Viterbo, Billiards & Mahler

Based on paper by Artstein-Avidan
Karasev
Ostrover

Def a capacity is a map

$$c : \{ U \subset \mathbb{R}^{2n} \text{ submfd} \} \to [0,\infty] \text{ st}$$

- $U \subset V \Rightarrow c(U) \leq c(V)$
- $c(\varphi(U)) = c(U)$ if φ symp ($\varphi^* \omega = \omega$)
- $c(\alpha U) = \alpha^2 c(U)$
- $c(B_2^{2n}) = c(Z^{2n}) = \pi$

ex $\underline{c}(U) = \sup \{ \pi r^2 \mid B_2^{2n}(r) \overset{s}{\hookrightarrow} U \}$

$\overline{c}(U) = \inf \{ \pi r^2 \mid U \overset{s}{\hookrightarrow} Z^{2n} \}$

$Z^{2n} = B_2^2 \times \mathbb{R}^{2n-2}$

def – for $U \subseteq \mathbb{R}^{2n}$ submfd w/ ∂U smooth

def $\Theta_U = \{ (x,\xi) \in T\partial U \mid \forall \eta \in T_x \partial U \ \ \omega(\eta,\xi) = 0 \}$

$= \ker \omega|_{\partial U}$

char. line bundle

- char. lines are sol^n of ODE
  Hamiltonian

  w/ H s.t. $\partial U = H^{-1}(1)$

- a closed char. is $\gamma : S^1 \to \partial U$

  s.t. $\dot\gamma \in \Theta_u$

  $\dot\gamma \parallel J n_u(\gamma)$ $\quad n_u : \partial U \to S^1$ normal

  $J = \begin{pmatrix} & I \\ -I & \end{pmatrix}$

- can extend to non-smooth:

  $\dot\gamma \in J N_u(\gamma)$

- Action of char. $\gamma : S^1 \to \partial U$

  $$A(\gamma) = \int_{\substack{D \\ \partial D = \gamma}} \omega = \int_\gamma p\,dq$$

- HZ capacity: $c_{HZ}(U) = \min\{|A(\gamma)| ; \gamma \text{ is cl. char. on } \partial U\}$

  U cvx $\Rightarrow$ c_{HZ} is 'capacity'

Conj (Viterbo)

for U cvx in $\mathbb{R}^{2n}$ & c capacity:

$$\frac{c(U)}{\mathrm{vol}(U)^{1/n}} \leq \frac{c(B_2)}{\mathrm{vol}(B_2)^{1/n}}$$

or $$\frac{c(U)}{c(B_2)} \leq \left(\frac{\mathrm{vol}(U)}{\mathrm{vol}(B_2)}\right)^{1/n}$$

Thm (Artstein-Avidan, Milman, Ostrover)

Viterbo holds up to universal const

Thm (Abbodandolo, Bramham, Hryniewicz, Salomão)

Viterbo holds in C^3-neigh of B_2

Mahler

Def Let $K \in \mathcal{K}^n_0$ (cmpct cvx sets K, $0 \in \mathrm{int}\, K$)

$$K^\circ = \{ y \in \mathbb{R}^n \mid \forall x \in K \ \langle x, y \rangle \leq 1 \}$$

props

- $K^{\circ\circ} = K$
- $K \subseteq T \Rightarrow T^\circ \subseteq K^\circ$
- $K \mapsto K^\circ$ is cont.
- $(AK)^\circ = A^{-*} K^\circ$, $A \in GL_n\mathbb{R}$

Thm (Boroczky, Schneider) $\quad \vee$ - cvx hull

If $\varphi : \mathcal{K}_0^n \to \mathcal{K}_0^n$ st

- $\varphi(K \cap L) = \varphi(K) \vee \varphi(L)$
- $\varphi(K \vee L) = \varphi(K) \cap \varphi(L)$

