Homotopy principle for subcritical isotropic embeddings

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Symplectic and contact homeomorphisms

C^0-topology on the group of compactly supported homeomorphisms on a manifold M is induced by a metric $d_{C^0}(\phi, \psi) = \sup_{x \in M} d(\phi(x), \psi(x))$, where d is a Riemannian distance on M (topology doesn't depend on d).

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Theorem (Gromov-Eliashberg rigidity)

A diffeomorphism which is a C^0-limit of symplectomorphisms is itslef symplectic. In other words, $Symp(M, \omega)$ is C^0-closed inside $Diff(M)$.

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Definition (symplectic and contact homeomorphisms)

Let M be a symplectic or a contact manifold. A homeomorphism $f : M \to M$ is called **symplectic/contact** if it is a C^0-limit of a sequence of symplectic/contact diffeomorphisms.

Gromov-Eliashberg $\implies$ smooth symplectic/contact homeomorphisms preserve symplectic/contact structure.

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Definition (Oh, Muller)

Let $(\phi^t)_{t \in [0,1]}$ be a compactly supported isotopy of M. We say that ϕ^t is a **hameotopy** if there exists a sequence of smooth and compactly supported Hamiltonians H_i, and a continious function $H : [0,1] \times M \to \mathbb{R}$ such that:

- $\max_{t \in [0,1]} d_{C^0}(\phi^t_{H_i}, \phi^t) \to 0$ as $i \to \infty$,
- $||H_i - H||_\infty \to 0$ as $i \to \infty$.

We say that H generates hameotopy ϕ^t, and call its time-1 map ϕ^1 a Hamiltonian homeomorphism or **Hameomorphism**.

If $\phi^t_H = \phi^t_G$, then $H - G$ is a function of time.

Example

- Let D^2 be a standard 2-disc with symplectic form $\omega = rdr \wedge d\theta$ in polar coordinates. For a smooth $f : (0, 1] \to \mathbb{R}$ define $\phi_f : D^2 \to D^2$

$$\phi_f(0) = 0, \quad \phi_f(r, \theta) = (r, \theta + f(r)).$$

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- Let $f(r) = \frac{1}{\sqrt{r}}$ near 0, and f_i any sequence of functions regular at 0 that uniformly converge to f. Then $\phi_f = \lim_{i \to \infty} \phi_{f_n}$ is a Hamiltonian homeomorphism which is not differentiable at 0.

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- Submanifold $L \subset (V, \xi)$ is called **isotropic** if $TL \subset \xi$. If $\xi = ker\,\alpha$ one can equivalently say $\alpha|_{TL} \equiv 0$.
- An isotropic submanifold of dimension n is called **Legendrian**, otherwise if dimension is less than n we call it **subcritical isotropic**.

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Theorem (Usher 2020)

Let $L \subset (V, \xi)$ be a Legendrian submanifold, and let $\varphi : V \to V$ be a contact homemorphism that have positive local lower bounds on the conformal factors of the approximating contactomorphisms. If $\varphi(L)$ is smooth, then it must be Legendrian.

h-principle for subcritical isotropic embeddings

Let (W, ξ) be contact manifold and V compact manifold of subcritical dimension $2 \cdot dim\, V + 1 \leq dim\, W$.

Definition

A **formal isotropic embedding** is a pair (f, F_s) where $f : V \to W$ is an embedding, and $F_s : TV \to TW$ is a homotopy of monomorphisms which cover f, such that $F_0 = df$ and $F_1(TV) \subset \xi$.

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Theorem (Relative, 1-parametric *h*-principle for isotropic embeddings)

Let $V_0 \subset V$ be a compact subset and $f_0, f_1 : V \to (W, \xi)$ isotropic embeddings such that $f_0|_{Op(V_0)} = f_1|_{Op(V_0)}$. Assume f_0 and f_1 are isotopic through formal isotropic embeddings $(f_t, F_{t,s})$ such that $f_t|_{Op(V_0)} = f_0$.
Then there exists an isotopy of isotropic embeddings $\widetilde{f}_t : V \to (W, \xi)$ between f_0 and f_1, such that $\widetilde{f}_t|_{Op(V_0)} = f_0$.

Extension of h-principle for isotropic discs

- Let $W \subset \mathbb{R}^{2n+1}$ be a contact manifold with the contact structure ξ. Then every embedding $f : D^k \to W$ is formally isotropic.
- Let $f_t : D^k \to W$ be an isotopy of isotropic embeddings. Then, there exists a contact isotopy $\phi_t : W \to W$, such that $\phi_t \circ f_0 = f_t$.

Proposition

Let $A \subset D^k$ be closed subset and $u_0, u_1 : D^k \to W$ isotropic embeddings which coincide on $Op(A)$. Assume u_0 is isotopic to u_1 relative to $Op(A)$. Then there exists contact isotopy ϕ^t such that $\phi^1 \circ u_0 = u_1$, $\phi^t|_{Op(A)} = Id$.

