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Game semantics for the Lambek-Calculus: capturing directionality and the absence of structural rules

Abstract. In this paper, we propose a game semantics for the (associative) *Lambek calculus*. Compared to the implicational fragment of intuitionistic propositional calculus, the semantics deals with two features of the logic: absence of structural rules, as well as directionality of implication. We investigate the impact of these variations of the logic on its game semantics.

Keywords: Lambek calculus, Game semantics.

1. Introduction

Game semantics have long been recognized as a successful means for providing semantics for logical calculi. In this paper, we propose a game semantics for the (associative) *Lambek calculus* **L** [3]. Compared to the implicational fragment of intuitionistic propositional calculus (*IIPC*), the semantics deals with two features of the logic.

Absence of structural rules: In the absence of *Weakening, Contraction*, and, most notably, *Exchange*, the logic becomes *resource-sensitive*, namely *linear*, and *non-commutative*. This calls for a significant modification of the intuitionistic game definition.

Directionality of implication: In the presence of two *directed implications*, an additional minor modification of the intuitionistic game is needed.

Because of these two features of **L**, its natural model-theoretic semantics is *not* via truth-values, but via *strings*, rendering the calculus as a basis for formal grammar. The effect of the above features is two-folded:

- The game is defined over a board of *sequents*, instead of the traditional board of formulae. There is a need to associate each meeting of a challenge with *a sequence* of offers (see below).

Presented by **Wojciech Buszkowski**; *Received* May 27th, 2008

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- The antecedent *split* (see below) is two-directional, attributing each offer-formula its direction of use.

While the game semantics proposed in this paper comprises an additional semantics for **L**, the main interest of this paper is rather different than proposing another semantics for **L**. Namely, this paper aims at investigating the game semantics of the intuitionistic propositional logic by examining the effect that the variations of the logic, encapsulated by **L**, have on it. The study reported in this paper demonstrates the correspondence between properties of the logic and properties of its game semantics.

The effect of the absence of structural rules (but without directionality) in terms of game semantics was carried out also for linear logic (e.g., [1]), but we are not aware of any study of its implicational fragment, in particular one that constitutes a direct and simple generalization of its intuitionistic counterpart.

2. Preliminaries

2.1. Game semantics for *IIPC*

The *IIPC* well-formed formulae (wffs) are the closure of a (countably infinite) set of *propositional variables*, ranged over by meta-variables $A, A_i, B, B_i, ...$, under *implication* '$\rightarrow$'. We let $\varphi, \varphi_i, \psi, \psi_i, ...$ be meta-variables ranging over wffs, and $\Gamma, \Gamma_i, \Sigma, \Sigma_i, \Pi, \Pi_i ...$ range over *contexts*, i.e., finite sets of pairs[1] of wffs and *simply-typed* λ-terms. As usual, $\Gamma_1\Gamma_2$ abbreviates $\Gamma_1 \cup \Gamma_2$, and Γ, φ abbreviates[2] $\Gamma \cup \{\varphi\}$. For $\Gamma = \{\varphi_1, \cdots, \varphi_n\}$, we let $\Gamma \rightarrow \psi$ abbreviate $(\varphi_1 \rightarrow (\varphi_2 \rightarrow \cdots (\varphi_n \rightarrow \psi) \cdots))$. For a sequent $\sigma = \Gamma \rhd \varphi$, denote the assumptions sequence Γ by $assump(\sigma)$, and the formula φ by $form(\sigma)$.

We adopt the presentation of [5] of the game-semantics for *IIPC*. The natural deduction proof system (in a Gentzen-like presentation with sequents, for ease of comparison) is presented in Figure 1. As usual $[\varphi]_i$ is a *discharged assumption*, and rule-names refer to the assumptions they discharge. We denote by $\vdash_{IIPC} \Gamma \rhd \varphi$ the provability[3] of $\Gamma \rhd \varphi$ in *IIPC*. The structural rules are present and remain implicit. The proof-terms embodying the Curry-

[1] We will in general leave to context the determination when the context is as described here, and when it consists of formulae only. In case we want to stress that the latter is the case, we use $\Gamma\downarrow$, etc.

[2] We also abbreviate $\{\varphi\}$ to just φ.

[3] We use this symbol ambiguously, for derivations either with or without term assignments.

$$\varphi : x \rhd \varphi : x \quad (ax)$$

$$\frac{\Gamma_1 \rhd \varphi : N \quad \Gamma_2 \rhd (\varphi \to \psi) : M}{\Gamma_1 \Gamma_2 \rhd \psi : (MN)} (\to E) \qquad \frac{\Gamma[\varphi]_i : x \rhd \psi : M}{\Gamma \rhd (\varphi \to \psi) : \lambda x.M} (\to I_i)$$

Figure 1. Natural Deduction rules for IIPC with term-assignments

Howard (CH) correspondence are the simply-typed λ-terms. The following is well-known (e.g., [2], [5]):

- The *subject-construction* theorem holds.
- *Weak normalization* holds (and strong-normalization holds, too).
- Each λ-term is β-reducible to a β-normal-form of the form

$$\lambda x_{n+1}, \cdots, \lambda x_{n+m}.x_k N_1 \cdots N_l;$$

  furthermore, it also has a β-η-normal-form, saturated with arguments for applications.

A game over (Γ, φ) consists of a *dialogue* (defined below) between two participants:

- A *Prover*, aiming to produce a witness derivation establishing $\vdash_{IIPC} \Gamma \rhd \varphi$.
- A *Skeptic*, aiming to establish non-derivability of φ from Γ (in *IIPC*).

In the sequel, the Prover, P, will be referred to as 'she', while the Skeptic, S, will be referred to as 'he'. The dialogue starts by the Prover making an *assertion*. Based on the Prover's assertion, the Skeptic can *challenge* the Prover's assertion, presenting an *offer*, a set of wffs the Prover may assume derivable. The Prover must *meet* the latest challenge using the offers offered so far, and in doing so she is allowed to introduce new assertions; the Skeptic, in turn, may challenge any of these assertions (but one only), and so on. The game *ends* when the player who is up has no response, in which case the other player *wins*, or it may go on *forever* in which case the Skeptic wins.

Definition: [5] A *dialogue* over (Γ, φ) is a (possibly infinite) sequence $(\Sigma_1, \alpha_1), (\Pi_1, \beta_1), (\Sigma_2, \alpha_2), (\Pi_2, \beta_2), ...$, where all β_i are propositional variables, s.t.:

1. $\Sigma_1 = \varphi$, and α_1 is an arbitrary propositional variable: Prover begins.

2. $\alpha_{i+1} = \beta_i$: Prover meets preceding challenge.
3. $\Sigma_{i+1} \rightarrow \alpha_{i+1} \in \Gamma \cup \cup_{j \leq i} \Pi_j$: Prover uses an available offer.
4. $\Pi_i \rightarrow \beta_i \in \Sigma_i$: Skeptic challenges a formula from preceding step.

Thus, a Prover step (Σ_i, α_i) means meeting the challenge α_i by introducing additional assertions Σ_i. A Skeptic step (Π_i, β_i) means challenging β_i while offering Π_i.

Example: Consider the claim $\vdash_{IIPC} (A \rightarrow B), (B \rightarrow C) \rhd (A \rightarrow C)$. Its witness derivation is

$$\dfrac{\dfrac{\dfrac{[A \rhd A]_1 \quad (A \rightarrow B) \rhd (A \rightarrow B)}{A, (A \rightarrow B) \rhd B} (\rightarrow E) \quad (B \rightarrow C) \rhd (B \rightarrow C)}{A, (A \rightarrow B), (B \rightarrow C) \rhd C} (\rightarrow E)}{(A \rightarrow B), (B \rightarrow C) \rhd (A \rightarrow C)} (\rightarrow_1)$$

Below is a game for this claim, where the Prover (P) wins.

P: $\Sigma_1 = \{(A \rightarrow C)\}$	-	
S: $\Pi_1 = \{A\}$	$\beta_1 = C$	justification: $(A \rightarrow C) \in \Sigma_1$ offering A, challenging C.
P: $\Sigma_2 = \{B\}$	$\alpha_2 = C$	justification: $(B \rightarrow C) \in \Gamma$ asserting B, meeting challenge C.
S: $\Pi_2 = \emptyset$	$\beta_2 = B$	justification: $B \in \Sigma_2$ challenging (atomic) B with no offer.
P: $\Sigma_3 = \{A\}$	$\alpha_3 = B$	justification: $(A \rightarrow B) \in \Gamma$ asserting A, meeting challenge B.
S: $\Pi_3 = \emptyset$	$\beta_3 = A$	justification: $A \in \Sigma_3$ challenging (atomic) A with no offer.
P: $\Sigma_4 = \emptyset$	$\alpha_4 = A$	justification: $A \in \Pi_1$ meeting challenge A with no assertion (using initial offer).
S:		no further move – loss.

Note that in this example, always $|\Sigma|,\ |\Pi| \leq 1$, because the implicant in φ is atomic. For compound implicant, we get $|\Sigma|,\ |\Pi| > 1$.

Before addressing the Lambek calculus, we analyze the role of the players in the game, relating them to the proof-system.
The Prover, in her turn, provides a set of formulas Σ_{i+1} such that[4] $\Sigma_{i+1} \rightarrow$

[4] The Prover actually uses α_{i+1} rather than β_i, but $\alpha_{i+1} = \beta_i$, thus it is easier to refer only to the β's.

