Convexity, dynamical convexity and the Ruelle invariant

Background:

- A contact manifold (Y,α) is an odd-dim mfld Y^{2n-1} with a 1-form α, called contact form, such that

  $\alpha\wedge(d\alpha)^{n-1}$ is a volume form on Y.

  $\xi=\ker\alpha\subset TY$ is a hyperplane distribution called contact distribution.

  $\ker(d\alpha)\subset TY$ is a rank-1 distribution. The Reeb vf $R\in\ker(d\alpha)$ is defined by $\alpha(R)=1$.

  $\rightsquigarrow$ flow φ^t is called "Reeb flow", gives the char. foliation tangent to $\ker(d\alpha)$.

Main example:

$X\subset\mathbb{R}^{2n}$ a star-shaped domain, such that the radial vf ∂_r is transverse to $\partial X=:Y$.

Given the std $\omega_0=\sum dx_i\wedge dy_i$, the contact form α on Y is: $\alpha_0=\omega_0\left(\frac{1}{2}r\partial_r,\cdot\right)\Big|_{TY}=\frac{1}{2}\sum_{i=1}^{n}(x_i\,dy_i-y_i\,dx_i)\Big|_{TY}$.

Notice that in this example $Y=S^{2n-1}$.

• A contact mfld (Y,α) is called convex if $\exists$ convex star-shaped domain $X \subset \mathbb{R}^{2n}$ and a strict contactomorphism $(Y,\alpha) \cong (\partial X, \alpha_0|_{T\partial X})$.

i.e. diffeo $\varphi: Y \xrightarrow{\sim} \partial X \cong S^{2n-1}$ s.t. $\varphi^*(\alpha_0|_{T\partial X}) = \alpha$.

Viterbo Conjecture: (Y,α) convex, then $sys(Y,\alpha) \leq 1$

where $sys(Y,\alpha) := \dfrac{T_{min}^n}{(n-1)!\, vol(Y,\alpha)}$ ← min period of a closed Reeb orbit.

Natural question: Is there an "intrinsic" / "dynamical" discription of "convex"?

Theorem (Hofer - Wysocki - Zehnder): If (Y^{2n-1},α) is [non-deg] convex then the CZ index of every periodic Reeb orbit is at least $n+1$.

Def. (Y,α) is dynamically convex if the CZ index of every periodic Reeb orbit is $\geq n+1$.

$$\boxed{\text{Convex} \Rightarrow \text{dyn. convex. Is the converse true?}}$$

Theorem (Abbondandolo - Bramham - Hryniewicz - Salamão): $\exists\, (Y^3,\alpha) \cong (\partial X, \alpha_0|_{\partial X})$ for X starshaped, s.t. $sys(Y^3,\alpha) > 1$.

This suggests that dyn convex $\neq$ convex.

Theorem (Chaidez - Edtmair): $\exists (Y^3, \alpha) \cong (\partial X, \alpha_0|_{\partial X})$ for star shaped $X \subset \mathbb{R}^{2n}$, that is dynamically convex but not convex.

The key ingredient in the proof is called "Ruelle invariant", denoted $Ru(Y, \alpha) \in \mathbb{R}$.

Roughly speaking, it is the integral over Y of a time-averaged rotation number that measures the degree to which Reeb trajectories twist counter-clockwise around each other.

Main steps in the proof:

① Normalize: $ru(Y, \alpha) := \dfrac{Ru^2(Y, \alpha)}{vol(Y, \alpha)}$ Invariant under rescaling!

② $\exists$ $C > c > 0$ s.t. for every convex (Y^3, α),

$$c < ru(Y, \alpha) \cdot sys(Y, \alpha) < C$$

Idea: • Show this holds for $(Y, \alpha) = (\partial E, \alpha_0|_{\partial E})$ where E is any ellipsoid.

• Use John's ellipsoid theorem and sandwich any convex domain by an ellipsoid and its rescaling: $E \subset X \subset 4 \cdot E$.

- The vol and sys. ratio are monotone.

Problem: Ruelle inv. is not monotone.

This is resolved using a formula relating 2^{nd} fund. form and the local rotation of Reeb flow.