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Theorem (Quantitative h-principle for isotropic discs)

Let $W \subset \mathbb{R}^{2n+1}$ be an open subset, $k < n$, $u_0, u_1 : D^k \to W$ be isotropic embeddings of closed discs. We assume there exists a homotopy $F : D^k \times [0,1] \to W$ between u_0 and u_1 of size less than ε $(\operatorname{diam} F(\{z\} \times [0,1]) < \varepsilon$ for all $z \in D^k)$. Then there exists a contact isotopy $(\Psi^t)_{t\in[0,1]}$ such that $\Psi^1 \circ u_0 = u_1$, of size less than ε.

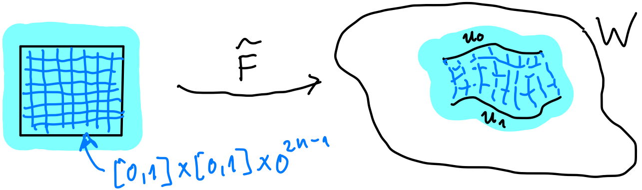
Proof for $k = 1$

Lemma

Let $\Sigma_1, \Sigma_2 \subset W$ be smooth submanifolds which are transverse in the neighbourhood of ∂W. Then there exists an arbitrarily small contact isotopy ϕ^t such that $\phi^1(\Sigma_1) \pitchfork \Sigma_2$.

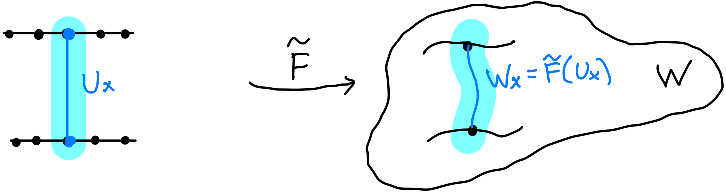
Let $u_0, u_1 : [0,1] \to W$ and assume $u_0([0,1]) \cap u_1([0,1]) = \emptyset$. Let $\widetilde{F}$ be approximation of F which is embedding, such that $size(\widetilde{F}) < \varepsilon$. Extend $\widetilde{F}$ to a smooth embedding

$$\widetilde{F} : [-\mu, 1+\mu] \times [-\mu, 1+\mu] \times [-\mu, \mu]^{2n-1} \hookrightarrow W$$

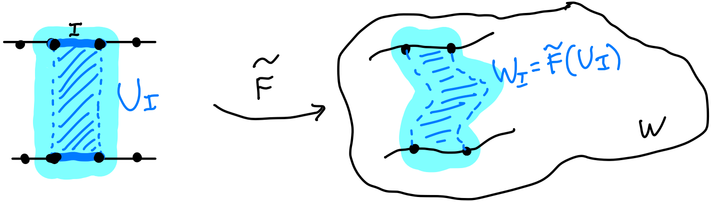


Proof

Let $\eta > 2\delta > 0$. Consider grid $\Gamma_0 = [0,1] \cap \eta\mathbb{Z}$ and for each $x \in \Gamma_0$ let U_x be δ-neighbourhood of $\{x\} \times [0,1] \times \{0\}^{2n-1}$ and define $W_x := \widetilde{F}(U_x)$.



For each 1-cell $I = [x_i, x_{i+1}]$ let U_I be δ-neighbourhood of $I \times [0,1] \times \{0\}^{2n-1}$, and put $W_I := \widetilde{F}(U_I)$.



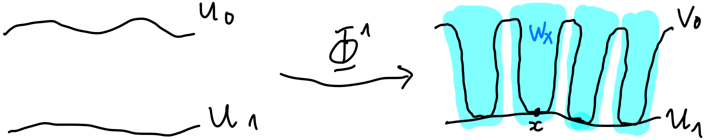
Proof

For each $x \in \Gamma_0$ let $I(x) = [x - \rho, x + \rho]$. Then $\widetilde{F}$ gives isotopy between $u_0|_{I(x)}$ and $u_1|_{I(x)}$ inside $W_x \implies \exists$ contact isotopy ϕ_x^t supported inside W_x such that $\phi_x^1 \circ u_0|_{I(x)} = u_1|_{I(x)}$.

Let $\phi_0^t := \circ\psi_x^t$, then $\psi_0^t \circ u_0|_{Op(\Gamma_0)} = u_1|_{Op(\Gamma_0)}$. Using transversality lemma, let $\widetilde{\phi}^t$ be contact isotopy which achieves

$$\widetilde{\phi}^1 \circ \phi_0^1 \circ u_0(I) \pitchfork u_1(I'),$$

for every pair of distinct 1-cells I, I'. Let $\Phi^t = \widetilde{\phi}^t \# \phi_0^t$ and $v_0 = \Phi^1 \circ u_0$.



