An Introduction to the H-Principle in Geometry & Symplectic Topology

Talk 1

Consider the ODE $\left(\frac{dy}{dx}\right)^2 + e^y = 0$

$y : \mathbb{R} \to \mathbb{R}$

does it have solutions? NO!

$u^2 + e^y = 0 \qquad u, y : \mathbb{R} \to \mathbb{R}$

this eqn has no solutions

"decoupled the derivative from f"

What about the reverse?

does solvability of the alg eqn tell us something about the diff eqn

In general NO:

$\left\{ \begin{array}{l} y' + y^2 = 0 \\ y' \neq 0 \end{array} \right\}$ – NO solutions

$y = \frac{1}{x+C}$

$y : \mathbb{R} \to \mathbb{R}$

$\left\{ \begin{array}{l} u + y^2 = 0 \\ u \neq 0 \end{array} \right\}$ – plenty

---

Gromov: In many underdetermined PDEs (or PDRs) Answer is YES

Thm (Hirsch-Smale)

Given V^n, W^q mnflds with $n<q$

If $\exists$ a bundle map $F: TV \to TW$

(covering some $f: V \to W$)

which is fiberwise injective

then $\exists\ g: V \to W$ s.t $Dg: TV \to TW$

is fib. inj. i.e. g is an immersion

(In fact more is true

$$\{\text{Immersions}\} \xhookrightarrow[\text{w.h.e}]{} \left\{\begin{matrix}\text{fib inj} \\ \text{bundle maps}\end{matrix}\right\}$$

iso on π_n

An immersion $g: V \to W$ is a map

s.t. $\operatorname{rank} Dg = n$

Can be expressed as nonvanishing of (at least one) determinants

$\Rightarrow$ a diff relation

each g gives a bundle map

$(g, Dg): TV \to TW$

In a general bundle map (f, F)

F need not equal Df

Jet Spaces & Jet Bundles

Let V, W be manifolds

The space of jets over a point $(v,w) \in V \times W$ is an arbitrary nbhd of $\{v\}$

$$\left\{ \text{local maps } \boxed{\mathcal{O}p\{v\}} \longrightarrow W \; , \; f(v) = w \right\}$$

$f_1 \sim f_2$ if their r-th Taylor polynomials agree (in some chart hence in all charts)

$J^r_{(v,w)}(V \times W) \cong$ q-tuples of polynomials of deg $\leq r$ in n variables

non canonically

They assemble to a fiber bundle over $V \times W$. $J^r(V,W)$ - Jet bundle

transition functions between different charts - determined by chain rule

---

Ex $J^1(V,\mathbb{R}) = T^*V \times \mathbb{R}$

differential value

In general $J^1(V,W) = \mathrm{Hom}(TV, TW)$

$$J^2(\mathbb{R}^n, \mathbb{R}) = \underbrace{\mathbb{R}^n}_{\text{src}} \times \underbrace{\mathbb{R}}_{\text{tgt}} \times \underbrace{\mathbb{R}^n}_{\text{1-jets}} \times \underbrace{\mathbb{R}^{n(n+1)/2}}_{\text{2-jets}}$$

$$\begin{array}{c} J^1(V,W) \\ \downarrow \uparrow J^1(f) \\ V \end{array}$$

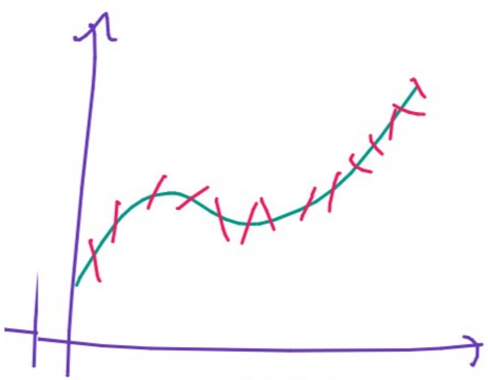
any $f: V \to W$ gives a section

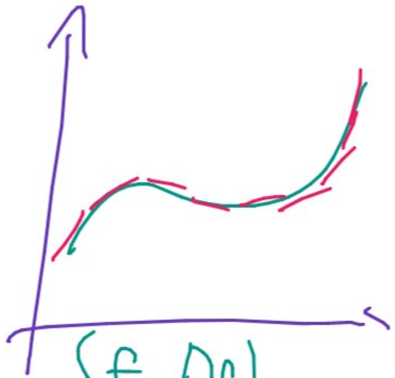
$J^1(f) = (f, Df)$

Note that most sections $\sigma = (g, G)$ do not satisfy $G = Dg$

<u>Def</u>: A differential relation of order r is a subset $R \subseteq J^r(V, W)$