$\Rightarrow$ $\varphi \equiv$ const or $\varphi(K) = AK^\circ$ $(A \in GL_n)$

def Let $K \in \mathcal{K}_0^n$

$h_K : \mathbb{R}^n \to \mathbb{R}$ support func

$$x \mapsto \max_{y \in K} \langle x, y \rangle$$

$$g_K(\cdot) = \|\cdot\|_K : \mathbb{R}^n \to \mathbb{R}$$

$$x \mapsto \min\{\lambda \mid \tfrac{x}{\lambda} \in K\}$$

If $K = -K \Rightarrow \|\cdot\|_K$ is a norm

$$\nu(K) = \mathrm{vol}(K) \cdot \mathrm{vol}(K^\circ)$$

volume prod

mahler vol

ν is GL_n-inv

$$\nu(AK) = \mathrm{vol}(AK)\,\mathrm{vol}(A^{-*}K^\circ) =$$

$$= \underbrace{\det A \cdot \det A^{-*}}_{1}\,\nu(K)$$

<u>Thm</u> (Blaschke - Santálo)

$\forall\ K \in \mathcal{K}_0^n$, $K = -K$ $\quad \nu(K) \leq \nu(B_2)$

<u>Conj</u> (Mahler) $\quad \nu(K) \geq \nu([-1,1]^n) = \frac{4^n}{n!}$

$K = -K$

$K \in \mathcal{K}_0^n$

conj. min. are Hanner poly

$\shortparallel$

prod. of cubes & X-poly

$\shortparallel$

$(\text{cube})^\circ$

<u>Thm</u> (Bourgain - Milman) $\quad \nu(K) \geq c^n \frac{4^n}{n!}$

(Kuperberg) $\quad c = \frac{\pi}{4}$

(Nazarov, Petrov, Ryabogin, Zvavich)
Mahler holds in BM-neigh of $[-1,1]^n$

Thm 1 Viterbo $\Rightarrow$ Mahler

in fact: Viterbo for c_{HZ} for bodies $K \times K^\circ$

Thm 2 $\forall\, K \in \mathcal{K}_0^n$, $K = -K$:

$$c_{HZ}(K \times K^\circ) = 4$$

Pf (Thm 1)

$$\frac{4^n}{\pi^n} \overset{\text{Thm 2}}{=} \left(\frac{c_{HZ}(K \times K^\circ)}{c_{HZ}(B_2^{2n})}\right)^n \overset{\text{Viterbo}}{\leq} \frac{\overbrace{\mathrm{vol}(K \times K^\circ)}^{v(K)}}{\mathrm{vol}(B_2)} =$$

$$= \frac{n!\, v(K)}{\pi^n} \qquad \square$$

$c_{HZ}(K \times T)$

$$\partial(K \times T) = \partial K \times \operatorname{int} T \sqcup \operatorname{int} K \times \partial T \sqcup \underbrace{\partial K \times \partial T}$$

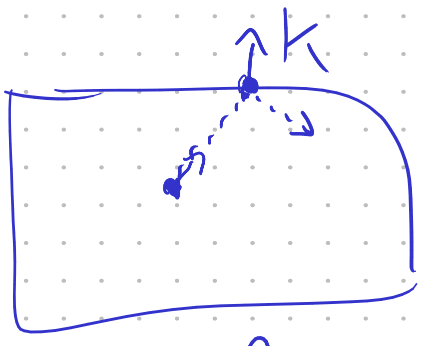
$$\dot{\gamma} \in J N_{K \times T}$$

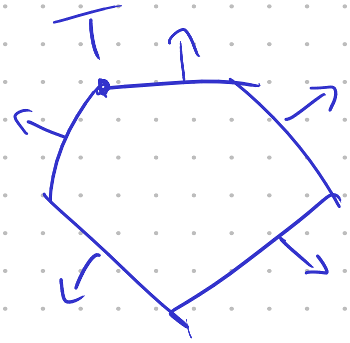
If $\gamma \in \mathrm{int}\, K \times \partial T$