Proof

For each 1-cell I we have $v_0(I), u_1(I) \subset W_I$ and $v_0|_{Op(\partial I)} = u_1|_{Op(\partial I)}$.
Consider now slightly smaller $\bar{I} \subset I$ and pick any homotopy

$$\sigma_I : \bar{I} \times [0,1] \to W_I, \quad \sigma_I(\cdot, 0) = v_0,\, \sigma_I(\cdot, 1) = u_1.$$

General position argument implies that we can moreover assume that the images $Im\,\sigma_I$ are disjoint for all 1-cells.
Relative (w.r.t. boundary ∂I) h-principle gives contact isotopies ψ_I^t supported in $Op(Im\,\sigma_I)$ such that $\psi_I^1 \circ v_0|_{\bar{I}} = u_1|_{\bar{I}}$.
Let $\psi^t = \circ \psi_I^t$, where composition runs over all 1-faces I. Finally we define

$$\Psi^t = \psi^t \# \Phi^t.$$

Flexibility of subcritical isotropic curves

Theorem (Flexibility of isotropic curves)

Let (V^{2n+1}, ξ) be a contact manifold and $n \geq 2$. Then there exists an isotropic curve $\gamma : [0,1] \to V$ and a contact homeomorphism $\psi : V \to V$ such that $\psi \circ \gamma$ is transverse to ξ.

Proof. Let $v : [0,1] \to V$ be any transverse curve. Contact neighbourhood theorem implies that $Op\, v([0,1])$ can be contactly embedded to

$$(\mathbb{R}^{2n+1}, ker(dz + \sum_{i=1}^{n}(x_i dy_i - y_i dx_i))),$$

such that $v(t)$ maps to $\widetilde{v}(t) := (t, 0, \ldots, 0)$. Define isotropic curves

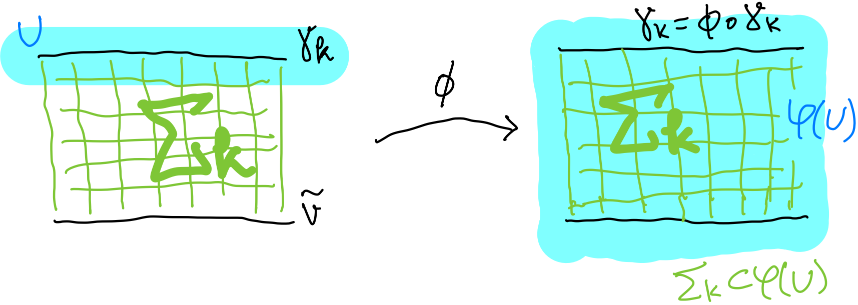
$$\gamma_k : [0,1] \to \mathbb{R}^{2n+1}, \quad t \mapsto \left(t, \frac{1}{k}\sin(k^2 t), \frac{1}{k}\cos(k^2 t), 0, \ldots, 0\right).$$

Note that $\gamma_k \xrightarrow{C^0} \widetilde{v}$ as $k \to \infty$.

Proof

Lemma (Stretching the neighbourhood)

Let $U \supset \gamma_k([0,1])$ be an open neighbourhood, and let $\Sigma_k : [0,1]^2 \to \mathbb{R}^{2n+1}$ be embedded surface defined as $\Sigma_k(s,t) = (1-s)\gamma_k(t) + s\widetilde{v}(t)$. Then there exists a contactomorphism ϕ supported in $Op\,\Sigma_k([0,1]^2)$ such that $\phi \circ \gamma_k = \gamma_k$, $\Sigma_k([0,1]^2) \subset \phi(U)$ and $d_{C^0}(\phi, Id) < \frac{3}{k}$.



Proof

Let $\{U_i\}_{i\geq 1}$ be decreasing sequence of open sets and $\bigcap_{i\geq 1} U_i = \gamma_{k_1}([0,1])$. We inductively construct increasing sequence $\{k_i\}_{i\geq 1}$ with $k_i \geq 2^i$, and sequence of contactomorphisms ψ_i with support in $\varphi_{i-1}(U_i)$ such that

$$\psi_i \circ \gamma_{k_i} = \gamma_{k_{i+1}}, \quad d_{C^0}(\psi_i, Id) < \frac{5}{2^i},$$

where $\varphi_i := \psi_i \circ \psi_{i-1} \circ \cdots \circ \psi_1$.
$\{\varphi_i\}$ is a Cauchy sequence, hence it converges to a continuous map φ. Moreover we have $\varphi \circ \gamma_{k_1} = \widetilde{v}$. Any $x \notin \gamma_{k_1}([0,1])$ satisfies $\varphi_i(x) = \varphi(x)$ for i large enough $\implies$ φ is injective, hence homeomorphism.

Induction step

- Let $U := \varphi_{i-1}(U_{i+1})$. Apply stretching Lemma to γ_{k_i} and U to get a contactomorphism ψ'_i. Let $r > 0$ such that $[0,1] \times B^{2n}(r) \subset \psi'_i(U)$.
- Pick k_{n+1} large enough such that $\gamma_{k_{n+1}}([0,1]) \subset [0,1] \times B^{2n}(r)$.
- Composition of homotopy $\Sigma_i(s,\cdot)$, $s \in [0,r]$ and linear homotopy between $\Sigma_i(r,\cdot)$ and $\gamma_{k_{n+1}}$ gives homotopy between γ_{k_i} and $\gamma_{k_{i+1}}$ of size less than $\frac{1}{2^i}$. Now apply quantitative h-principle to get ψ''_i, and finally define $\psi_i := \psi''_i \circ \psi'_i$.