How to picture section of $J^1(\mathbb{R}, \mathbb{R})$





<u>Def</u> A <u>formal solution</u> of R to be any section σ with image in R

<u>Def</u> A <u>holonomic section</u> σ is a section of the form $\sigma = J^1(f)$ for some $f : V \to W$

<u>Def</u> A <u>genuine solution</u> is a holonomic formal solution

---

note $Sol(R) \hookrightarrow Sol^f(R)$

<u>Def</u>: (1) we say R satisfies an <u>h-principle</u> if any formal solution is homotopic (through formal solutions) to a genuine solution

(2) a parametric h-principle holds, if $i : Sols(R) \hookrightarrow Sols^f(R)$ is a weak htpy equivalence

In particular if two genuine solutions are homotopic via formal sols then they are homotopic via genuine solutions

---

<u>Thm</u> [Gromov 69']

Let V be an <u>open manifold</u>,

let $R \subset J^r(V, W)$ be a diff relation

s.t.
- R is open
- R is $Diff(V)$ - invariant

then a parametric h-principle holds for R

<u>Cor:</u> Hirsh - Smale

Immersion $\Rightarrow$ some det $\neq 0$

can be expressed as a diff relation in $J^1(V, W)$

$Diff(V)$ - invariant

Open - ($\neq 0$ open condition)

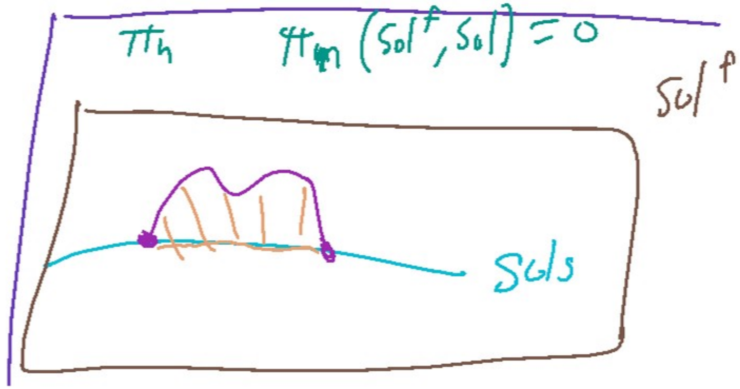
Smale Hirsh for open V

<u>Closed</u> V - "micro extension"

$\dim V < \dim W$ due to normal bundle and std nbhd theorems

Immersion of $V \Leftrightarrow$ Immersion of $V \times (-\varepsilon, \varepsilon)$