$\beta_i \in \Gamma \cup \bigcup_{j \leq i} \Pi_j$. If the Prover succeeds to do so, she meets the challenge β_i. Looking at the $\vdash_{IIPC}$ proof system (Figure 1), such a move corresponds to providing the premisses Σ_{i+1} of (a series of) applications of elimination rules that derive β_i. To understand why, suppose $\Sigma_{i+1} = (\sigma_1, \ldots, \sigma_j)$. Then, applying $(\rightarrow E)$ to $\Sigma_{i+1} \rightarrow \beta_i \in \Gamma \cup \bigcup_{j \leq i} \Pi_j$ and σ_1, results in $(\sigma_2, \ldots, \sigma_j) \rightarrow \beta_i$. Applying $(\rightarrow E)$ to the latter and to σ_2, results in $(\sigma_3, \ldots, \sigma_j) \rightarrow \beta_i$. Denote $(\sigma_k, \ldots, \sigma_j)$ by $suf_k(\Sigma_{i+1})$ (the suffix of the sequence starting from the kth place). Then in the kth step applying $(\rightarrow E)$ on the result of the previous step, and on σ_k results in $suf_{k+1}(\Sigma_{i+1}) \rightarrow \beta_i$. Eventually (for $k = j$) the result of the derivation is β_i. That is, the assertions that the Prover makes are such that $\Sigma_{i+1} \cup \Gamma \cup \bigcup_{j \leq i} \Pi_j$ reduces to β_i in a series of $(\rightarrow E)$ steps.

Thus, we can say that the Prover "uses" elimination rules in her moves in the sense of providing the premisses needed for the application of a series of elimination rules that will derive the challenge [5] β_i.

Similar arguments show that the Skeptic "uses" the introduction rules of the proof system. He provides a sequence Π_i of offers and a challenge β_i such that $\Pi_i \rightarrow \beta_i = \sigma \in \Sigma_i$. Thus the move of the Skeptic corresponds to making the Prover derive the premiss β_i for a series of applications of introduction rules that derive σ. This is because given such a challenge the Prover will in fact use $\Gamma' = \Gamma \cup \bigcup_{k \leq i} \Pi_k$ as a set of assumptions to derive β_i from. That is, the Prover will try to show $\Gamma' \rhd \beta_i$. Suppose $\Pi_i = \{\pi_1, \ldots, \pi_j\}$. Then, applying $(\rightarrow I)$ to $\Gamma' \rhd \beta_i$ results in $\Gamma' \setminus \pi_j \rhd \pi_j \rightarrow \beta_i$, and continuing this iteratively results in $\Gamma' \setminus \Pi_i \rhd (\Pi_i \rightarrow \beta_i) = \sigma$, meaning that $\Gamma' \rhd \beta_i \Rightarrow^*_{(\rightarrow I)} \Gamma'' \rhd \sigma$, where $\Gamma'' = \Gamma \cup \bigcup_{k<i} \Pi_k$. Thus we can say that the Skeptic chooses an assertion σ of the Prover and uses the introduction rules (by "reversing" them) to "construct" a challenge β_i (and offers Π_i) for the Prover, such that the derivation of β_i (that may use the offers Π_i) can be followed by a series of $(\rightarrow I)$ steps to derive σ. In this sense, the Skeptic makes the Prover "prove" σ, or in other words, challenges the assertion σ of the Prover.

To sum up, the Prover uses elimination rules in order to prove the challenge. The Skeptic uses introduction rules in order to make the Prover prove one of the assertions she (the Prover) made. This sequence of events and distribution of labor fits the well-known structure of a *normal* derivation in $\vdash_{IIPC}$.

[5] As each play in the game starts with the sequent that needs to be derived, and goes "up" the derivation tree, it is convenient to look at derivations and derivation rules bottom-up.

Definition: ([5]) A *winning strategy* for the Prover over game $(\Gamma,\ \varphi)$ is a finite labelled tree, where:

1. The root is a P-node.
2. Every branch is a dialogue over $(\Gamma,\ \varphi)$.
3. Every node labelled $(\Sigma,\ \alpha)$ (a P-node) with $|\Sigma| = n \geq 0$ (i.e., n assertions) has n distinct S-nodes descendants .
4. Every node labelled $(\Pi,\ \beta)$ (an S-node) has one P-node descendant.

Consequently, every leaf is a P-node with $|\Sigma| = 0$.

To see the need of the requirement that the challenge formula be atomic, consider the following example violating it, a game for $\vdash_{IIPC} (\varphi_1 \rightarrow \varphi_3) \rightarrow ((\varphi_3 \rightarrow \varphi_2) \rightarrow (\varphi_1 \rightarrow \varphi_2))$ (transitivity).

The Prover starts with $\Sigma_1 = (\varphi_1 \rightarrow \varphi_3) \rightarrow ((\varphi_3 \rightarrow \varphi_2) \rightarrow (\varphi_1 \rightarrow \varphi_2))$. If the Skeptic is allowed to provide a non-propositional formula as the challenge, then he can choose $\Pi_1 = \{(\varphi_1 \rightarrow \varphi_3), (\varphi_3 \rightarrow \varphi_2)\}$ and $\beta_1 = (\varphi_1 \rightarrow \varphi_2)$.

The Prover cannot respond to this move, since $(\varphi_1 \rightarrow \varphi_2)$ is not a subformula of any offer. As a result, when we allow the Skeptic to play as above, the Prover has no winning strategy. Clearly, if the Skeptic is required to provide a propositional challenge, then the Prover has a winning strategy as the only possible play is the following (assume φ_2 atomic):

The Prover starts as before.
Skeptic: $\Pi_1 = \{(\varphi_1 \rightarrow \varphi_3), (\varphi_3 \rightarrow \varphi_2), \varphi_1\}$ and $\beta_1 = \varphi_2$.
Prover: $\Sigma_2 = \{\varphi_3\}$ and $\alpha_2 = \varphi_2$ (note that $(\varphi_3 \rightarrow \varphi_2) \in \Pi_1$).
Skeptic: $\Pi_2 = \emptyset$, $\beta_2 = \varphi_3$.
Prover: $\Sigma_2 = \{\varphi_1\}$ and $\alpha_2 = \varphi_3$ (note that $(\varphi_1 \rightarrow \varphi_3) \in \Pi_1$).
Skeptic: $\Pi_3 = \emptyset$, $\beta_3 = \varphi_1$.
Prover: $\Sigma_3 = \{\}$ and $\alpha_3 = \varphi_1$ (since $\varphi_1 \in \Pi_1$), and wins.

Theorem (IIPC games): ([5])[6] There is a winning Prover strategy for game $(\Gamma,\ \varphi)$ iff $\vdash_{IIPC} \Gamma \rhd \varphi$.

Proof:

⇒: Let D be a winning Prover strategy for (Γ, φ). Consider a P-node labelled (Σ, α), and let $\hat{\Gamma}$ be the union of Γ and all the offers made by the Skeptic along the partial dialogue from the root to the considered node. We show by induction on the sub-tree rooted at the considered node that for every $\sigma \in \Sigma$, $\vdash_{IIPC} \hat{\Gamma} \rhd \sigma$. Thus, in particular in the root of D, where

[6] Actually, [5] considers only the case $\Gamma = \emptyset$.

$\Sigma = \{\varphi\}$ and $\hat{\Gamma} = \Gamma$, we conclude that $\vdash_{IIPC} \Gamma \rhd \varphi$.
Basis: The base case is when the P-node (Σ, α) has no descendants. This means that S has no response, i.e., could not challenge Σ, which is, therefore, the empty set, and the claim follows vacuously.
Induction step: Let $\sigma \in \Sigma$. The P-node (Σ, α) has an S-node descendant that corresponds to the Skeptic challenging σ. This S-node is labelled (Π, β), such that $\sigma = \Pi \rightarrow \beta$. The descendant S-node (Π, β) has a descendant P-node, (Σ', β), with $\hat{\Gamma}' = \hat{\Gamma} \cup \Pi$ as its accumulated offers, where $\Sigma' \rightarrow \beta \in \hat{\Gamma}'$. Therefore, $\vdash_{IIPC} \hat{\Gamma}' \rhd \Sigma' \rightarrow \beta$. In addition, by the induction hypothesis, $\vdash_{IIPC} \hat{\Gamma}' \rhd \sigma'$, for every $\sigma' \in \Sigma'$. Therefore, $\vdash_{IIPC} \hat{\Gamma}' \rhd \beta$, and so $\vdash_{IIPC} \hat{\Gamma} \rhd \Pi \rightarrow \beta = \sigma$, proving the claim.

$\Leftarrow$**:** Assume $\vdash_{IIPC} \Gamma \rhd \varphi$, for $\Gamma = \{\varphi_1, \cdots, \varphi_n\}$. By the Curry-Howard correspondence (the subject-construction theorem ([5, 2])), there is a (simply-typed) λ-term M with free variables $x_1, \cdots, x_n$, s.t.

$$\vdash_{IIPC} \varphi_1 : x_1, \cdots, \varphi_n : x_n \rhd \varphi : M$$

The proof is by induction on M. W.l.o.g., assume M is in β-η long normal form, of the form $\lambda x_{n+1}, \cdots, \lambda x_{n+m}.x_k N_1 \cdots N_l$, where $1 \leq k \leq m + n$. Then, for every $1 \leq i \leq l$, $\vdash_{IIPC} \varphi_1 : x_1, \cdots, \varphi_{n+m} : x_{n+m} \rhd \rho_i : N_i$, where $\varphi_k = \rho_1 \rightarrow \cdots \rightarrow \rho_l \rightarrow \alpha$, and $\varphi = \varphi_{n+1} \rightarrow \cdots \rightarrow \varphi_{n+m} \rightarrow \alpha$, for some propositional variable α, the type of x_k.
By the induction hypothesis, there exists a winning strategy for each game $(\{\varphi_1, \cdots, \varphi_{n+m}\}, \rho_i)$, in which D_i, say, is the continuation for the Skeptic challenge (Π_i, β_i), $1 \leq i \leq l$. These strategies can be combined, to form a P-winning strategy for (Γ, φ), by letting the Skeptic challenge $(\{\varphi_{n+1}, \cdots, \varphi_{n+m}\}, \alpha)$ be answered by the P-assertion $(\{\rho_1, \cdots, \rho_l\}, \alpha)$ (which is an appropriate P-assertion, since φ_k is in the offer), where for each subsequent Skeptic challenge (Π_i, β_i) $(1 \leq i \leq l)$, the continuation is D_i, winning by the induction hypothesis.

