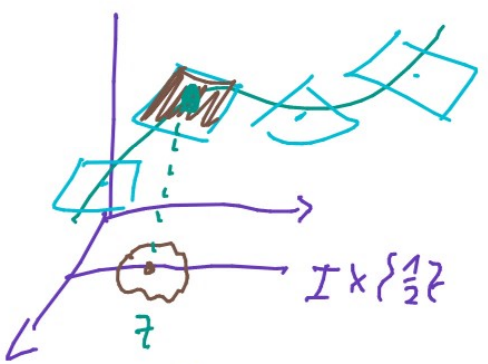
s.t $\begin{cases} |f-\sigma|_{C^0} < \varepsilon \\ f|_{\mathcal{O}p(\partial I \times \{\frac{1}{2}\})} = \sigma|_{\mathcal{O}p(\partial I \times \{\frac{1}{2}\})} \end{cases}$

<u>Proof</u>: at each point $z \in I \times \{\frac{1}{2}\}$

we can find $F_z : \mathcal{O}p(z) \to J^1(I^2,\mathbb{R})$,

s.t F_z holonomic & $F_z(z) = \sigma(z)$

F_z = linear ~~approx~~



<u>Note</u>: "F_z continuous in z"

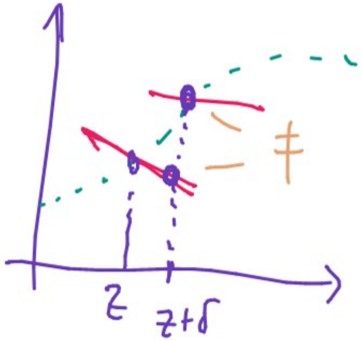
$F_\bullet : (I \times \{\frac{1}{2}\}) \times I^2 \to J^1(I^2,\mathbb{R})$

(defined in $\mathcal{O}p(\underbrace{\Delta}_{\text{graph of inc}})$)

<u>Problem</u>: they dont line up!

$$F_z(z+\delta) \neq F_{z+\delta}(z+\delta)$$

Sideways:



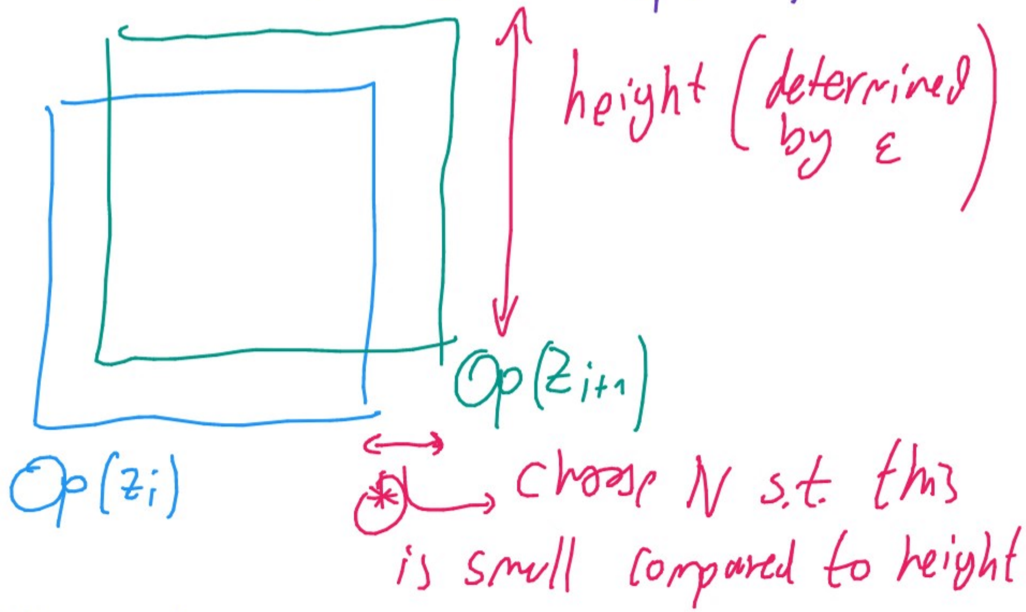
Need a way to glue them together

Take $N \gg 0$ (to be specified later)