open



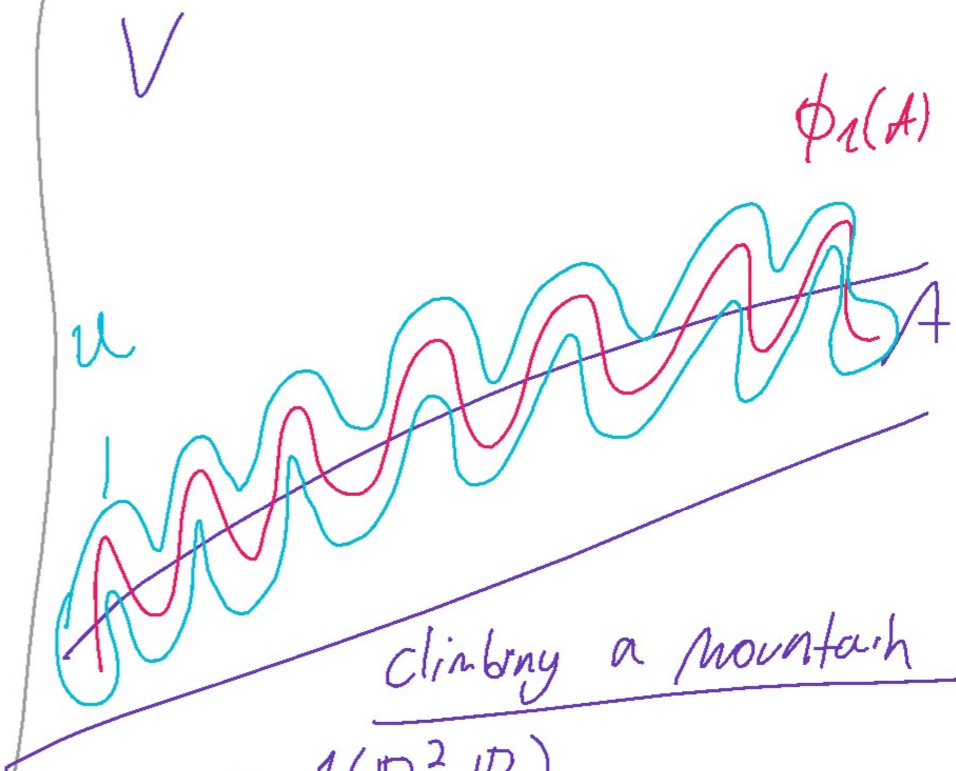
How to prove Gromov's Thm?

Theorem (holonomic approximation)

[Eliashberg - Mishachev]

let $F : V \to J^r(V,W)$ be a section and let $A \subseteq V$ be a submanifold (or some stratified complex) of codim ≥ 1

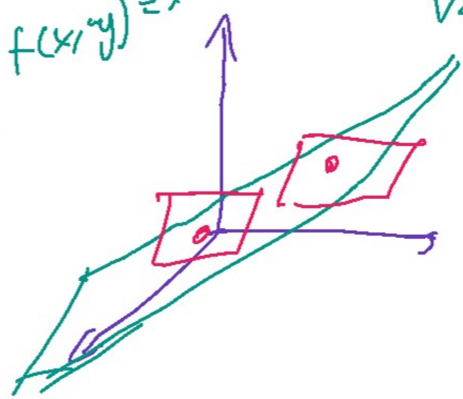
Then there exists a C^0-small isotopy $\phi_t : V \to V$, a nbhd $\mathcal{U} \supseteq \phi_1(A)$ and a holonomic section $f : \mathcal{U} \to W$ s.t $J^r(f) \approx_{C^0} F|_{\mathcal{U}}$