3. Game semantics for L

3.1. The L calculus

To keep in mind the grammatical significance of this calculus, whose wffs are known as *categories*, we use meta-variables c, c_i to range over them. The wffs are the closure of the propositional variables under *two directed implications*, '$\rightarrow$' and '$\leftarrow$'. Occasionally, the infix notation $(c \rightarrow c')$, $(c' \leftarrow c)$ is replaced by $\rightarrow(c, c')$, $\leftarrow(c, c')$, respectively.

$$\varphi : x \rhd \varphi : x \quad (ax)$$

$$\frac{\Gamma_1 \rhd \varphi : N \quad \Gamma_2 \rhd (\varphi \rightarrow \psi) : M}{\Gamma_1 \Gamma_2 \rhd \psi : (MN)_{\rightarrow}} (\rightarrow E) \qquad \frac{\Gamma_2 \rhd (\varphi \leftarrow \psi) : M \quad \Gamma_1 \rhd \varphi : N}{\Gamma_1 \Gamma_2 \rhd \psi : (MN)_{\leftarrow}} (\leftarrow E)$$

$$\frac{[\varphi]_i : x \Gamma \rhd \psi : M}{\Gamma \rhd (\varphi \rightarrow \psi) : (\overrightarrow{\lambda}\, x.M)} (\rightarrow I_i) \qquad \frac{\Gamma [\varphi]_i : x \rhd \psi : M}{\Gamma \rhd (\psi \leftarrow \varphi) : (\overleftarrow{\lambda}\, x.M)} (\leftarrow I_i)$$

$$\Gamma \neq \emptyset, \quad x \notin Subjects(\Gamma)$$

Figure 2. Natural Deduction rules for **L** with (directed) term-assignments

In Figure 2 we present the natural-deduction rules with (directed) term-assignments for **L**.

Here Γ is a *sequence* of assumptions. In the elimination rules, the order of the combination of Γ_1, Γ_2 in the conclusion is determined by the direction of the eliminated arrow. In the introduction-rules, the position of the discharged assumption relative to Γ determines the direction of the introduced arrow. There are no structural rules. Note the restriction on the introduction-rules not to discharge the *last* assumption. Thus, there are no analogons of tautologies in this calculus, and in particular all proof-terms are *linear*, where each binder has exactly one occurrence of the bound variable in its scope. More details about **L** can be found in [4]. Below we use some additional notation:
$\Rightarrow^*_{(E)}$ and $\Rightarrow^*_{(I)}$ stand for finite series of applications of elimination rules and introduction rules, respectively, where both directions of the rules can be used. We use the following notational variation for directed λ-terms, originating in [6].

- $(MN)_{\leftarrow}$ and $(MN)_{\rightarrow}$: directed applications.
- $(\overleftarrow{\lambda}\, x.M)$ and $(\overrightarrow{\lambda}\, x.M)$: directed abstractions.
- Directed β-reductions:
  - $((\overleftarrow{\lambda}\, x.M)N)_{\leftarrow} \leadsto_\beta M[x := N]$
  - $((\overrightarrow{\lambda}\, x.M)N)_{\rightarrow} \leadsto_\beta M[x := N]$

Theorem (directed subject-construction): [6] If $\vdash_{\mathbf{L}} \psi_1, \ldots, \psi_n \rhd \varphi$, then there exists a directed λ-term M, with $free(M) = \{x_1, \cdots, x_n\}$, such that $\vdash_{\mathbf{L}} \psi_1 : x_1, \ldots \psi_n : x_n \rhd \varphi : M$.

4. The Lambek Game: first attempt

Consider now the game for **L**. It seems natural to keep the general scheme, where the Prover uses elimination rules and the Skeptic uses introduction rules. This time there are two schemes of introduction rules to be used by the Skeptic and two schemes of elimination rules to be used by the Prover. This is a minor difference. The main difference is that here Γ is a sequence and we have no structural rules (such as weakening) on it. To deal with that, the game has to refer to *sequents* instead of formulas. That is, we will no longer separate the formula that we want to derive and the set of assumptions that can be used in its derivation. We will now consider each formula with its exact sequence of assumptions, in the form of a sequent. In particular, we will define the *challenge* (previously, β_i) as a sequent, rather than a formula.

Remark: Henceforth, all sequents *have a non-empty antecedent*.

By carefully inspecting **L**, one can see that in fact the need for using sequents arises from the elimination rules only. This is because the introduction rules have a *single* premiss. This means that the Skeptic that uses the introduction rules to "construct" a challenge for the Prover by "reversing" the introduction rules, starting from some $\sigma \in \Sigma_i$, can construct the sequence of offers Π_i as a *single* sequence. Thus, we can separate the sequence of offers from the challenge formula, without causing confusion (having exactly one of each). However, when it comes to the elimination rules, used by the Prover, the sequence used as an assumption is split in the premisses of the rule. Since no weakening exists in this proof system (unlike $\vdash_{IIPC}$), it is crucial to keep each premiss as a sequent, rather than separating the formulas from the assumptions. This means that the Prover in his assertions has to provide *sequents*, rather than formulas.

To conclude, Σ_i has to be a sequence (or a set) of *sequents*. Π_i, on the other hand, could remain a sequence of *formulas*, in which case β_i remains a *formula*. We prefer, however, to work with sequents only. In this case, the challenge also will be a *sequent* β_i. This means that Π_i is no longer needed as a separate component. Thus the remaining components will be Σ_i and β_i only, with the following meaning: β_i is a sequent that the Skeptic challenges the Prover to derive (the offers are given as the assumption sequence within the sequent β_i), and Σ_i is a set of assertions (sequents) that the Prover uses in his derivation of β_i. The challenge β_i is the result of "reverse" application of the introduction rules on some assertion σ from Σ_i. Thus, deriving β_i gives a derivation of σ (by using the derivation of β_i and then applying introduction rules). Note that in the absence of structural rules, the antecedent of a sequent may have several copies of the same formula, positioned at different places in the antecedent. We henceforth refer to the succedent formula in a challenge sequent as 'the formula of the challenge'.

More specifically, here is a general preliminary description of the game. This description will be refined further later on.

Definition: [first attempt] A *dialogue* over (Γ, φ) is a possibly infinite sequence $\Sigma_1, \beta_1, \Sigma_2, \beta_2, \ldots$, where Σ_i are (possibly empty) sets of sequents and β_j are sequents, satisfying:

1. $\Sigma_1 = \{\Gamma \rhd \varphi\}$ (Prover begins).
2. $\Sigma_{i+1} \Rightarrow^*_{(E)} \beta_i$ (Prover meets the challenge by introducing new assertions that derive the challenge).
3. $\beta_i \Rightarrow^*_{(I)} \sigma$ for some $\sigma \in \Sigma_i$, which is *not* an axiom (Skeptic challenges sequent σ from preceding Prover move).

Comments:

1. Note that we explicitly exclude axioms from the assertions that the Skeptic can challenge. In the game for $\vdash_{IIPC}$ the use of axioms required no special care. This is because there the Prover was allowed to use previous assumptions (as axioms), and they were not considered assertions. However, here *every* sequent that the Prover uses to derive β_i is considered an assertion. Clearly axioms require no further proof. We thus explicitly exclude them from the assertions that the Skeptic can challenge. This can be handled differently by changing the definition of Σ_{i+1} to exclude axioms.
2. Winning conditions and strategies are defined as before.

Our goal now is to give a more concrete description of the game. In particular, we would like to find a syntactic characterization of Σ_{i+1} and β_i by following the definitions of the introduction and elimination rules.

4.1. The Skeptic's moves

The move of the Skeptic is defined by $\beta_i \Rightarrow^*_{(I)} \sigma$ for some $\sigma \in \Sigma_i$, which is *not* an axiom, with the meaning that the Skeptic challenges the sequent σ from the preceding Prover move. That is, the Skeptic chooses a sequent $\sigma \in \Sigma_i$ (which is not an axiom) and has to construct a challenge out of it by "inverting" the introduction rules. Recall that in the game for $\vdash_{IIPC}$, the formula of the challenge that the Skeptic provided had to be a propositional variable. For similar reasons, we require the same here for the formula of the challenge. This requirement leads to the following definition of β_i.

If the sequent $\sigma \in \Sigma_i$ that the Skeptic chose is of the form $\Gamma' \rhd \alpha'$, then the challenge that he constructs is as follows. Let c be the right-most propositional variable in α', when α' is presented using the above mentioned prefix notation. Then, α' can be presented as $*_1(c_1, \ldots *_{n-2} (c_{n-2}, *_{n-1}(c_{n-1}, c_n)) \ldots)$, where $*_i \in \{\rightarrow, \leftarrow\}$, and $c_n = c$. Let Γ_r denote the sequence constructed from the c_i's such that $*_i = \rightarrow$, ordered by their indices from the highest to the lowest. Let Γ_l denote the sequence constructed from the c_i's such that $*_i = \leftarrow$, ordered by their indices from the lowest to the highest (opposite order). Then the Skeptic gives as a challenge $\Gamma_r \Gamma' \Gamma_l \rhd c_n$.

The move can also be defined recursively as follows. The Skeptic starts from $\sigma = \Gamma' \rhd \alpha' \in \Sigma_i$, and as long as α' is not a propositional variable, he "reverses" the introduction rule that matches its main connective, to get a new sequent.

Note that formulas in the offer part of a challenge sequent come with the indication as to the *direction of use* of this offer formula. This is determined by which introduction-rule was "inverted" to generate this offer. In particular, different instances of the same formula in an offer sequent may be used in different directions; see examples in the sequel.

Looking at it this way, we see that the Skeptic is obligated to keep using the introduction rules *as long as they are applicable* (in reverse). The intuition behind this maximal application is that if the derivation of the challenge consists of introduction rules only, while the Skeptic ceases applying them although application is still possible, then we cannot expect the Prover to be able to derive the conclusion, using *elimination* rules. That is, we have to make sure that the challenge given by the Skeptic can be derived (in case it is derivable) using elimination rules. To achieve this, we make the Skeptic apply introduction rules (in reverse) as long as it is possible, or in other words, as long as the remaining formula is not a propositional variable.