$$N_{K\times T}(\gamma) = N_K(\pi(\gamma)) \times \{0\}$$

$$J N_{K\times T} = \{0\} \times N_K(\pi(\gamma))$$

$J\mathbb{R}^n_q = \mathbb{R}^n_p$





$T = B_2 \Rightarrow$ billiard in K

<u>def</u> a (K,T)-billiard traj is

$\gamma : S^1 \to \partial(K\times T)$

- $\forall t : \gamma(t) \notin \partial K \times \partial T \Rightarrow \dot\gamma(t) = \overset{\overset{0}{\wedge}}{d}(t)\, X(\gamma(t))$

$$X(q,p) = \begin{cases} (-\nabla g_T(p), 0) & \text{if } (q,p) \in \mathrm{int}\, K \times \partial T \\ (0, \underbrace{\nabla g_K(q)}) & \text{if } (q,p) \in \partial K \times \mathrm{int}\, T \end{cases}$$

$g_K(\cdot) = \|\cdot\|_K$

- $\forall t : \underbrace{\gamma(t) \in \partial K \times \partial T}$

$\exists \; \dot\gamma^I(t)$ and

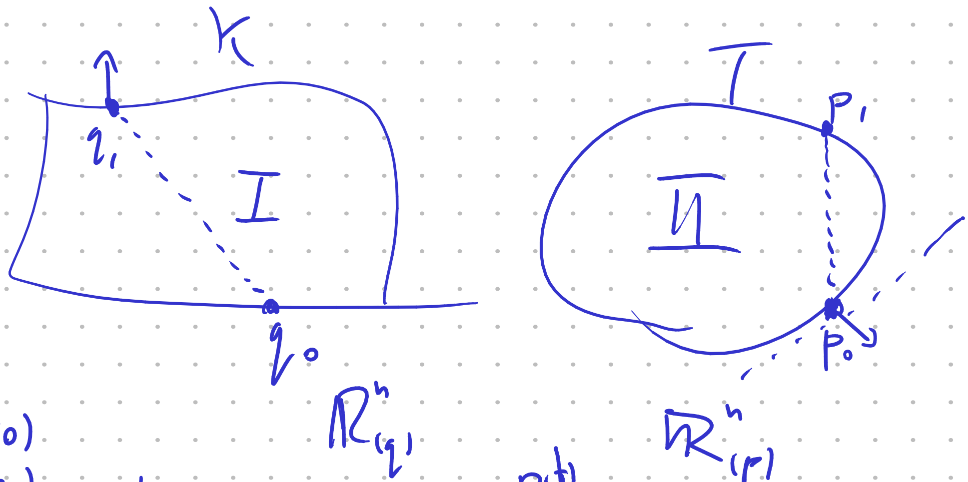
$$\dot\gamma^I(t) \in \underbrace{\left\{ \left(\alpha(-\nabla g_T(p), 0) + \beta(0, \nabla g_K(q)) \;\middle|\; \begin{array}{l} \alpha, \beta \geq 0 \\ (\alpha,\beta) \neq (0,0) \end{array} \right\} \right.}_{[J N_{K \times T}(q,p)}$$

Thm 3 $\{$closed (K,T)-billiards$\}$ $= \{$cl. chars$\}$ $K \times T$

$K = -K$

$T = -T$

(q_0, p_0)



$\gamma(0) = (q_0, p_0)$

$\gamma(t_0) = (q_1, p_0)$

$$\text{I} \quad |A(\gamma|\;)| = \left| \int_0^{t_0} p(t)\, \dot q(t)\, dt \right| = |\overset{p(t)}{\overset{\shortparallel}{p_0}} \cdot (q_1 - q_0)| =$$

$\langle p_0, q_1 - q_0 \rangle$

$\gamma(t_1) = (q_1, p_1)$ $\quad = \max\limits_{p \in \partial T} |\langle p, q_1 - q_0 \rangle| \leq |h_T(q_1 - q_0)|$

$$\text{II} \quad A(\gamma|_{t_0}^{t_1}) = \int_{t_0}^{t_1} p(t)\, \underbrace{\dot q(t)}_{\shortparallel\atop 0}\, dt = 0$$