(3) For every $\varepsilon > 0$, construct dyn. convex (Y, α) in the spirit of ABHS s.t.:

$\mathrm{vol}(Y, \alpha) = 1$, $\mathrm{sys}(S^3, \alpha) \geq 1 - \varepsilon$, $\mathrm{ru}(Y, \alpha) \leq \varepsilon$

Idea: "Open book methods" cf ABHS. Namely:

Out of a Hamiltonian disk map $\phi : \mathbb{D} \to \mathbb{D}$, $\exists$ way to construct a contact form α on S^3.

Say something about open book decomp?

properties of ϕ $\longleftrightarrow$ properties of (S^3, α).

So, construct ϕ s.t. (S^3, α) is dyn convex & satisfies the above estimates. This map ϕ is obtained by composing ϕ_H = rotation by $2\pi(1 + \frac{1}{n})$ and ϕ_G = supported in disj union of disks $D_i \subset \mathbb{D}$ and rotates most of each disk D_i clockwise by an angle $\lesssim 4\pi$.

The Ruelle invariant:

$\boxed{1}$ The rotation number:

In this section we define and study rot. no. $q:\widetilde{Sp}(2)\to\mathbb{R}$.

Notice $Sp(2)\overset{h.e.}{\approx}U(1)\approx S^1$

Def. Let $A\in Sp(2)$ and $\tilde{A}\in\widetilde{Sp}(2)$ a lift represented by a path $\{A_t\}_{t\in[0,1]}$ in $Sp(2)$ with $A_0=\mathbb{1}$, $A_1=A$.

* For $0\neq v\in\mathbb{R}^2$ denote by $2\pi\rho(\tilde{A},v)$ the angle in which $\{A_t v\}_{t\in[0,1]}$ rotates, namely:

  $\Theta:[0,1]\to S^1$, $\Theta(t):=\arg(A_t v)$, $\tilde{\Theta}:[0,1]\to\mathbb{R}$ the lift.

  Then $\boxed{2\pi\rho(\tilde{A},v)=\tilde{\Theta}(1)-\tilde{\Theta}(0)}$.

* The rotation number q is a homogenization of ρ:

$$\boxed{q(\tilde{A}):=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\rho(A^{k-1}v)}$$

Theorem: $q(\tilde{A})$ is independent of v.

Lemma: Fix $v\in\mathbb{R}^2$, then $\rho(\cdot,v):\widetilde{Sp}(2)\to\mathbb{R}$ is a quasimorphism:

$$|\rho(\tilde{A}\tilde{B},v)-\rho(\tilde{A},v)-\rho(\tilde{B},v)|<1.$$

Proof. Write $2\pi\rho(\tilde{A}, v) = \tilde{\Theta}_{\tilde{A},v}(1) - \tilde{\Theta}_{\tilde{A},v}(0) = \tilde{\Theta}_{\tilde{A},v}(1) - \arg(v)$.

The path $\tilde{A}\cdot\tilde{B}$ is homotopic to the concatenation of $\tilde{A}$ and $\tilde{B}$, hence:

$$\tilde{\Theta}_{\tilde{A}\tilde{B},v}(1) = \tilde{\Theta}_{\tilde{A},B,v}(1) - \underbrace{\tilde{\Theta}_{\tilde{A},B,v}(0)}_{=\arg(B,v)} + \tilde{\Theta}_{\tilde{B},v}(1).$$

Therefore:

$$|\rho(\tilde{A}\tilde{B}, v) - \rho(\tilde{A}, v) - \rho(\tilde{B}, v)| = \frac{1}{2\pi}|\tilde{\Theta}_{\tilde{A}\tilde{B},v}(1) - \tilde{\Theta}_{\tilde{A},v}(1) - \tilde{\Theta}_{\tilde{B},v}(1) + \arg(v)|$$

$$= \frac{1}{2\pi}|\tilde{\Theta}_{\tilde{A},B,v}(1) - \arg(B,v) - \tilde{\Theta}_{\tilde{A},v}(1) + \arg(v)|$$

$$= |\rho(\tilde{A}, B, v) - \rho(\tilde{A}, v)| \leq \max_{w,v\in S^1} |\rho(\tilde{A}, w) - \rho(\tilde{A}, v)| =: (*)$$

Let us show that the latter is < 1. Fix $w, v \in S^1$ and denote $d(t) := \frac{1}{2\pi}|\tilde{\Theta}_{\tilde{A},w}(t) - \tilde{\Theta}_{\tilde{A},v}(t)|$. We want $d(1) < 1$.