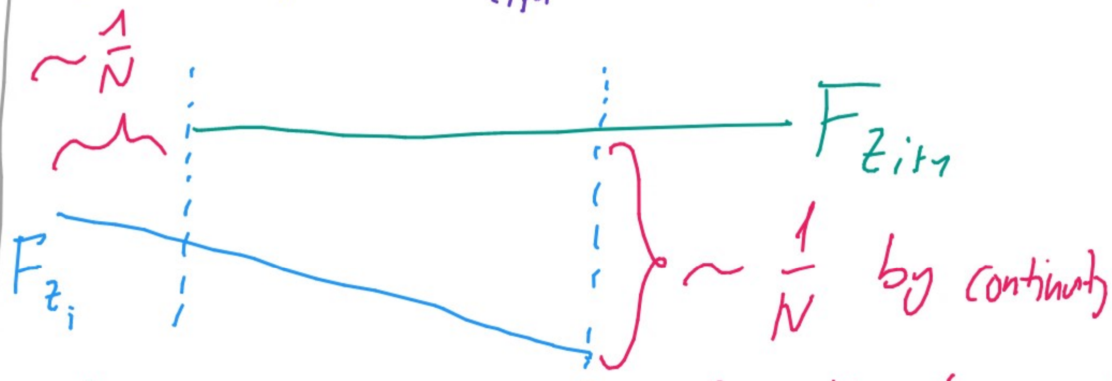
Pick $z_i = \frac{i}{N}$, recall that each F_z is defined $\mathcal{O}p(\{z\}\times\frac{1}{2})$

take them all to be small enough s.t. $\left|F_z - \sigma|_{\mathcal{O}p}\right| < \frac{\varepsilon}{10}$

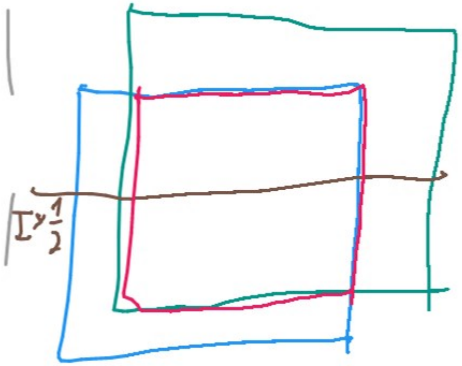
look at $\mathcal{O}p(z_i)$ and $\mathcal{O}p(z_{i+1})$:



How F_{z_i} and $F_{z_{i+1}}$ look like?

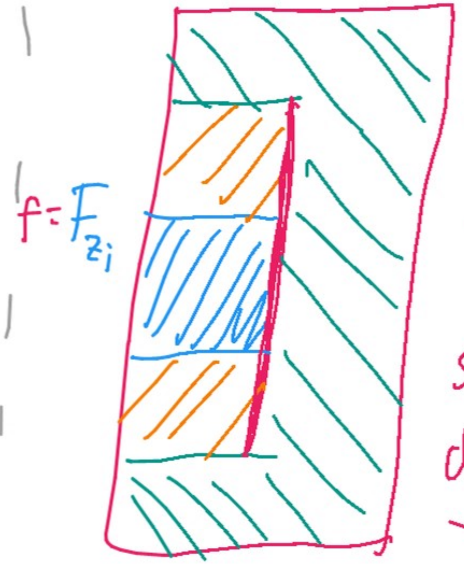


slope difference $\sim \frac{1}{N}$ (continuity w.r.t. J^1 sections)



restrict attention to the red part;

cut a slit in the middle



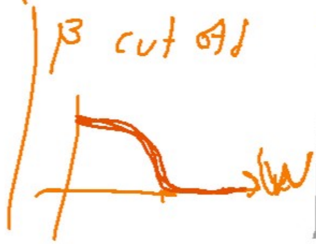
define f:

$f = F_{z_{i+1}}$

slit = line of discontinuity

in the orange part interpolate:

$f = \beta(w) F_{z_i} + (1-\beta(w)) F_{z_{i+1}}$



Indeed f is holonomic. in $\boxed{1}$

$$|J^1(f) - \sigma|_{C^0} < \varepsilon$$

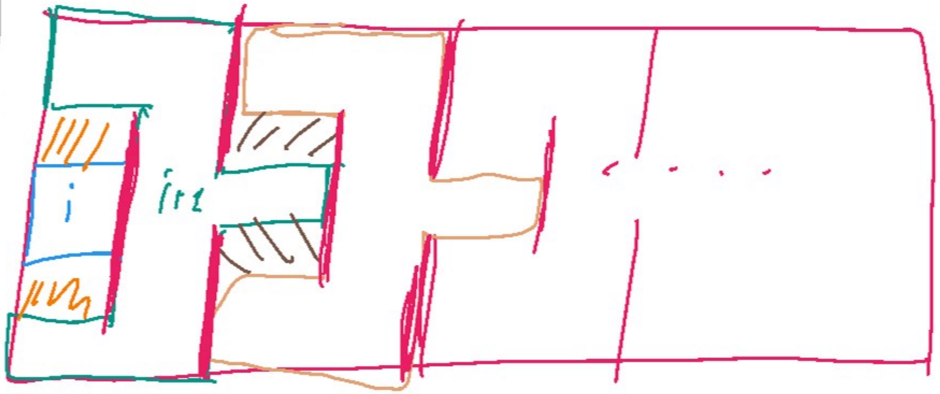
say $\sigma = (g, G)$

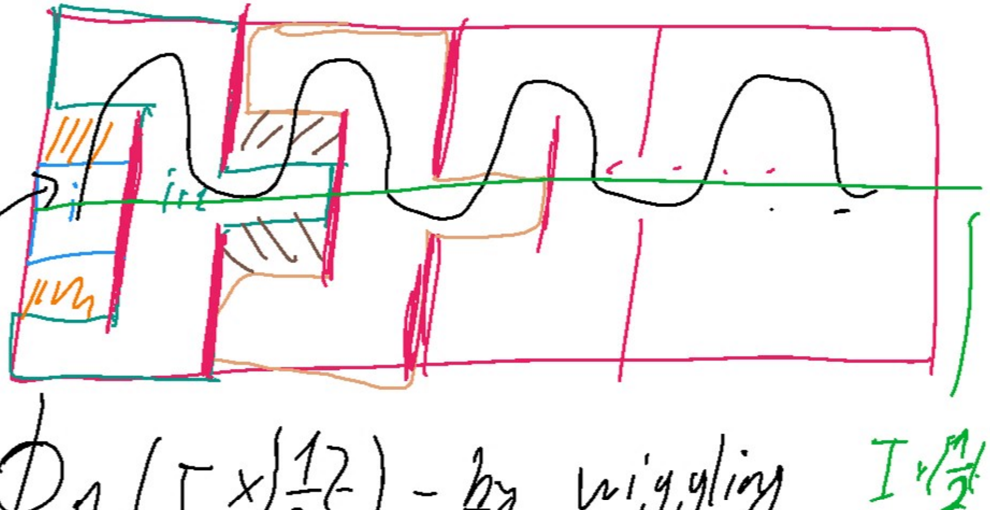
$f \approx g$ because $g \approx F_{z_i}$ $g \approx F_{z_{i+1}}$ and we interpolated