To demonstrate the latter argument, consider the following example, proving that $\vdash_{\mathbf{L}} c_1, ((c_1 \rightarrow c_2) \leftarrow c_1) \rhd (c_2 \leftarrow c_1)$. When the Skeptic applies introduction rules as much as possible and gives a propositional variable as the formula of the challenge, we get the following dialogue, where the Prover wins:

P (1): $c_1, ((c_1 \to c_2) \leftarrow c_1) \rhd (c_2 \leftarrow c_1)$
S (1): $c_1, ((c_1 \to c_2) \leftarrow c_1), c_1 \rhd c_2$
P (2): $c_1 \rhd c_1,$
$\big(((c_1 \to c_2) \leftarrow c_1) \rhd ((c_1 \to c_2) \leftarrow c_1),$
$c_1 \rhd c_1\big)$

To understand the last move: note that applying $(\leftarrow E)$ on $((c_1 \to c_2) \leftarrow c_1) \rhd ((c_1 \to c_2) \leftarrow c_1)$ and $c_1 \rhd c_1$ results in $((c_1 \to c_2) \leftarrow c_1), c_1 \rhd (c_1 \to c_2)$. Applying $(\to E)$ on $c_1 \rhd c_1$ and the latter results in the challenge. This gives us a derivation of the challenge, starting from the assertions of the Prover, using elimination rules only. P wins the dialogue since she uses only axioms in the last move, thus S has no response.

Furthermore, if we require the Skeptic to always provide a propositional variable as the formula of the challenge, then the Skeptic has no other possible move in his first (and only) move. This means that the latter is also a winning strategy for the Prover.

Similarly to the $\vdash_{IIPC}$ games, if the Skeptic could choose to provide a compound formula as the challenge formula, the following dialogue becomes possible.

S (1'): $c_1, ((c_1 \to c_2) \leftarrow c_1) \rhd (c_2 \leftarrow c_1)$

Since the formula of the challenge, $(c_2 \leftarrow c_1)$, does not appear in the assumption sequence of the challenge, $c_1, ((c_1 \to c_2) \leftarrow c_1)$, it is clear that the challenge cannot be derived using elimination rules only, and thus the Prover cannot respond and she loses. This means that the Prover does not have a winning strategy, although the sequent $c_1, ((c_1 \to c_2) \leftarrow c_1) \rhd (c_2 \leftarrow c_1)$ is derivable in the Lambek calculus. We conclude that the requirement of the Skeptic to provide a propositional variable as the formula of the challenge is essential for the correctness of the game.

Given this definition of β_i, where its formula has to be a propositional variable, we get that the following property of IIPC-games is preserved for **L**-games too: once the Skeptic chooses the assertion that he wants to "tackle" from Σ_i, the challenge is automatically determined. Thus, the Skeptic's choice, in his turn, boils down to choosing which assertion he wants to challenge.

4.2. The Prover's Moves

The goal of the Prover is to show that the challenge β_i is **L**-derivable. For this purpose P provides a set of assertions Σ_{i+1} that derive β_i, using elimination rules *only*. That is, $\Sigma_{i+1} \Rightarrow^*_{(E)} \beta_i$.

One way to look at it, is to see Σ_{i+1} as a sequence that represents the *leaves* of a binary tree (where every internal node has exactly two descendants), defined as follows. The nodes of the tree are sequents. The root is the challenge, and each internal node (sequent) in the tree is the result of an application of $(\rightarrow E)$ or $(\leftarrow E)$ on its two descendants. In particular, this implies that the concatenation of all the assumption sequences of the sequents in Σ_{i+1} results in the assumption sequence of the challenge β_i. (This is true provided that in the tree we always write the Γ_1 component of the elimination rule $(\rightarrow E)$ at the left and the Γ_2 component at the right, and vice versa for the elimination rule $(\leftarrow E)$). This is demonstrated by the second move of the Prover in the above example.

The problem with this view of Σ_{i+1} is that it is not explicit enough. Furthermore, it matches the intuition that the Prover can base her move on *any* set of applications of elimination rules. A closer inspection of the game for $\vdash_{IIPC}$ shows that there the Prover is more limited than that. We thus wish to limit in a similar way the moves of the Prover in our case as well.

First, note that if we allow the Prover to only simulate *one* application of an elimination rule, then we limit P too much. Intuitively, this is because it is possible that the proof from a certain stage uses only elimination rules, and letting the Skeptic interfere in the middle, can result in an infinite dialogue. To demonstrate it, consider again the same example above, proving that $\vdash_{\mathbf{L}} c_1, ((c_1 \rightarrow c_2) \leftarrow c_1) \rhd (c_2 \leftarrow c_1)$. As we saw before, when we allowed the Prover to use *any* set of applications of elimination rules, we got a dialogue, where the Prover won (and also had a winning strategy). Yet, if the Prover can only use one elimination rule, then she will not be able to respond as above. We will get the following (only possible) dialogue.

P (2'):	$c_1 \rhd c_1$,
	$((c_1 \rightarrow c_2) \leftarrow c_1), c_1 \rhd (c_1 \rightarrow c_2)$
S (2'):	$c_1, ((c_1 \rightarrow c_2) \leftarrow c_1), c_1 \rhd c_2$ [=S(1)]
	.
	.
	.

That is, the (only) dialogue is infinite, and the Prover does not win, although the sequent $c_1, ((c_1 \rightarrow c_2) \leftarrow c_1) \rhd (c_2 \leftarrow c_1)$ is **L**-derivable. Therefore, limiting the Prover this way is not possible. Let us now see how we can limit her, based on the analogy of our game to the $\vdash_{IIPC}$ game.

In the game for $\vdash_{IIPC}$, the Prover chooses an offer and "splits" it to $\Sigma_{i+1} \rightarrow \beta_i$. We would like to find the analog definition for our case, where

β_i is a *sequent* rather than a formula. In other words, all the "offers" are gathered in the assumptions sequence (i.e., the antecedent) of β_i, denoted $assump(\beta_i)$. Thus the analogon here should be to "choose" one of the formulas α in $assump(\beta_i)$ and split it similarly to $form(\Sigma_{i+1}) \rightarrow form(\beta_i)$. There are a few differences. First, in **L** we have two directed implications, and the "split" should consider both of them. Second, in **L** the rest of the offers (formulas) in $assump(\beta_i)$ should also be "partitioned" to subsequences. Each such subsequence should be considered as an assumptions sequence of one of the formulas of $form(\Sigma_{i+1})$. This "matching" between subsequences of $assump(\beta_i)$ and formulas of $form(\Sigma_{i+1})$ should be done in a way that indeed enables application of elimination rules on $\alpha \rhd \alpha$ and the resulting sequents to derive the sequent β_i. This part (of partitioning $assump(\beta_i)$ and matching to $form)\Sigma_{i+1})$ was unnecessary for $\vdash_{IIPC}$, because there *all* the offers could be used all the time, in any order. This is not the case here.

Therefore, the Prover chooses $\alpha \in assump(\beta_i)$, s.t. $\alpha = *_1(\alpha_1, \ldots *_{n-2} (\alpha_{n-2}, *_{n-1}(\alpha_{n-1}, form(\beta_i))) \ldots)$, where $*_i \in \{\rightarrow, \leftarrow\}$. She returns the α_i's as the formulas in Σ_{i+1}.

The assumption sequences that match each formula in Σ_{i+1} obey the following rule. Let Γ_r denote the sequence constructed from the assumption sequences of the α_i's such that $*_i = \rightarrow$, ordered by their indices from the highest to the smallest. Let Γ_l denote the sequence constructed from the assumption sequences of the α_i's such that $*_i = \leftarrow$, ordered by their indices from the smallest to the highest (opposite order). Then, it should be the case that $assump(\beta_i) = \Gamma_r \alpha \Gamma_l$.

To sum up, we get the following definition of the Prover's moves. The Prover returns $\Sigma_{i+1} = \{\sigma_1, \ldots \sigma_{n-1}\}$, such that $assump(\beta_i) = \Gamma_r \alpha \Gamma_l$, where:

- $\alpha = *_1(form(\sigma_1), \ldots *_{n-2} (form(\sigma_{n-2}), *_{n-1}(form(\sigma_{n-1}), form(\beta_i))) \ldots)$.
- Γ_l is the sequence constructed from $assump(\sigma_i)$ such that $*_i = \leftarrow$, ordered by their indices from the smallest to the highest.
- Γ_r is the sequence constructed from $assump(\sigma_i)$ such that $*_i = \rightarrow$, ordered by their indices from the highest to the smallest.

Note, that α is not part of $form(\Sigma_{i+1})$. In this sense we changed the definition of Σ_{i+1}, because now its sequents alone do not necessarily derive β_i. Instead, now $\Sigma_{i+1} \cup \{\alpha \rhd \alpha\}$ derive β_i.

This can be verified via a close inspection of the above definition. It can be seen that an application of $(*_1 E)$ on $\alpha \rhd \alpha$ and σ_1, and then application of $(*_2 E)$ on the result and σ_2, and so on until an application of $(*_{n-1} E)$ on the result of the previous step and σ_{n-1}, results in β_i. Thus, Σ_{i+1} indeed consists of a set of sequents that derive β_i, with the "help" of the axiom $\alpha \rhd \alpha$.

When using the new definition, there is no need to consider Σ_{i+1} as a sequence or a tree. A set is sufficient.

5. The Lambek Game Definition

We now present the actual definition for the Lambek calculus game.

Definition: A *dialogue* over (Γ, φ) is a possibly infinite sequence $\Sigma_1, \beta_1, \Sigma_2, \beta_2, \ldots$, where Σ_i are (possibly empty) sets of sequents and β_i are sequents, satisfying:

1. $\Sigma_1 = \{\Gamma \rhd \varphi\}$ (Prover begins).
2. $\beta_i = \Gamma_r \Gamma' \Gamma_l \rhd c$, where:
   - $\Gamma' \rhd \alpha' \in \Sigma_{i+1}$ (which is not an axiom).
   - $\alpha' = *_1(c_1, \ldots *_{n-1}(c_{n-1}, *_n(c_n, c)) \ldots)$, where $*_i \in \{\rightarrow, \leftarrow\}$, and c is a propositional variable (the rest of the c_i's can be more complicated formulas).
   - Γ_r denotes the sequence constructed from the c_i's such that $*_i = \rightarrow$, ordered by their indices from the highest to the smallest.
   - Γ_l denotes the sequence constructed from the c_i's such that $*_i = \leftarrow$, ordered by their indices from the smallest to the highest (opposite order).