$\gamma = q_0, \dots, q_m \in K$, $q_m = q_0$

$p_0, \dots, p_m \in T$

$$\boxed{\text{ex} \quad g_K = h_{K^\circ}}$$

$$A(\gamma) = \sum_{i=0}^{m-1} h_T(q_{i+1} - q_i) = \sum \| q_{i+1} - q_i \|_{T^\circ}$$

def If $K = -K$, $K \in \mathcal{K}^n_0$; P poly. loop in $\mathbb{R}^n$

$P \sim x_0, \dots, x_m = x_0$

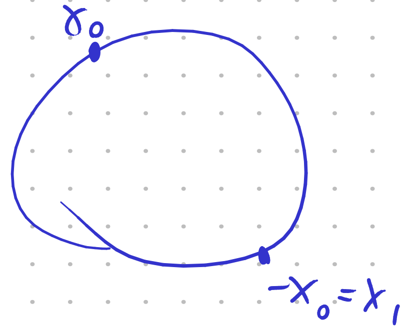
$$\mathrm{Len}_K(P) = \mathrm{Len}_K(x_0, \dots, x_m) = \sum_{i=0}^{m-1} \| x_{i+1} - x_i \|_K$$

Thm 4 Let $K \in \mathcal{K}^n_0$, $K = -K$, $x_0, \dots, x_m = x_0 \in \mathbb{R}^n \setminus \mathrm{int}(K)$

$\forall i: \ x_{i+1} \neq x_i$

$$\boxed{0 \in \underbrace{\bigvee \{x_i\}_{i=0}^{m-1}}_{\text{cvx hull}}}$$

$\Rightarrow \mathrm{Len}_K(x_0, \dots, x_m) \geq 4$



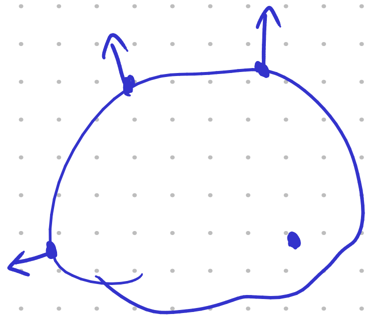
Pf Later

Thm 5 Let $K \in \mathcal{K}^n_0$, $K=-K$, ∂K smooth

$q_0, \dots, q_m = q_0 \in \partial K \qquad (q_{i+1} \neq q_i)$

$0 \in V\{n_K(q_i)\}_{i=1}^{m}$

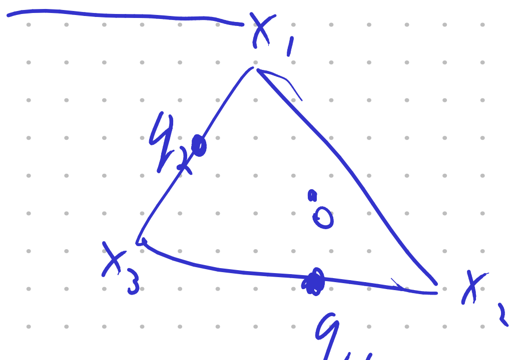
$\Rightarrow \quad \mathrm{Len}_K(q_0, \dots, q_m) \geq 4$



Lemma Let S a non-deg simplex in $\mathbb{R}^n$

$\mathrm{ext}(S) = \{x_i\}_{i=1}^{n+1}$, $\quad 0 \in S$

Let $q_i \in V\{x_j\}_{j \neq i}$



$\Rightarrow \exists\, \emptyset \neq I \subseteq \underset{\{1,\dots,n+1\}}{\underset{\shortparallel}{[n+1]}}$:

$0 \in V\{q_i\}_{i \in I} \vee V\{x_i\}_{i \notin I}$

Pf Set $C = V\{\pm e_i\}_{i=1}^{n+1}$ X-poly

Define $f : \partial C \to S$ piecewise-linear

$f(+e_i) = q_i$

$f(-e_i) = x_i$

$\partial C \simeq S^n \quad \Rightarrow \quad \deg f = 0$

$$0 \in S = F\left(\bigvee \{-e_i\}_{i=1}^{n+1}\right)$$

$\Rightarrow \exists$ another facet $\hat{F}$ of C : $0 \in f(F)$ $\square$

<u>pf</u> (Thm 5) $\quad q_0, \dots, q_m = q_0$

WLOG $m \le n+1$
(Caratheodory)

WLOG $\mathrm{Span}_{\mathbb{R}_{\ge 0}} \{n_K(q_i)\} = \mathbb{R}^n$

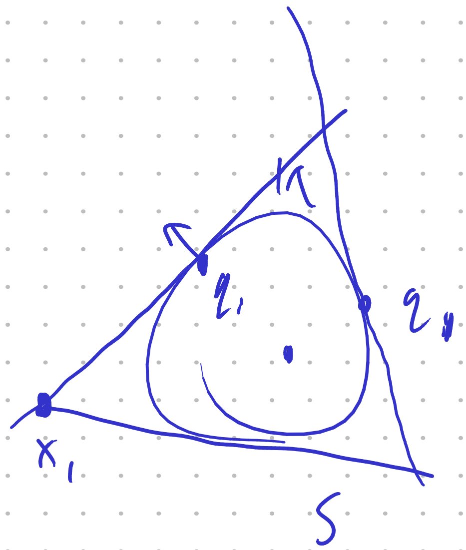
$\Rightarrow m \ge n+1$

$m = n+1$

$\{q_i\} = \partial K \cap \partial S$

Denote by $\{x_i\} = \mathrm{ext}\, S$

$q_i \in \bigvee \{x_j\}_{j \ne i}$

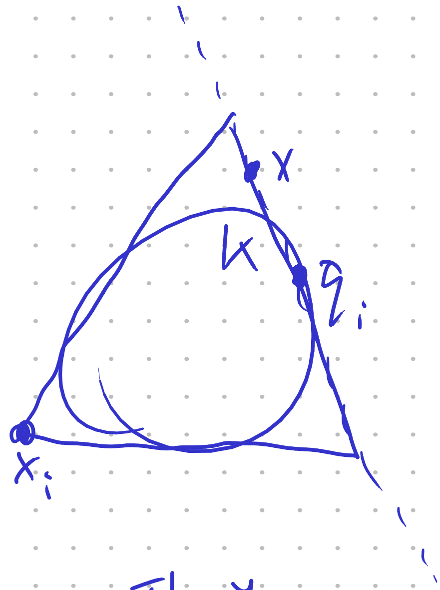


Lemma
$\Rightarrow \exists\ \emptyset \ne I \subseteq [n+1]$: $\boxed{0 \in \bigvee \{q_i\}_{i \in I} \vee \bigvee \{x_i\}_{i \notin I}}$

If $I = [n+1] \underset{\text{Thm4}}{\Rightarrow} \mathrm{Len}_K(q_0, \dots, q_m) \ge 4$

<u>else</u> $\quad \exists\ q \in \bigvee \{q_i\}_{i \in I}\ ,\ x \in \bigvee \{x_i\}_{i \notin I}$
$\lambda \in [0,1]$

$$\boxed{0 = \lambda x + (1-\lambda) q}$$



for $i \in I$ $\quad q_i' = (1-\lambda) q_i + \lambda \underline{\underline{x}}$

- $0 \in V\{q_i'\}_{i \in I}$
- $\forall\ \underline{\underline{i}} \in I : \underbrace{q_i' \notin \operatorname{int} K}$

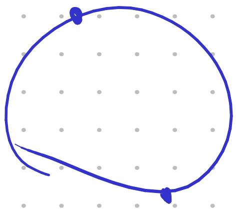
$$\mathrm{Len}_K(\{q_i\}_{i=0}^{m-1}) \geq \mathrm{Len}_K(\{q_i'\}_{i \in I}) \overset{\text{Thm 4}}{\geq} 4$$