$df \approx G$ because $G \approx dF_{z_i}$, $G \approx dF_{z_{i+1}}$

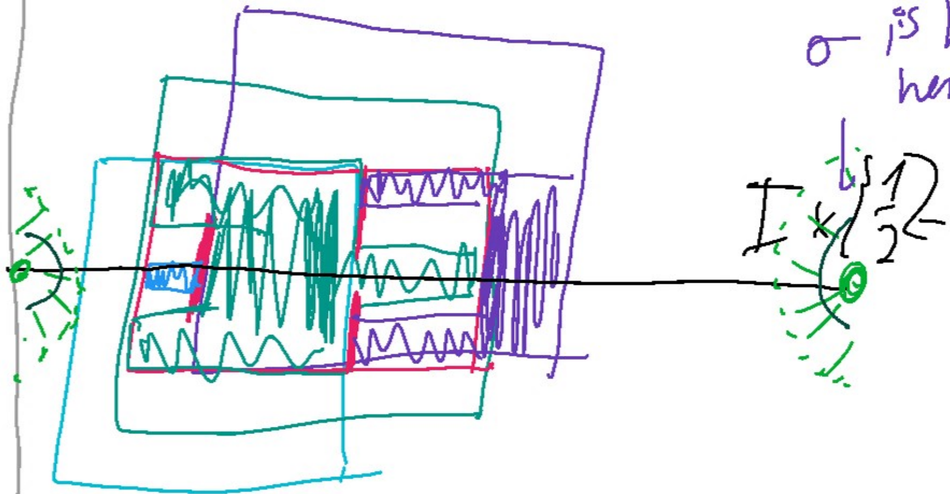
β' – constant – doesn't depend on N

$|F_{z_i} - F_{z_{i-1}}| \xrightarrow{N \to \infty} 0$ so by chain rule one checks that $df \approx G$





$\phi_1(I\times\{\tfrac12\})$ - by wiggling
one avoids the slits of discontinuity



and f is defined on $\mathcal{O}p(\phi_1(I\times\tfrac12))$

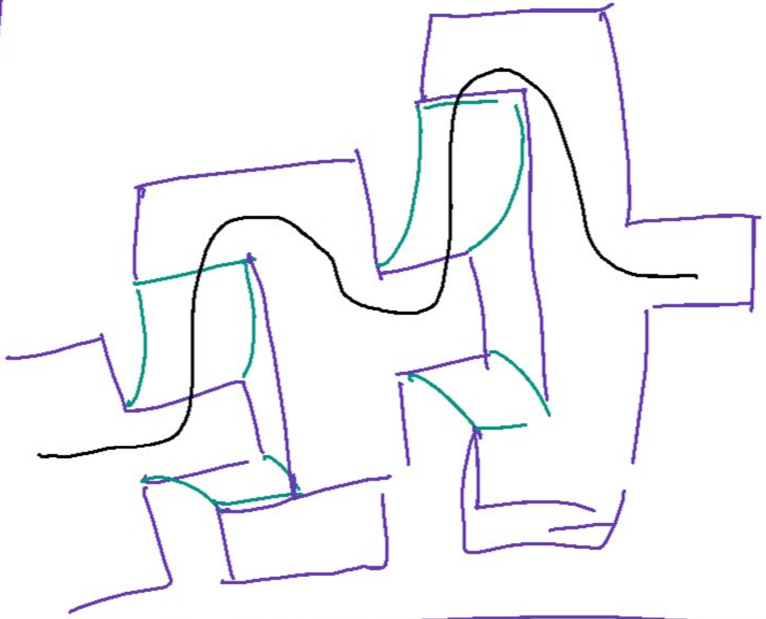
relative version: pick:

$$F_{z_0} = \sigma\big|_{\mathcal{O}p(z_0)}$$

$$F_{z_N} = \sigma\big|_{\mathcal{O}p(z_N)}$$

by assumption σ is holomorphic there

instead of picking linear approximation



Step 2 - general Case - Sketch

define suppose $X \to V$ fib. bundle and $\boxed{\pi : V \to B}$ is also fib bundle we say that a section

$$F : V \to J^r(X)$$

$$\left(\begin{array}{l} J^r(X) \text{ - jets of sections } V \to X \\ J^r(V,W) = J^r\left(\begin{array}{c} V\times W \\ \downarrow \\ V \end{array}\right) \quad X = V\times W \end{array} \right)$$

is fiberwise holonomic (w.r.t π)

if $\exists \tilde{F} : \mathcal{O}p\, \tilde{V} \to J^r(X)$

with $\tilde{V} = \{(v, \pi(v))\} \subset V \times B$

s.t. $\begin{cases} \tilde{F}_b(v) = F(v) & \text{for } v \in \pi^{-1}(b) \\ \tilde{F}_b(\cdot) \text{ is holonomic for fixed } b \end{cases}$