   That is, $\beta_i \Rightarrow^*_{(I)} \sigma$ for $\sigma = \Gamma' \rhd \alpha' \in \Sigma_i$, which is *not* an axiom (Skeptic challenges assertion σ from preceding Prover move).
3. $\Sigma_{i+1} = \{\Gamma_1 \rhd \alpha_1, \ldots, \Gamma_k \rhd \alpha_k\}$, where:
   - $assump(\beta_i) = \Gamma_r \alpha \Gamma_l$.
   - $\alpha = *_1(\alpha_1, \ldots *_{k-1}(\alpha_{k-1}, *_k(\alpha_k, form(\beta_i))) \ldots)$.
   - Γ_r denotes the sequence constructed from the Γ_i's such that $*_i = \rightarrow$, ordered by their indices from the highest to the smallest.
   - Γ_l denotes the sequence constructed from the Γ_i's such that $*_i = \leftarrow$, ordered by their indices from the smallest to the highest.

   That is, $\Sigma_{i+1} \cup \{\alpha \rhd \alpha\} \Rightarrow^*_{(E)} \beta_i$ (Prover meets the challenge by introducing new assertions that derive the challenge).

Winning conditions and strategies are defined as in the game for $\vdash_{IIPC}$. That is,

Definition: A *winning Prover strategy* for an **L**-game over (Γ, φ) is a finite labelled tree where:

- The root is a P-node.
- Every branch is a dialogue over (Γ, φ).
- Every node labelled with a Prover step (P-node) with n assertions (excluding the axioms) has n S-nodes descendants labelled with distinct steps.
- Every node labelled β (an S-node) has one P-node descendant.

Consequently, in a leaf with a Prover step all the assertions are axioms.

Note, that in this case the Skeptic has no response if all of the assertions that the Prover made are axioms.

Theorem (L-games): There is a winning Prover strategy for (Γ, φ) iff $\vdash_{\mathbf{L}} \Gamma \rhd \varphi$.

Before proving the theorem, we present several examples of **L**-games.

5.1. Game Examples

Example 1

Based on $\vdash_{\mathbf{L}} c_1, ((c_1 \rightarrow c_2) \leftarrow c_1) \rhd (c_2 \leftarrow c_1)$.

P: $c_1, ((c_1 \rightarrow c_2) \leftarrow c_1) \rhd (c_2 \leftarrow c_1)$
S: $c_1, ((c_1 \rightarrow c_2) \leftarrow c_1), c_1 \rhd c_2$
P: $c_1 \rhd c_1$,
$c_1 \rhd c_1$
$[\ \alpha = ((c_1 \rightarrow c_2) \leftarrow c_1)\ ]$

Note that this is the example we used before. However, since we changed the definition of the game (especially the moves of the Prover), the dialogue is now different. Namely, in the last move of the Prover, the sequent (axiom) $((c_1 \rightarrow c_2) \leftarrow c_1) \rhd ((c_1 \rightarrow c_2) \leftarrow c_1)$, becomes a "pivot", around which the other assumptions are split. The same happens and in all subsequent examples.

To understand the last move: the Prover chooses $\alpha = ((c_1 \rightarrow c_2) \leftarrow c_1)$. Since $form(\beta) = c_2$, this makes $\alpha_1 = \alpha_2 = c_1$ and $*_1 = \leftarrow$, $*_2 = \rightarrow$. Therefore, $\Gamma_r = \Gamma_1$ and $\Gamma_l = \Gamma_2$. Furthermore, the assumption sequence of the challenge, which is $c_1, ((c_1 \rightarrow c_2) \leftarrow c_1), c_1$ should be equal to $\Gamma_r \alpha \Gamma_l$.

This means that $\Gamma_r = \Gamma_l = c_1$. Together with our knowledge that $\Gamma_r = \Gamma_1$, $\Gamma_l = \Gamma_2$, and that $\alpha_1 = \alpha_2 = c_1$, we get $\Sigma = \{\sigma_1, \sigma_2\}$ for $\sigma_1 = \Gamma_1 \rhd \alpha_1 = c_1 \rhd c_1$, $\sigma_2 = \Gamma_2 \rhd \alpha_2 = c_1 \rhd c_1$.

Clearly, since Σ is a *set* of sequents, we should avoid repetitions. We did not remove them from the examples only to make them more understandable.

P wins since she uses only axioms in the last move, thus S has no response. This is also a winning strategy for P in this case since the Skeptic has no other possible move in his first (and only) move.

Example 2

Based on $\vdash_{\mathbf{L}} c_1, ((c_1 \to c_2) \leftarrow c_1), (c_2 \leftarrow c_1) \to c_3) \rhd c_3$.

P: $c_1, ((c_1 \to c_2) \leftarrow c_1), (c_2 \leftarrow c_1) \to c_3) \rhd c_3$
S: $c_1, ((c_1 \to c_2) \leftarrow c_1), (c_2 \leftarrow c_1) \to c_3) \rhd c_3$
P: $c_1, ((c_1 \to c_2) \leftarrow c_1) \rhd (c_2 \leftarrow c_1)$
[$\alpha = (c_2 \leftarrow c_1) \to c_3)$]
S: $c_1, ((c_1 \to c_2) \leftarrow c_1), c_1 \rhd c_2$
[challenges the first assertion, the second is an axiom.
From now on the dialogue proceeds as in the previous example.]
P: $c_1 \rhd c_1$,
$c_1 \rhd c_1$
[$\alpha = ((c_1 \to c_2) \leftarrow c_1)$]

Once again P wins, and this is also a winning strategy for her.

Example 3

In the previous examples the Prover always used at most one assertion that is not an axiom in her moves. Thus, the Skeptic had only one possible response, and the notion of a winning strategy became degenerate.

We now give an example where this not the case. The example is based on $\vdash_{\mathbf{L}} c_4, ((c_1 \to c_2) \to c_3) \leftarrow (c_4 \to c_5), c_3, c_3 \to (c_4 \to c_5) \rhd ((c_1 \to c_2) \leftarrow c_4) \to c_3$.

P (1): $c_4, ((c_1 \rightarrow c_2) \rightarrow c_3) \leftarrow (c_4 \rightarrow c_5), c_3, c_3 \rightarrow (c_4 \rightarrow c_5)$
$\rhd ((c_1 \rightarrow c_2) \leftarrow c_4) \rightarrow c_3$

S (1): $(c_1 \rightarrow c_2) \leftarrow c_4, c_4, ((c_1 \rightarrow c_2) \rightarrow c_3) \leftarrow (c_4 \rightarrow c_5), c_3,$
$c_3 \rightarrow (c_4 \rightarrow c_5) \rhd c_3$

P (2): $(c_1 \rightarrow c_2) \leftarrow c_4, c_4 \rhd c_1 \rightarrow c_2,$
$c_3, c_3 \rightarrow (c_4 \rightarrow c_5) \rhd c_4 \rightarrow c_5$
[$\alpha = ((c_1 \rightarrow c_2) \rightarrow c_3) \leftarrow (c_4 \rightarrow c_5)$]

S (2): $c_1, (c_1 \rightarrow c_2) \leftarrow c_4, c_4 \rhd c_2$
[challenges the first assertion]

P (3): $c_1 \rhd c_1,$
$c_4 \rhd c_4$
[$\alpha = (c_1 \rightarrow c_2) \leftarrow c_4$]

P wins.

Another possibility:

S (2'): $c_4, c_3, c_3 \rightarrow (c_4 \rightarrow c_5) \rhd c_5$
[challenges the second assertion]

P (3'): $c_4 \rhd c_4,$
$c_3 \rhd c_3$
[$\alpha = c_3 \rightarrow (c_4 \rightarrow c_5)$]

P wins as well. Since there are no more possibilities for the Skeptic to respond, the Prover has a winning strategy.

Example 4

Based on a sequent that is *not* derivable in the proof system, namely
$\nvdash_{\mathbf{L}} (c_4 \leftarrow ((c_1 \rightarrow c_2) \rightarrow c_3)), (c_3 \leftarrow (c_1 \rightarrow c_5)) \rhd (c_4 \leftarrow (c_2 \rightarrow c_5))$.

P: $(c_4 \leftarrow ((c_1 \rightarrow c_2) \rightarrow c_3)), (c_3 \leftarrow (c_1 \rightarrow c_5)) \rhd (c_4 \leftarrow (c_2 \rightarrow c_5))$
S: $(c_4 \leftarrow ((c_1 \rightarrow c_2) \rightarrow c_3)), (c_3 \leftarrow (c_1 \rightarrow c_5)), (c_2 \rightarrow c_5) \rhd c_4$

That is, $\beta = (c_4 \leftarrow ((c_1 \rightarrow c_2) \rightarrow c_3)), (c_3 \leftarrow (c_1 \rightarrow c_5)), (c_2 \rightarrow c_5) \rhd c_4$. The only assumption in $assump(\beta)$ that contains c_4 (which is $form(\beta)$), is $(c_4 \leftarrow ((c_1 \rightarrow c_2) \rightarrow c_3))$. Therefore, the only possibility of the Prover is to "split" this assumption as $(c_4 \leftarrow \alpha_1)$, where $\alpha_1 = ((c_1 \rightarrow c_2) \rightarrow c_3)$. This means that $\Gamma_r = \emptyset, \Gamma_l = (c_3 \leftarrow (c_1 \rightarrow c_5)), (c_2 \rightarrow c_5)$. Furthermore, since $*_1 = \leftarrow$, the assumption sequence of α_1 is Γ_l. The result is the following move (where the Prover makes only one assertion):

P: $(c_3 \leftarrow (c_1 \rightarrow c_5)), (c_2 \rightarrow c_5) \rhd ((c_1 \rightarrow c_2) \rightarrow c_3)$
$[\ \alpha = (c_4 \leftarrow ((c_1 \rightarrow c_2) \rightarrow c_3))\ ]$
S: $(c_1 \rightarrow c_2), (c_3 \leftarrow (c_1 \rightarrow c_5)), (c_2 \rightarrow c_5) \rhd c_3$

That is, $\beta = (c_1 \rightarrow c_2), (c_3 \leftarrow (c_1 \rightarrow c_5)), (c_2 \rightarrow c_5) \rhd c_3$. The only assumption in the assumption sequence of β that contains c_3 (which is $form(\beta)$), is $(c_3 \leftarrow (c_1 \rightarrow c_5))$. Therefore, the only possibility of the Prover is to “split” this assumption as $(c_3 \leftarrow \alpha_1)$, where $\alpha_1 = (c_1 \rightarrow c_5)$. This means that $\Gamma_r = (c_1 \rightarrow c_2), \Gamma_l = (c_2 \rightarrow c_5)$. Yet, the Prover asserts only one formula (α_1), which makes it impossible to have two assumption sequences. The conclusion is that the Prover has no response that obeys the rules, and thus she loses. Since this is the only possible dialogue, we conclude that the Prover has no winning strategy.