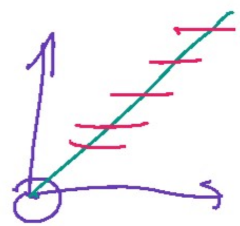


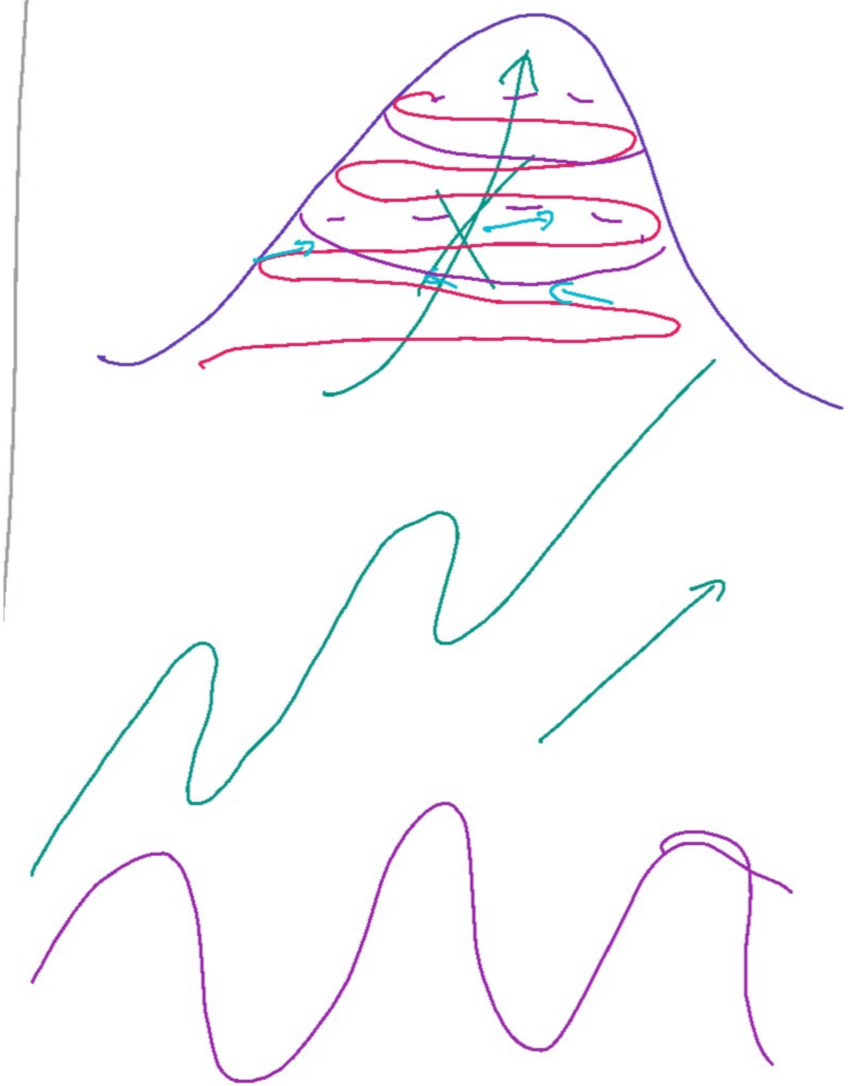
climbing a mountain

Take $J^1(\mathbb{R}^2, \mathbb{R})$ as section $F = (x, 0, 0)$

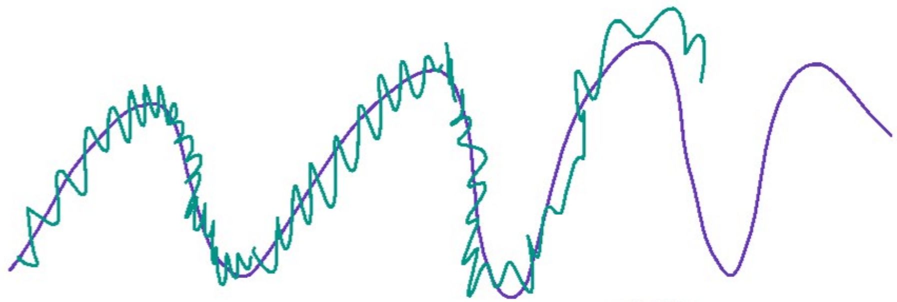
val ↗ ↘ "∂_x" "∂_y"