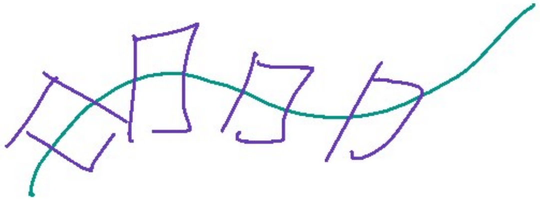
Note $\tilde{F}_b(v) \neq F(v)$ for v with $\pi(v) \neq b$

That is $F\big|_{\pi^{-1}(b)}$ extends to a hol section in $\mathcal{O}p(\pi^{-1}(b))$

Ex: $\mathbb{R}^2 \longrightarrow \mathbb{R}$

proj on 1st coord

Section of 1st jet bundle



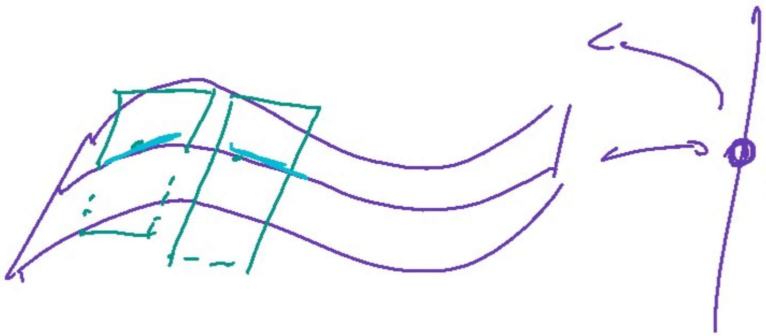
$\times \mathbb{R}$

fib dir

proj to bound.