Example 5 - Bi-directionality

Based on $\vdash_{\mathbf{L}} c_2 \leftarrow c_1, c_1, c_1 \rightarrow c_2, c_2 \rightarrow (c_1 \rightarrow c_3), c_3 \rhd c_2$.

P: $c_2 \leftarrow c_1, c_1, c_1 \rightarrow c_2, c_2 \rightarrow (c_1 \leftarrow c_3), c_3 \rhd c_2$
S: $c_2 \leftarrow c_1, c_1, c_1 \rightarrow c_2, c_2 \rightarrow (c_1 \leftarrow c_3), c_3 \rhd c_2$
P: $c_1, c_1 \rightarrow c_2, c_2 \rightarrow (c_1 \leftarrow c_3), c_3 \rhd c_1$
$[\ \alpha = c_2 \leftarrow c_1\ ]$
S: $c_1, c_1 \rightarrow c_2, c_2 \rightarrow (c_1 \leftarrow c_3), c_3 \rhd c_1$
P: $c_1, c_1 \rightarrow c_2 \rhd c_2$
$c_3 \rhd c_3$
$[\ \alpha = c_2 \rightarrow (c_1 \leftarrow c_3)\ ]$
S: $c_1, c_1 \rightarrow c_2 \rhd c_2$
P: $c_1 \rhd c_1$
$[\ \alpha = c_1 \rightarrow c_2\ ]$

P wins, and since the Skeptic has no other possibilities in his moves, this is also a winning Prover strategy.

This example demonstrates how the rules of the game guide the Prover in finding the correct derivation. Consider the first move of the Skeptic. The Skeptic challenges the Prover to derive c_2 using the assumptions sequence $c_2 \leftarrow c_1, c_1, c_1 \rightarrow c_2, c_2 \rightarrow (c_1 \leftarrow c_3), c_3$, which contains $c_2 \leftarrow c_1, c_1, c_1 \rightarrow c_2$ as a subsequence. In this subsequence, c_1 can be used either with the assumption to its left, or with the assumption to its right to derive c_2 (via elimination). However, the derivation that leads to success in the overall task is one where c_1 is used with the assumption to its right, $c_1 \rightarrow c_2$, at a later round. This is reflected in the game by the fact that if the Prover had chosen

$\alpha = c_1 \to c_2$ as a response to the current Skeptic move, then she would not have been able to "match" the remaining subsequence $c_2 \to (c_1 \leftarrow c_3), c_3$ to its right with a formula. This indicates that this is not the correct derivation.

Example 6

Based on $\vdash_{\mathbf{L}} c_4, c_4 \to (c_1 \to c_2), (c_3 \to c_4) \leftarrow c_1, c_1, (c_1 \to c_2) \to ((c_3 \to c_4) \to c_4) \rhd c_4$

P (1): $c_4, c_4 \to (c_1 \to c_2), (c_3 \to c_4) \leftarrow c_1, c_1,$
$(c_1 \to c_2) \to ((c_3 \to c_4) \to c_4) \rhd c_4$
S (1): $c_4, c_4 \to (c_1 \to c_2), (c_3 \to c_4) \leftarrow c_1, c_1,$
$(c_1 \to c_2) \to ((c_3 \to c_4) \to c_4) \rhd c_4$
P (2): $c_4, c_4 \to (c_1 \to c_2) \rhd c_1 \to c_2$
$(c_3 \to c_4) \leftarrow c_1, c_1 \rhd c_3 \to c_4$
[$\alpha = (c_1 \to c_2) \to ((c_3 \to c_4) \to c_4)$]
S (2): $c_1, c_4, c_4 \to (c_1 \to c_2) \rhd c_2$
[challenges the first assertion]
P (3): $c_4 \rhd c_4$
$c_1 \rhd c_1$
[$\alpha = c_4 \to (c_1 \to c_2)$]

P wins.

Another possibility:

S (2'): $c_3, (c_3 \to c_4) \leftarrow c_1, c_1 \rhd c_4$
[challenges the second assertion]
P (3'): $c_3 \rhd c_3,$
$c_1 \rhd c_1$
[$\alpha = (c_3 \to c_4) \leftarrow c_1$]

P wins as well. Since there are no more possibilities for the Skeptic to respond, the Prover has a winning strategy.

In the previous examples, once the Prover chose α, it immediately determined how $assump(\beta)$ and α should be split to result in the new assertions. This example demonstrates that this is not always the case. Consider the second move of the Prover. There, $assump(\beta) = c_4, c_4 \to (c_1 \to c_2), (c_3 \to c_4) \leftarrow c_1, c_1, (c_1 \to c_2) \to ((c_3 \to c_4) \to c_4)$. Thus, after choosing $\alpha = (c_1 \to c_2) \to ((c_3 \to c_4) \to c_4)$, the Prover still needed to make a choice of how to divide the subsequence $c_4, c_4 \to (c_1 \to c_2), (c_3 \to c_4) \leftarrow c_1, c_1$, consisting of four formulas, into two subsequences: one that would become the assumption sequence of $(c_1 \rightarrow c_2)$ and another that would become the assumption sequence of $(c_3 \rightarrow c_4)$.

6. Correctness Proof

We start with some background that is needed for the proof.

6.1. Preliminaries for $\Rightarrow$

For the direction from left to right we state two simple properties of the Players' moves.

Property: [Skeptic] Let $\sigma' = \Gamma' \rhd \alpha'$ be a sequent (which is not an axiom) and let $\beta_i = \Gamma_r \Gamma' \Gamma_l \rhd c$, where:

- $\alpha' = *_1(c_1, \ldots *_{n-1} (c_{n-1}, *_n(c_n, c)) \ldots)$, where $*_i \in \{\rightarrow, \leftarrow\}$, and c is a propositional variable (the rest of the c_i's can be compound formulas).
- Γ_r denotes the sequence constructed from the c_i's such that $*_i = \rightarrow$, ordered by their indices from the highest to the smallest.
- Γ_l denotes the sequence constructed from the c_i's such that $*_i = \leftarrow$, ordered by their indices from the smallest to the highest (opposite order).

Then, $\beta_i \Rightarrow^*_{(I)} \sigma'$.

It follows that the challenge β_i that the Skeptic chooses based on some assertion σ' of the Prover, fulfills $\beta_i \Rightarrow^*_{(I)} \sigma'$.

Property: [Prover] Let β_i be a sequent and let $\Sigma_{i+1} = \{\Gamma_1 \rhd \alpha_1, \ldots, \Gamma_k \rhd \alpha_k\}$. Suppose that $assump(\beta_i) = \Gamma_r \alpha \Gamma_l$, where

- $\alpha = *_1(\alpha_1, \ldots *_{k-1} (\alpha_{k-1}, *_k(\alpha_k, form(\beta_i))) \ldots)$.
- Γ_r denotes the sequence constructed from the Γ_i's such that $*_i = \rightarrow$, ordered by their indices from the highest to the smallest.
- Γ_l denotes the sequence constructed from the Γ_i's such that $*_i = \leftarrow$, ordered by their indices from the smallest to the highest.

Then, $\Sigma_{i+1} \cup \{\alpha \rhd \alpha\} \Rightarrow^*_{(E)} \beta_i$.

It follows that the assertions Σ_{i+1} that the Prover supplies in response to the challenge β_i fulfill $\Sigma_{i+1} \cup \{\text{axiom}\} \Rightarrow^*_{(E)} \beta_i$.

Both properties can be proved by induction on n (the number of introduction rules that the Skeptic simulates), and k (the number of elimination rules that the Prover simulates, which is also the number of assertions in Σ_i), respectively.

Proof: (of the Skeptic-Property)
We prove that for every $n \geq 0$, the sequent β_i that is defined by $\Gamma_r\Gamma'\Gamma_l \rhd c$ for $\alpha' = *_1(c_1, \ldots *_{n-1} (c_{n-1}, *_n(c_n, c)) \ldots)$, and *any* sequence Γ', fulfills $\beta_i \Rightarrow^*_{(I)} \Gamma' \rhd \alpha'$.

Base case: $n = 0$: in this case $\alpha' = c$ and therefore $\Gamma_r = \Gamma_l = \emptyset$, meaning that $\beta_i = \Gamma' \rhd c$ and the claim is clear.

Induction step: $n > 0$: denote $*_2(c_2, \ldots *_{n-1} (c_{n-1}, *_n(c_n, c)) \ldots)$ by η. We distinguish between two cases.

$*_1 =\rightarrow$**:** i.e. $\alpha' = c_1 \rightarrow \eta$. By the induction hypothesis, $\beta' = \Gamma'_r c_1 \Gamma' \Gamma'_l \rhd c$, where Γ'_r, Γ'_l are defined w.r.t. η, fulfills $\beta' \Rightarrow^*_{(I)} c_1\Gamma' \rhd \eta$. Furthermore, $c_1\Gamma' \rhd \eta \Rightarrow_{(\rightarrow I)} \Gamma' \rhd c_1 \rightarrow \eta \equiv \Gamma' \rhd \alpha'$. Thus $\beta' \Rightarrow^*_{(I)} \Gamma' \rhd \alpha'$. In addition, since $\Gamma'_r c_1 = \Gamma_r$ and $\Gamma'_l = \Gamma_l$ (for $\alpha' = c_1 \rightarrow \eta$), the sequent $\beta' = \Gamma'_r c_1 \Gamma' \Gamma'_l \rhd c$, equals β_i, and we conclude that $\beta_i \Rightarrow^*_{(I)} \Gamma' \rhd \alpha'$.