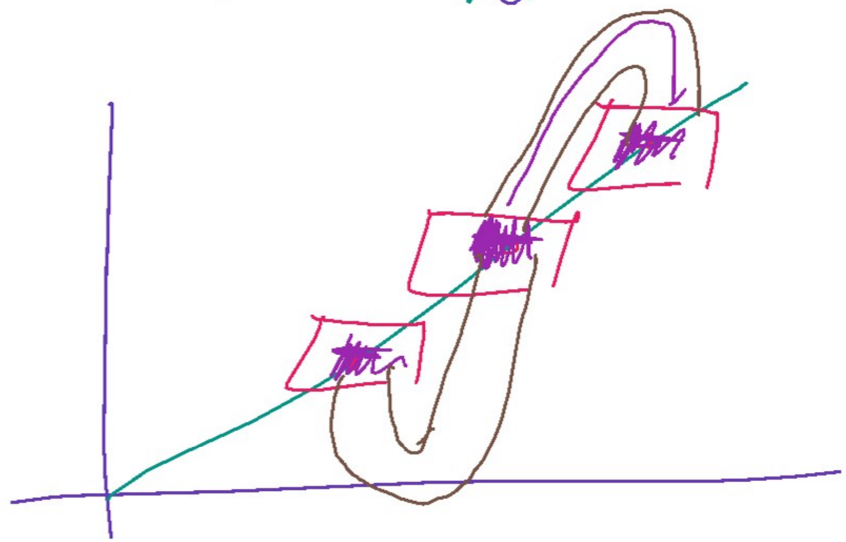






Proof is an induction on values

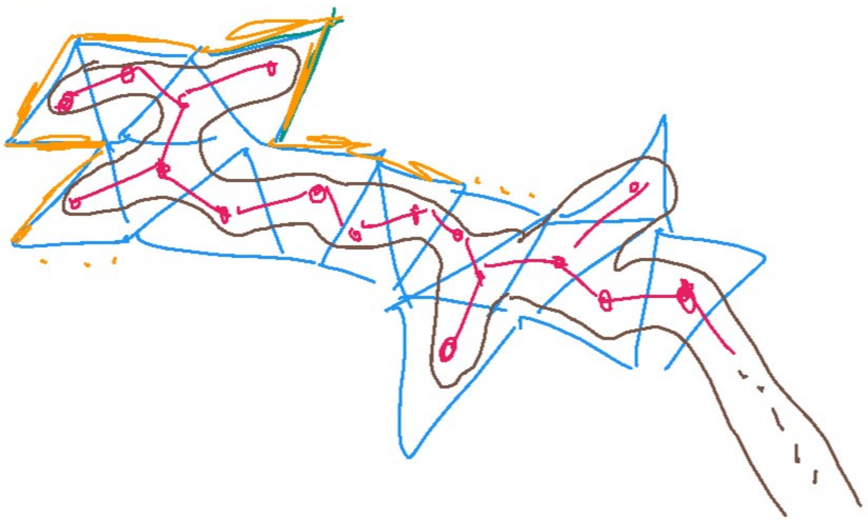




Proof "holonomic approx ⟹ Gromov"

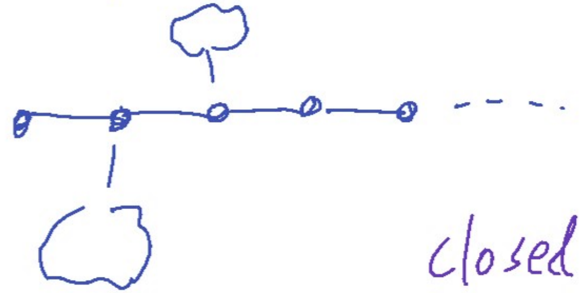
Lemma: If M is an open manifold then there exists a set $A \subseteq M$ of codim ≥ 1 s.t. there exists an isotopy ψ_t of M, relative to A where $\psi_0 = Id$, $Im\ \psi_1 = Op(A)$

Idea of proof: (1) Triangulate
(2) consider the dual graph



(3) cover this dual graph by a union of "special trees"

Special tree is a one sided infinite ray with some finite trees attached



Special trees have nbhds which are diffeomorphic $\mathbb{D}^n \setminus \{bdry\ pt\}$ they retract to $S^{n-1} \setminus \{pt\}$

⟹ get retraction to a nbhd of some codim ≥ 1 complex let $F: V \to J^r(V,W)$ with image in R. let A to be the codim ≥ 1 skeleton on which V retracts

by holonomic approx $\exists\, U \supseteq \phi_1(A)$ with ϕ isotopy and a hol section $J^r(f)$ s.t. $J^r(f) \simeq_{C^0} F|_U$

since R open $\Rightarrow J^r(f)$ also has image in R

take a diffeo $\psi_1 : V \to U$

the desired section is

$\tilde{f} = f \circ \psi_1 : V \to W$

still in R because R is $\mathrm{Diff}(V)$ invariant.

How to get the parametric version!

There is a parametric version of holonomic approx where one approximates families of sections parametrized by some set (be it a path, a disc, a sphere, etc)

Lets Classify Immersions!