Fibwise hol, not hol

$\pi^{-1}(b)$



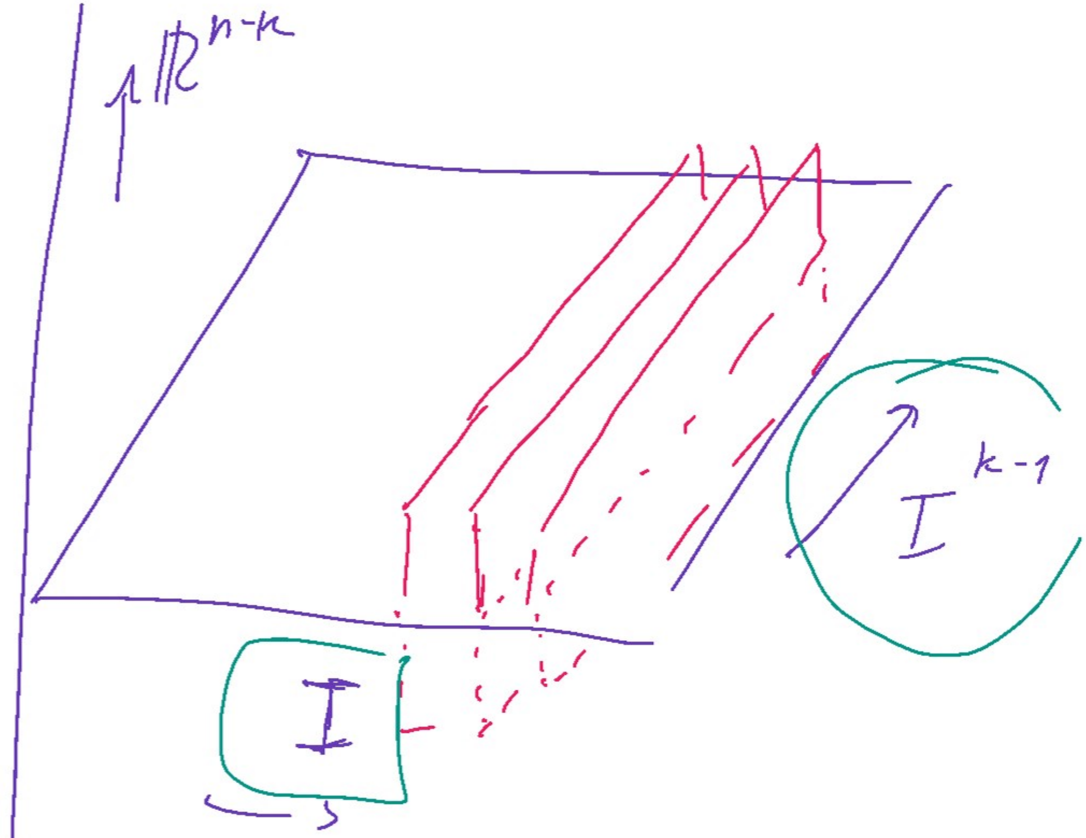
$\mathbb{R}$

Idea $I^k \subset \mathbb{R}^n$

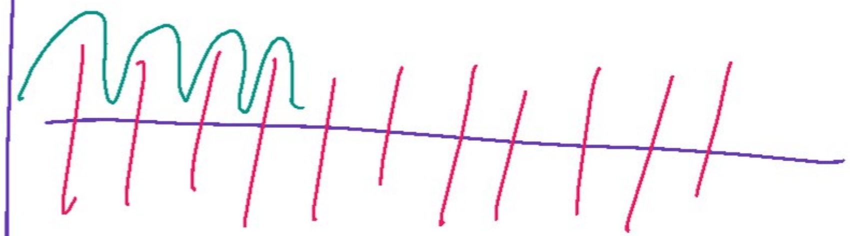
consider $I^k = \underbrace{I^{k-l} \times I^l}$

$$\begin{tikzcd} I^{k-l} \times I^l \arrow[d, "\pi"] \arrow[dd, bend right, "\pi'"'] \\ I^l \\ I^{l-1} \end{tikzcd}$$

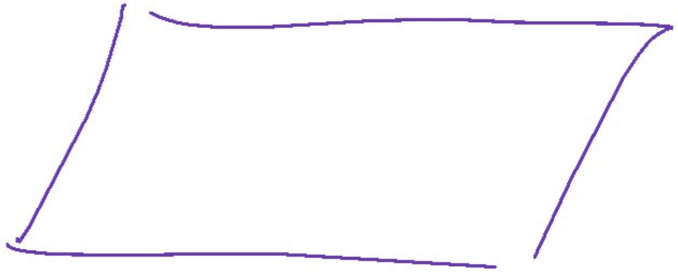
If I start with σ which is fib-hol w.r.t π upgrade to fibwise hol w.r.t π'



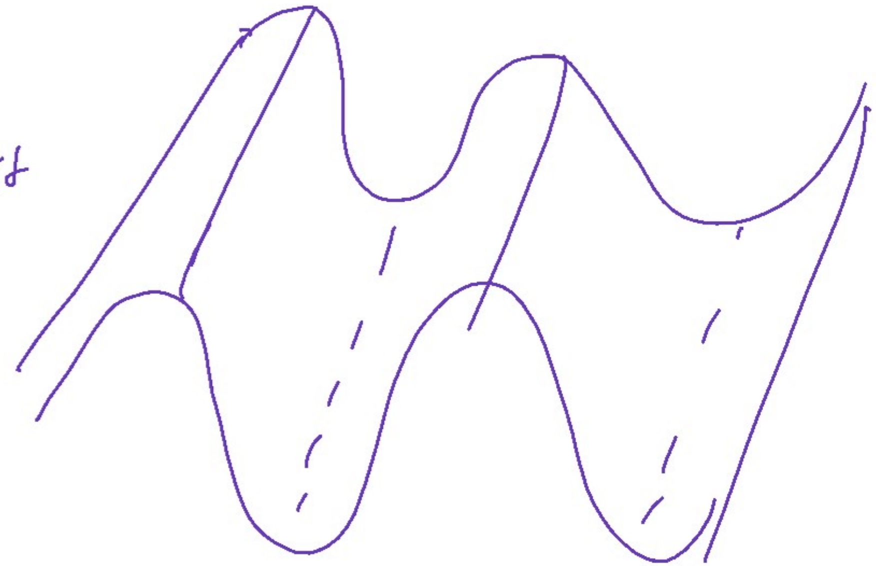
from the side



0th



1st



2nd