$\square$

Want: $\quad \gamma : S^1 \to \partial(K \times K^\circ)$ is (K, K°)-billiard

$$\Rightarrow \quad 0 \in V\{n_K(\pi_q(\gamma) \cap \partial K)\}$$

$$\pi_q : \mathbb{R}^{2n} \to \mathbb{R}^n_{(q)}$$



Lemma $\quad \gamma : S^1 \to \partial(K \times T)$ is a (K, T)-billiard, $\quad K \subset \mathbb{R}^n_{(q)},\ T \subset \mathbb{R}^n_{(p)}$

denote $\pi_q : \mathbb{R}^{2n} \to \mathbb{R}^n_{(q)}$ proj.

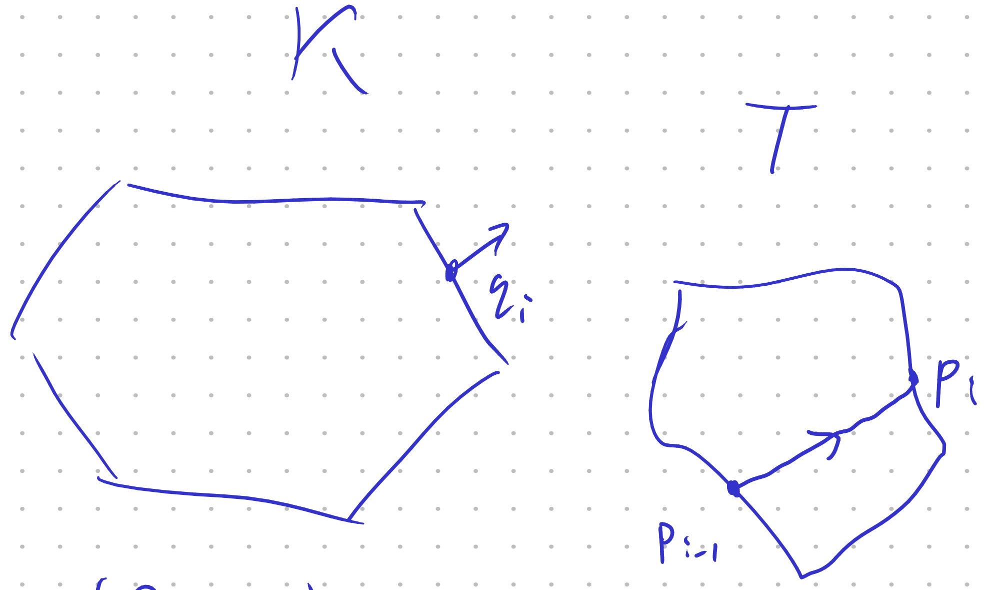
$$\Rightarrow \quad 0 \in V\{n_K(\pi_q(\gamma) \cap \partial K)\}$$

pf

(q_i, p_i)

$\uparrow$

(q_i, p_{i-1})



$\underset{\substack{\vee \\ 0}}{\lambda_i} \, n_K(q_i) = (p_i - p_{i-1})$

$$\sum_i \lambda_i n_K(q_i) = \sum p_i - p_{i-1} = 0$$

$\square$

Thm 4

$K \in \mathcal{K}^n_0$, $K = -K$, $x_0, \dots, x_m = x_0 \in \mathbb{R}^n \setminus \mathrm{int}(K)$

$$0 \in V\{x_i\}$$

$$\Rightarrow \mathrm{Len}_K(x_0, \dots, x_m) \geq 4$$

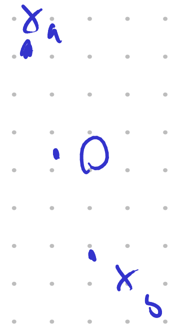
Pf

$$\sum_{i=0}^{m-1} \lambda_i x_i = 0 \qquad \sum \lambda_i = 1$$

First step: find $i_0, j_0 \in [m]$ and $p \in \mathbb{R}^n$:

$$p \in V\{x_i\}_{i_0}^{j_0} \quad \text{and} \quad -p \in V\{x_i\}_{j_0}^{i_0 - 1}$$