$*_1 =\leftarrow$**:** i.e. $\alpha' = \eta \leftarrow c_1$: similar

Proof: (of the Prover-Property)
We prove that for every $k \geq 0$, and for every sequent β_i, the set $\Sigma_{i+1} = \{\Gamma_1 \rhd \alpha_1, \ldots, \Gamma_k \rhd \alpha_k\}$, that satisfies $assump(\beta_i) = \Gamma_r\alpha\Gamma_l$, for $\alpha = *_1(\alpha_1, \ldots *_{k-1} (\alpha_{k-1}, *_k(\alpha_k, form(\beta_i))) \ldots)$, fulfills $\Sigma_{i+1} \cup \{\alpha \rhd \alpha\} \Rightarrow^*_{(E)} \beta_i$.

Base case: $k = 0$: in this case $assump(\beta_i) = \alpha = form(\beta_i)$, which means that β_i is an axiom and the claim holds trivially. Note, that this case is not possible in the game, since the Skeptic cannot provide an axiom as the challenge β_i.

Induction step: $k > 0$: denote $*_k(\alpha_k, form(\beta_i))$ by η. That is, $\alpha = *_1(\alpha_1, \ldots *_{k-1} (\alpha_{k-1}, \eta) \ldots)$. By the induction hypothesis, the set $\Sigma' = \{\Gamma_1 \rhd \alpha_1, \ldots, \Gamma_{k-1} \rhd \alpha_{k-1}\}$ that satisfies $assump(\beta') = \Gamma'_r\alpha\Gamma'_l$ for $\beta' = \Gamma'_r\alpha\Gamma'_l \rhd \eta$, fulfills $\Sigma' \cup \{\alpha \rhd \alpha\} \Rightarrow^*_{(E)} \beta'$ (*). We distinguish between two cases.

$*_k =\rightarrow$**:** i.e. $\eta = \alpha_k \rightarrow form(\beta_i)$. In this case, $\Gamma_k \rhd \alpha_k,\ \beta'\ [\equiv \Gamma'_r\alpha\Gamma'_l \rhd \alpha_k \rightarrow form(\beta_i)]\ \Rightarrow_{(\rightarrow E)} \Gamma_k\Gamma'_r\alpha\Gamma'_l \rhd form(\beta_i)$. Since $\Gamma_k\Gamma'_r = \Gamma_r$ and $\Gamma'_l = \Gamma_l$ (for $*_k =\rightarrow$), we get that $\Gamma_k \rhd \alpha_k, \beta' \Rightarrow_{(\rightarrow E)} \Gamma_r\alpha\Gamma_l \rhd form(\beta_i)$, and the latter sequent is exactly β_i, meaning that $\Gamma_k \rhd \alpha_k, \beta' \Rightarrow_{(\rightarrow E)} \beta_i$. By replacing the role of β' in the derivation by the derivation guaranteed by (*), we get that $\{\Gamma_k \rhd \alpha_k\} \cup \Sigma' \cup \{\alpha \rhd \alpha\} \Rightarrow^*_{(E)} \beta_i$, and thus $\Sigma_{i+1} \cup \{\alpha \rhd \alpha\} \Rightarrow^*_{(E)} \beta_i$.

$*_k =\leftarrow$**:** i.e. $\eta = form(\beta_i) \leftarrow \alpha_k$: similar.

6.2. Preliminaries for $\Leftarrow$

For the direction from right to left of the theorem we need some background about directed λ-terms [6] and their CH-correspondence with **L**.

It is known ([6], appendix 5.8) that every typed λ-term[7] can be reduced to a λ-term of the same type in β normal form. That is,

Theorem (L weak normalization): If $\vdash_{\mathbf{L}} \Gamma \rhd \varphi : M$, then there exists M' in β normal form such that $\vdash_{\mathbf{L}} \Gamma \rhd \varphi : M'$.

Our goal now is to analyze the β normal form of a typed term.

Theorem (shape of β normal form): Any typed λ-term in β normal form is of the form
$\lambda_{*_1} x_1 \ldots \lambda_{*_n} x_n.(\ldots((xN_1)_{*_1} N_2)_{*_2} \ldots N_l)_{*_l}$, where

- $n \geq 0, l \geq 0$
- for every $1 \leq i \leq n$: $\lambda_{*_i} \in \{\overrightarrow{\lambda}, \overleftarrow{\lambda}\}$
- for every $1 \leq i \leq l$: $()_{*_i} \in \{()_\rightarrow, ()_\leftarrow\}$
- for every $1 \leq i \leq l$: N_i is in β-normal form.

Proof: For the purpose of analyzing the β normal form, first note that by the subject construction theorem, a term of the form $((\overleftarrow{\lambda}\ x.M)N)_\rightarrow$ $(((\overrightarrow{\lambda}\ x.M)N)_\leftarrow$ resp.) does not have a type. Suppose to the contrary that it does. Then, by subject construction, the last step of the derivation is $\Gamma_2 \rhd (c_2 \leftarrow c_1) : (\overleftarrow{\lambda}\ x.M),\ \Gamma_1 \rhd c_1 : N \ \Rightarrow_{(\leftarrow E)}\ \Gamma_2\Gamma_1 \rhd ((\overleftarrow{\lambda}\ x.M)N)_\rightarrow$. However, by subject construction, $(\overleftarrow{\lambda}\ x.M)$ can only have type $c_1' \rightarrow c_2'$, thus $\Gamma_2 \rhd (c_2 \leftarrow c_1) : (\overleftarrow{\lambda}\ x.M)$ cannot be derived, in contradiction.

Now, any term can be written as

$$\lambda_{*_1} x_1 \ldots \lambda_{*_n} x_n.(\ldots((NN_1)_{*_1} N_2)_{*_2} \ldots N_l)_{*_l}$$

(with N instead of x). If we "split" the leftmost term as much as possible, then we get that N is either (1) a variable x, or (2) of the form $\lambda_* x.P$, where $l \geq 1$. If we refer to a typed term, then by the previous observation, in case (2) it must be the case that λ_* and $()_{*_1}$ have the same directions.

This is true in particular for a typed term in β normal form. Only that when talking about a term in β normal form, it is impossible to have $((\lambda_* x.P)N_1)_{*_1}$, where λ_* and $()_{*_1}$ share the same direction. Therefore, case (2) is impossible when considering a term in normal form. Furthermore, the

[7] From now on, whenever we refer to λ-terms, or even just terms, we mean *directed* λ-terms.

N_i's are of course also in normal form in this case. We conclude that the normal form is as claimed above.

For our purpose, it will be convenient to refer to typed terms which are in η long normal form.

Definition (directed η long normal form): Assume that types of variables are fixed in some way, thus every term can be assigned only one type (Church style). The set of λ-terms in η *long normal form* is defined inductively as follows.

- If x is a variable and $M_1, \ldots M_n$ are in η long normal form, where the type of $(\ldots((xM_1)_{*_1}M_2)_{*_2}\ldots M_n)_{*_n}$ is A for some propositional variable A, then $(\ldots((xM_1)_{*_1}M_2)_{*_2}\ldots M_n)_{*_n}$ is in η long normal form.
- If M is in η long normal form, and $\lambda_* x.M$ has a type, then $\lambda_* x.M$ is also in η long normal form.

Note that a term in η long normal form is also in β normal form, and thus has the form stated in the shape of β normal form theorem.

Furthermore, any typed term in β normal form can be transformed to a term in η long normal form, with the same type. That is,

Theorem: If $\Gamma \rhd \varphi : M$, for M in β normal form then there exists M' in η long normal form such that $\Gamma \rhd \varphi : M'$.

Proof: We only give the idea of the proof. The main point here is the following. Suppose the subterm $P \equiv (\ldots((xN_1)_{*_1}N_2)_{*_2}\ldots N_l)_{*_l}$ of the term in β normal form has a more complicated type c. Let $c = *_1(c_1, \ldots *_{k-1}(c_{k-1}, *_k(c_k, A))\ldots)$, where $k \geq 1$, $*_i \in \{\rightarrow, \leftarrow\}$ for each $1 \leq i \leq k$, and A is a propositional variable (the c_i's can be compound formulas). Then we introduce new variables $y_1, \ldots, y_k$, with types corresponding to $c_1, \ldots c_k$. That is, for each $1 \leq i \leq k$, $c_i : y_i$. We then transform the above subterm P to $P' \equiv \lambda_{*_k} y_k \lambda_{*_{k-1}} y_{k-1} \ldots \lambda_{*_1} y_1.((\ldots(Py_1)_{*_1}\ldots y_{k-1})_{*_{k-1}} y_k)_{*_k}$. This way, on the one hand the subterm P' remains of type c, and thus the type of the entire term is maintained. On the other hand, the type of $((\ldots(Py_1)_{*_1}\ldots y_{k-1})_{*_{k-1}} y_k)_{*_k}$ is a propositional variable (A), which ensures that P' is in η long normal form. By applying this procedure recursively on the N_i's a well, we get a term in η long normal form (which is therefore also in β normal form), with the same type.

Corollary: If $\Gamma \rhd \varphi : M$, then there exists M' in η long normal form such that $\Gamma \rhd \varphi : M'$.

We now recall the CH correspondence (directed subject construction theorem), which ensures that whenever $\vdash_{\mathbf{L}} \psi_1, \ldots, \psi_n \rhd \varphi$, then there exists a directed λ-term M such that $\vdash_{\mathbf{L}} \psi_1 : x_1, \ldots \psi_n : x_n \rhd \varphi : M$. By the above corollary, we conclude that there exists M' in η long normal form such that $\vdash_{\mathbf{L}} \psi_1 : x_1, \ldots \psi_n : x_n \rhd \varphi : M'$. We get the following concluding theorem.