Notice $d(0) = \frac{1}{2\pi}|\arg(w) - \arg(v)| < 1$. Thus, if $d(1) \geq 1$ $\exists$ t_0 with $d(t_0) = 1 \Rightarrow \arg(A_{t_0} w) = \arg(A_{t_0} v)$ ($\arg$ is mod 2π)

$\Rightarrow \exists\, c \in \mathbb{R}$ s.t. $A_t w = c \cdot A_t v \xRightarrow[A_t^{-1}]{} w = c\cdot v \xRightarrow[w,v\in S^1]{} w = v$

$\Rightarrow d(t) \equiv 0$ in contrad.

We conclude that $d(t) < 1$ for all t and in particular $(*) < 1$ which finishes the proof. $\blacksquare$

Proof of thm: We saw $\rho(\cdot, v)$ is a quasimorphism. The rotation number q is the homogenization of ρ $\Rightarrow$ q is a homogeneous quasi morphism.

Theorem [Simon-Salamon]: $\exists!$ homogeneous q.m. $\widetilde{Sp}(2) \to \mathbb{R}$ that sends $\{e^{2\pi i L t}\}_{t \in [0,1]} \longmapsto L \quad \forall L \in \mathbb{R}$.

By the above thm q is unique $\to$ indep. of v. $\square$

[2] Defining the Ruelle invariant:

Def. (Upshot) Let $(Y = S^3, \alpha)$ contact manifold.

- Choose a trivialization $\tau : \xi \xrightarrow{\sim} Y \times \mathbb{R}^2 = S^3 \times \mathbb{R}^2$
- Linearized Reeb flow $\Phi_\tau : S^3 \times \mathbb{R} \longrightarrow Sp(2)$
- Lift: $\widetilde{\Phi}_\tau : S^3 \times \mathbb{R} \to \widetilde{Sp}(2)$.
- Rotation density: $\mathrm{rot} : Y \longrightarrow \mathbb{R}$ def by: $\forall y \in Y$

$$\boxed{\mathrm{rot}(y) := \lim_{T \to \infty} \frac{q(\widetilde{\Phi}_\tau(y,T))}{T}}$$

- Ruelle invariant: $\boxed{\mathrm{Ru}(Y,\alpha) := \int_Y \mathrm{rot}\ \alpha \wedge d\alpha.}$

! Need to show this is well defined!

Proposition: The 1-parameter family $f_T : Y \to \mathbb{R}$ given by $f_T(y) := \dfrac{q \circ \widetilde{\Phi}_\tau(y,T)}{T}$ converges in L^1 to a function $f : Y \to \mathbb{R}$ depending on α and τ.

The proof uses the following thm from measure theory:

Kingman's subadditive ergodic thm: Let φ be a measure preserving trans. on $(Y, \mu = \alpha\wedge d\alpha)$ and let g_k be a sequence of L^1 functions s.t. $g_{k+\ell} \leq g_k + (\varphi^k)^* g_\ell$ (subadditivity).

Then, $\lim\limits_{k\to\infty} \frac{g_k(x)}{k} =: g(x) \geq -\infty$ for μ-a.e. x and g is ϕ-invariant. Moreover, if φ is ergodic then g is constant.

Proof of Prop. Consider $g_T := T\cdot f_T + \underset{\text{quasimorphism constant.}}{\underbrace{1}} = q\circ\tilde{\Phi}_\tau(\cdot, T) + 1$.

Since $\tilde{\Phi}_\tau(y, S+T) = \tilde{\Phi}_\tau(\varphi^S y, T)\cdot\tilde{\Phi}_\tau(y, S)$, we have

$$g_{S+T} = q\circ\tilde{\Phi}_\tau(\cdot, S+T) + 1 \leq q\circ\tilde{\Phi}_\tau(\cdot, S) + q\circ\tilde{\Phi}_\tau(\varphi^S(\cdot), T) + 2$$

quasimorphism property for q.

$$= g_S + (\varphi^S)^* g_T \quad =:(*)$$

This proves the subadditivity property and by Kingman's thm $\exists \lim\limits_{T\to\infty} f_T = \frac{g_T}{T} =: g \geq -\infty$ almost everywhere. Now we want to show $g > -\infty$ a.e. and $g\in L^1$.

Similarly to the above computation, $g_{S+T} \geq g_S + (\varphi^S)^* g_T - 2$. $(**)$

In part. $g_n \geq g_1 + \varphi^* g_{n-1} - 2 \geq \cdots \geq g_1 + \varphi^* g_1 + \cdots + (\varphi^{n-1})^* g_1 - 2n$

and $\frac{g_n}{n} \geq \min\limits_Y g_1 - 2 > -\infty$.