If $\lambda_a = \lambda_b = \frac{1}{2}$:



✓

Else : $\lambda_1, \lambda_m < \frac{1}{2}$

$$\boxed{\begin{array}{l} \lambda_a = \frac{1}{2} \\ x_a = p \\ 0 \\ -x_a \\ -p \in V\{x_i\}_{i \neq a} \end{array}}$$

set $i_0 = 1$

start w/ $j_0 = 1$

increase j_0 until

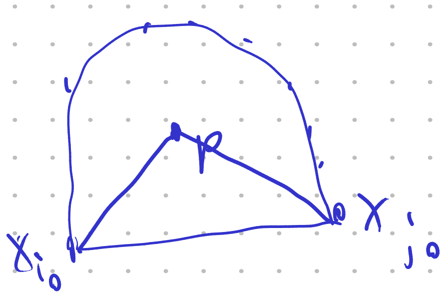
$\left[\sum_{i=i_0}^{j_0} \lambda_i \geq \frac{1}{2} \right.$

Denote $\lambda'_{j_0}, \lambda''_{j_0} > 0$

$$\underbrace{\sum_{i=i_0}^{j_0 - 1} \lambda_i}_{\leq \frac{1}{2}} + \lambda'_{j_0} = \frac{1}{2}, \quad \lambda''_{j_0} = \lambda_{j_0} - \lambda'_{j_0}$$

$$0 = \sum \lambda_i x_i = \overbrace{\sum_{i=i_0}^{j_0 - 1} \lambda_i x_i + \lambda'_{j_0} x_{j_0}}^{p \in V\{x_i\}_{i=i_0}^{j_0}} +$$

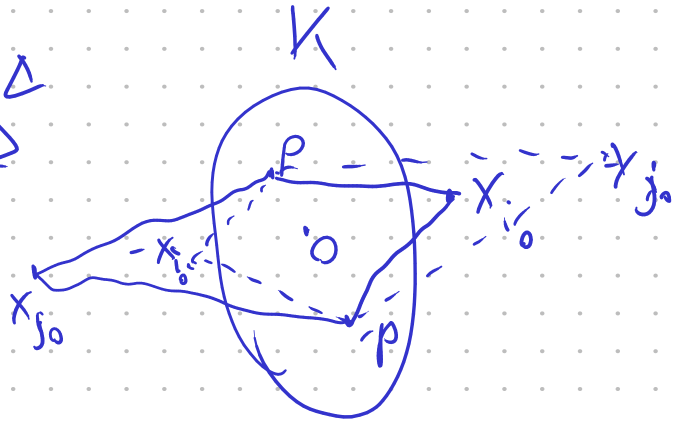
$$+ \underbrace{\sum_{i=j_0+1}^{i_0 - 1} \lambda_i x_i + \lambda''_{j_0} x_{j_0}}_{-p \in V\{x_i\}_{i=j_0}^{i_0 - 1}}$$



$$\mathrm{Len}_K(\{x_i\}) \overset{\Delta}{\geq} \mathrm{Len}_K(x_{i_0}, p, x_{j_0}, -p) \geq$$

WLOG

$$\geq \mathrm{Len}_K(x_{i_0}, p, -x_{i_0}, -p) \overset{\Delta}{\geq}$$



$$\geq \mathrm{Len}_K(x_{i_0}, -x_{i_0}) =$$

$$= \|x_{i_0} - (-x_{i_0})\|_K + \|-x_{i_0} - x_{i_0}\|_K = 4\|x_{i_0}\| \geq 4$$

$x_{i_0} \notin \mathrm{int}$

$\square$