---

we have seen that

$$\pi_0(\text{Immersions}) = \pi_0\left(\begin{array}{c}\text{formal}\\ \text{immersions}\end{array}\right)$$

start with $S^1 \looparrowright \mathbb{R}^2$

what is a formal immersion?

fib inj $\quad TS^1 \longrightarrow T\mathbb{R}^2$

$$\begin{array}{ccc} TS^1 & \longrightarrow & T\mathbb{R}^2 \\ \wr\wr & & \wr\wr \\ S^1 \times \underline{\mathbb{R}^1} & \xrightarrow{\text{inj}} & \mathbb{R}^2 \times \underline{\mathbb{R}^2} \end{array}$$

a formal immersion a pair f, F

lives in a contractible space

$\boxed{f: S^1 \to \mathbb{R}^2}$ $\quad F: S^1 \to \underbrace{\left\{ \begin{matrix} \text{inj linear} \\ \mathbb{R} \to \mathbb{R}^2 \end{matrix} \right\}}_{\mathbb{R}^2 \setminus 0}$

$\pi_0(\text{Immersions})$

$\parallel$

So $\pi_0(\text{formal } S^1 \looparrowright \mathbb{R}) =$

$$\underbrace{[S^1 : \mathbb{R}^2 \setminus 0]}_{\text{htpy classes of } S^1 \to \mathbb{R}^2 \setminus 0} = [S^1 : S^1] = \pi_1(S^1) = \boxed{\mathbb{Z}}$$

$\parallel$

htpy classes of maps $S^1 \to S^1$

$S^2 \looparrowright \mathbb{R}^3$

TS^2 not parallelizable

<u>Workaround:</u> immerse $S^2 \times (-\varepsilon, \varepsilon)$

This is a 3-mnfld (<u>Orientable</u>)

they are all known to be Parallelizable (Stiefel's theorem)

even better $S^2 \times (-\varepsilon, \varepsilon) \subseteq \mathbb{R}^3$
open set

a formal immersion f, F

again f lives in a contractible space

$$F: S^2 \times (-\varepsilon, \varepsilon) \longrightarrow \underbrace{\left\{ \begin{matrix} \text{inj linear} \\ \mathbb{R}^3 \to \mathbb{R}^3 \end{matrix} \right\}}_{GL_{\mathbb{R}}(3)} \underset{\text{htpy}}{\simeq} SO_3(\mathbb{R}) \cong \mathbb{RP}^3$$

π_0(space of formal immersions $S^2 \looparrowright \mathbb{R}^3$) $=$

$$= [\underbrace{S^2 \times (-\varepsilon, \varepsilon)}_{\text{oriented}}, \mathbb{R}P^3] =$$

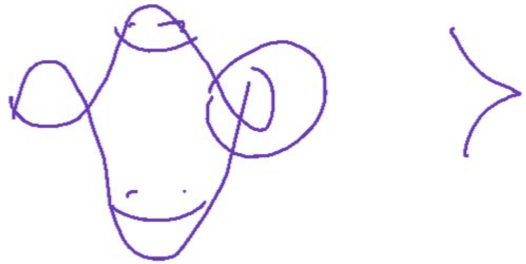
$$= \pi_2(\mathbb{R}P^3) = \pi_2(S^3) \boxed{= 0}$$

Univ cover

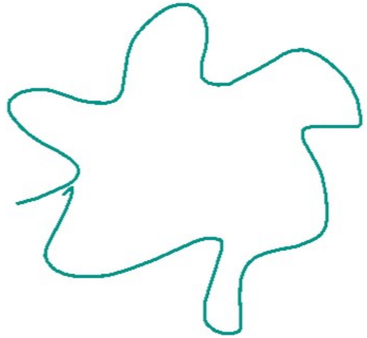
π_0(Immersions $S^2 \looparrowright \mathbb{R}^3$) $= 0$

$\Downarrow$

all Immersions of $S^2 \looparrowright \mathbb{R}^3$ are homotopic!



not immersed any more



$(-\varepsilon, \varepsilon) \times S^2 \hookrightarrow \mathbb{R}^3$

$(-\varepsilon, \varepsilon) \times S^2 \hookrightarrow \mathbb{R}^3 \xrightarrow{\text{antipodal}} \mathbb{R}^3$