Moreover, $(**)$ for $T = S+n$, $n\in\mathbb{N}$, $S\in[0,1]$, we have:

$$\int_Y g_T\, \alpha\wedge d\alpha \geq \sum_{k=0}^{n-1}\int_Y (\varphi^k)^* g_1\, \alpha\wedge d\alpha + \int_Y (\varphi^n)^* g_S\, \alpha\wedge d\alpha - 2T \geq$$

φ is measure preserving

$$-(n+1)\cdot\min_{S\in[0,1]}\int_Y g_S\, \alpha\wedge d\alpha - 2T \geq -AT$$

for any $A \geq 2$, $-\min_{s \in [0,1]} \left\{ \int_Y g_s \, \alpha \wedge d\alpha \right\}$ finite.

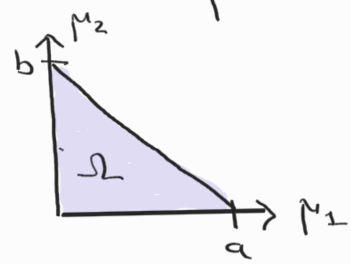
We conclude that g is finite a.e. and is indeed L^1.

Here we're using Vitali convergence theorem, which is a "stronger" version of the dominated convergence theorem. $\blacksquare$

$\boxed{3}$ Computing Ru for ellipsoids.

Consider $Y = \partial E(a,b) \subset \mathbb{C}^4$ boundary of the ellipsoid

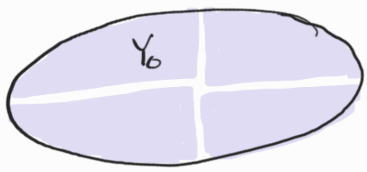
$$E(a,b) := \left\{ z \;\middle|\; \pi \left(\frac{|z_1|^2}{a} + \frac{|z_2|^2}{b} \right) \leq 1 \right\} \longrightarrow$$



$\mu(z) = \pi(|z_1|^2, |z_2|^2)$

We want to show

$$\mathrm{Ru}(Y = \partial E, \alpha = \alpha_0|_{\partial E}) = a + b.$$



On $Y_0 := \{ z \in Y \mid z_1, z_2 \neq 0 \} \subset Y$:

- Coordinates: $\mu_i = \pi |z_i|^2$, $\theta_i = \arg(z_i)$
- $\alpha_0 = \frac{1}{2\pi} (\mu_1 d\theta_1 + \mu_2 d\theta_2)$
- $T_z Y = \mathrm{span}(\partial_{\theta_1}, \partial_{\theta_2}, a\partial_{\mu_1} - b\partial_{\mu_2})$
- $\xi_z = \mathrm{span}(V := \mu_2 \partial_{\theta_1} - \mu_1 \partial_{\theta_2}, W := a\partial_{\mu_1} - b\partial_{\mu_2})$
- $R_z = 2\pi \left(\frac{1}{a} \partial_{\theta_1} + \frac{1}{b} \partial_{\theta_2} \right)$ Notice $\frac{\mu_1}{a} + \frac{\mu_2}{b} = 1 \Rightarrow \alpha_0(R) = 1$.
- Symp. trivialization τ' of $\xi|_{Y_0}$: $(\tau')^{-1} = (V, -\frac{2\pi}{ab} W)$.

Denoting by φ^t the Reeb flow, want to compute $d\varphi_t$ in τ'. In coord μ_i, θ_i, $\varphi_t(z) = (\mu_1, \theta_1 + \frac{2\pi t}{a}, \mu_2, \theta_2 + \frac{2\pi t}{b})$.

Starting with $V = (\tau')^{-1}(e_1)$, since μ_i are constant under φ_t and the flows of ∂_{θ_1} and ∂_{θ_2} commute, we have $[R, V] = 0$. Therefore:

$$\frac{d}{dt'}\Big|_{t'=0} d\varphi^{t-t'}(V_{\varphi^{t+t'}(z)}) = [R, V]_{\varphi^t(z)} = 0 \Rightarrow d\varphi^t(V) = d\varphi^0(V) = V.$$

This implies that in the triv. τ', $d\varphi^t$ does not rotate the vector $e_1 = \tau'(V)$ and so $\rho(\tilde{\Phi}, e_1) = 0 \Rightarrow$

$$q(\tilde{\Phi}_{\tau'}(y, T)) = 0 \quad \forall T, y \in Y_0 \Rightarrow \mathrm{rot}_{\tau'}(y) \equiv 0 \quad \forall y \in Y_0.$$

But this does NOT imply $R_u(\partial E) = 0$, since τ' does not extend to a trivialization of ξ over $Y \setminus Y_0$.