Theorem (directed subject-construction, η-long form): If $\vdash_{\mathbf{L}} \psi_1, \ldots, \psi_n \rhd \varphi$, then there exists a directed λ-term M, with $free(M) = \{x_1, \cdots, x_n\}$, such that $\vdash_{\mathbf{L}} \psi_1 : x_1, \ldots \psi_n : x_n \rhd \varphi : M$.

6.3. Proof of the L-Game theorem

*$\Rightarrow$: Let D be a winning Prover strategy for $\Gamma \rhd \varphi$. For any subtree D' with root Σ (a Prover's node), we now show, by induction, that every $\sigma \in \Sigma$ is derivable in **L**. Thus, in particular in the root of D, labelled by $\Gamma \rhd \varphi$, we conclude that $\vdash_{\mathbf{L}} \Gamma \rhd \varphi$.

The base case is when Σ consists of axioms, in which case the claim holds. Otherwise, by the definition of a Prover's winning strategy, for every assertion in Σ that is not an axiom, there exists a distinct descendant in D', which is a Skeptic's node.

Let $\sigma_i \in \Sigma$ (which is not an axiom) and let β_i be the labelling of the distinct descendant that corresponds to the Skeptic challenging σ_i. Again, by the definition of a winning Prover strategy, this node, labelled by β_i, has a descendant, which is a Prover's node, labelled by Σ_i.

By the induction hypothesis, $\vdash_{\mathbf{L}} \sigma'$, for every $\sigma' \in \Sigma_i$. Furthermore, it follows from the Prover-Property, that $\Sigma_i \cup \{\text{axiom}\} \Rightarrow^*_{(E)} \beta_i$. All together, by replacing the assertions of Σ_i by their derivations in the latter derivation of β_i from Σ_i, we get a derivation of β_i (from axioms). Now, by the Skeptic-Property it holds that $\beta_i \Rightarrow^*_{(I)} \sigma_i$. Therefore, by replacing the role of β_i in this derivation by its derivation, we get a derivation of σ_i. We conclude that $\vdash_{\mathbf{L}} \sigma_i$.

*$\Leftarrow$: Suppose $\vdash_L \Gamma \rhd \varphi$, where $\Gamma = \psi_1, \ldots \psi_n$ $(n \geq 1)$. Then, by the directed subject-construction theorem, η-long form, there exists a λ-term M in η long normal form, such that $\vdash_{\mathbf{L}} \psi_1 : x_1, \ldots \psi_n : x_n \rhd \varphi : M$. Therefore it suffices to show that if

$$\vdash_{\mathbf{L}} \psi_1 : x_1, \ldots \psi_n : x_n \rhd \varphi : M$$

for M in η long normal form, then there is a winning Prover strategy for the game for $(\psi_1 \ldots \psi_n, \varphi)$. The proof is by induction on the size of M. By the shape of the β normal-form theorem, M has the following form:

$$\lambda_{*_{n+1}} x_{n+1} \ldots \lambda_{*_{n+m}} x_{n+m}.(\ldots((xN_1)_{*_1} N_2)_{*_2} \ldots N_l)_{*_l}$$

where $m \geq 0, l \geq 0$. Let

$$\psi_{n+1} : x_{n+1}, \ldots \psi_{n+m} : x_{n+m}$$

(Church style). Then by subject construction (by induction on m),

$$\vdash_{\mathbf{L}} \Gamma_r \psi_1 : x_1 \ldots \psi_n : x_n \Gamma_l \rhd A : (\ldots((xN_1)_{*_1} N_2)_{*_2} \ldots N_l)_{*_l} \tag{1}$$

for some propositional variable A (since M is in η long normal form) where

- Γ_r denotes the sequence constructed from the $\psi_{n+i} : x_{n+i}$'s such that $\lambda_{*_{n+i}} = \overrightarrow{\lambda}$, ordered by their indices from the highest to the smallest.
- Γ_l denotes the sequence constructed from the $\psi_{n+i} : x_{n+i}$'s such that $\lambda_{*_{n+i}} = \overleftarrow{\lambda}$, ordered by their indices from the smallest to the highest (opposite order).

This implies, in particular, that

$$\varphi = *_{n+1}(\psi_{n+1}, \ldots *_{n+m-1} (\psi_{n+m-1}, *_{n+m}(\psi_{n+m}, A)) \ldots) \tag{2}$$

where $*_{n+i} = \begin{cases} \rightarrow & \text{if } \lambda_{*_{n+i}} = \overrightarrow{\lambda} \\ \leftarrow & \text{if } \lambda_{*_{n+i}} = \overleftarrow{\lambda} \end{cases}$

Denote $\Gamma_r \psi_1 : x_1 \ldots \psi_n : x_n \Gamma_l$ by Γ'. Then by (1) we get

$$\vdash_{\mathbf{L}} \Gamma' \rhd A : (\ldots((xN_1)_{*_1} N_2)_{*_2} \ldots N_l)_{*_l} \tag{3}$$

and by the structure of φ (see (2)), the Skeptic in his move, given the assertion $(\psi_1 \ldots \psi_n \rhd \varphi)$, has only one possibility for the challenge that he chooses, which is

$$\beta = \Gamma' \downarrow \rhd A \tag{4}$$

Let $N_{k_1}, \ldots, N_{k_t}$ be the sequence of N_i's such that $()_{*_i} = ()_{\rightarrow}$, ordered by their indices from the highest to the smallest (that is $k_1 < \ldots < k_t$). Similarly, let $N_{j_1}, \ldots, N_{j_{l-t}}$ be the sequence of N_i's such that $()_{*_i} = ()_{\leftarrow}$, ordered by their indices from the smallest to the highest (that is $j_1 > \ldots > j_{l-t}$). Note that $\{k_i : 1 \leq i \leq t\} \cup \{j_i : 1 \leq i \leq l-t\} = \{1, \ldots, l\}$. Starting from $\vdash_{\mathbf{L}} \Gamma' \rhd A : (\ldots((xN_1)_{*_1} N_2)_{*_2} \ldots N_l)_{*_l}$ (see (3)), again by subject construction (by induction on l), we have that Γ' can be split to $\Gamma_{k_1} \ldots \Gamma_{k_t} \Gamma'' \Gamma_{j_1} \ldots \Gamma_{j_{l-t}}$ (all of which are nonempty) such that

$$\vdash_{\mathbf{L}} \Gamma_{k_i} \rhd \rho_{k_i} : N_{k_i} \quad \text{for every } 1 \leq i \leq t \tag{5}$$

$$\vdash_{\mathbf{L}} \Gamma_{j_i} \rhd \rho_{j_i} : N_{j_i} \quad \text{for every } 1 \leq i \leq l-t \tag{6}$$

$$\vdash_{\mathbf{L}} \Gamma'' \rhd *_1(\rho_1, \ldots *_{l-1} (\rho_{l-1}, *_l(\rho_l, A)) \ldots) : x \tag{7}$$

where

$$\forall 1 \leq i \leq t: \quad *_{k_i} = \rightarrow \tag{8}$$

$$\forall 1 \leq i \leq l-t: \quad *_{j_i} = \leftarrow \tag{9}$$

By (7), and due to subject construction, we conclude that Γ'' itself has to be of the form $*_1(\rho_1, \ldots *_{l-1}(\rho_{l-1}, *_l(\rho_l, A)) \ldots) : x$. Therefore, $\Gamma' = \Gamma_{k_1} \ldots \Gamma_{k_t} \ *_1(\rho_1, \ldots *_{l-1}(\rho_{l-1}, *_l(\rho_l, A)) \ldots) : x \ \Gamma_{j_1} \ldots \Gamma_{j_{l-t}}$, and

$$\Gamma' \downarrow = \Gamma_{k_1} \downarrow \ldots \Gamma_{k_t} \downarrow *_1(\rho_1, \ldots *_{l-1}(\rho_{l-1}, *_l(\rho_l, A)) \ldots) \\ \Gamma_{j_1} \downarrow \ldots \Gamma_{j_{l-t}} \downarrow \tag{10}$$

If we denote $*_1(\rho_1, \ldots *_{l-1}(\rho_{l-1}, *_l(\rho_l, A)) \ldots)$ by α, then by (10), we get that $\Gamma' \downarrow = \Gamma_{k_1} \downarrow \ldots \Gamma_{k_t} \downarrow \ \alpha \ \Gamma_{j_1} \downarrow \ldots \Gamma_{j_{l-t}} \downarrow$. Therefore, based on (8), (9) and the latter, and since $\beta = \Gamma' \downarrow \rhd A$ (by (4)), we conclude that by choosing $\Sigma = \{\Gamma_i \downarrow \rhd \rho_i : 1 \leq i \leq l\}$ in response for β, the Prover obeys the rules of the game. It remains to show that this is also a winning strategy. This is true because by the induction hypothesis w.r.t. (5) and (6), there is a winning Prover strategy for each $(\Gamma_i \downarrow, \rho_i)$.

7. Conclusions

The paper introduces a game semantics for the Lambek calculus **L**, by naturally extending the familiar game semantics for the well-known implicational fragment of the intuitionistic propositional calculus. The effect of the absence of structural rules is captured by lifting the game board to the level of (ordered!) sequents, instead of the level of formulae, as in the intutionistic game. In particular, the absence of the 'exchange' structural rule, together with the presence of two *directed implications*, reshapes the way the Skeptic offers are analysed.

Since **L**, known also as the *syntactic calculus*, is the underlying base logic for categorial type-logical grammar (TLG), it would be interesting to lift **L**-games even further, turning them to what might be called "*grammar games*", whereby the Prover is trying to establish $w \in L[\![G]\!]$, for some type-logical grammar G, while the Skeptic tries to refute this membership. Assertions/offers can be interpreted in terms of "constituency" and subcategorization. The axioms, as usual for TLG, represent the lexicon.
We leave this to future work.

Acknowledgement This research was supported by the Israel Science Foundation (grant No. 2009055).

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