Let τ be a triv over all of Y. We claim that as we move around a circle in Y_0 in which θ_i rotates once around S^1, the vector V in τ rotates once around S^1. Namely $\rho(\{\tau(V_{\gamma(t)})\}_{t\in[a,1]}) = 1$.

Claim: $\quad \mathrm{rot}_\tau(y) = \frac{1}{a} + \frac{1}{b}$

Proof. Compute wrt V in τ. Denote $\tilde{\Phi}_\tau$ and $\tilde{\Phi}_{\tau'}$ in $\widetilde{Sp}(2)$ the paths corr to $d\varphi^t$ in τ and τ'. Fixing $y \in Y_0$,

$$\Phi_{\tau'}(y,t)\cdot e_1 \overset{\tau'}{=} d\varphi^t_y(V) = V \overset{\tau'}{=} e_1 . \quad \text{On the other hand,}$$

$\Phi_\tau = \tau \circ d\varphi^t \circ \tau^{-1} = \tau\circ(\tau')^{-1}\circ \Phi_{\tau'}\circ\tau'\circ\tau^{-1}$. Let $v \in \mathbb{R}^2$ s.t. $\tau'\circ\tau^{-1}(v) = e_1$, then $\Phi_\tau(y,T)\cdot v = \tau\circ(\tau')^{-1}\Phi_{\tau'} V = \tau\circ(\tau')^{-1} V$.

$\varphi^t(y=(z_1,z_2)) = (e^{2\pi i t/a} z_1, e^{2\pi i t/b} z_2)$ so along $\{\varphi^t(y)\}_{t\in[0,T]}$ θ_1 rotates by T/a and θ_2 rotates by T/b. Therefore

$\varrho(\tilde{\Phi}_\tau, v) = \frac{T}{a} + \frac{T}{b}$. This implies

$$\mathrm{rot}_\tau(y) = \lim_{T\to\infty} \frac{1}{T}\cdot T \cdot\left(\frac{1}{a}+\frac{1}{b}\right) = \frac{1}{a}+\frac{1}{b}. \qquad \boxed{//}$$

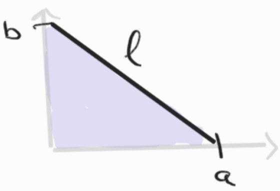
To compute $\mathrm{Ru}(\partial E)$ it remains to integrate over Y:

$$\mathrm{Ru}(\partial E) = \int_{\partial E} \mathrm{rot}_\tau(y)\ \alpha_0\wedge d\alpha_0 = \left(\frac{1}{a}+\frac{1}{b}\right)\int_{\partial E} \alpha_0\wedge d\alpha_0 .$$

$$\alpha_0\wedge d\alpha_0 = \frac{1}{2\pi}(\mu_1 d\theta_1 + \mu_2 d\theta_2)\wedge \frac{1}{2\pi}(d\mu_1\wedge d\theta_1 + d\mu_2\wedge d\theta_2)$$

$$= \frac{1}{(2\pi)^2}(\mu_1 d\theta_1\wedge d\mu_2\wedge d\theta_2 + \mu_2 d\theta_2\wedge d\mu_1\wedge d\theta_1)$$

$$\int_{\partial E}\alpha_0\wedge d\alpha_0 = \frac{1}{(2\pi)^2}\cdot(2\pi)^2\int_{\mu(\partial E)} \mu_1 d\mu_2 - \mu_2 d\mu_1 .$$



$$\mu(\partial E) = \left\{ \left(a(1-t),\ bt \right) \mid t \in [0,1] \right\} \Rightarrow$$

$$R_u(\partial E) = \left(\frac{1}{a} + \frac{1}{b}\right) \cdot \int_0^1 a(1-t) \cdot b\,dt - bt \cdot (-a)\,dt$$

$$= \left(\frac{1}{a} + \frac{1}{b}\right) \int_0^1 ab\ dt = a + b .$$