Purely Subjective Maxmin Expected Utility

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ABSTRACT
The Maxmin Expected Utility decision rule suggests that the decision maker can be characterized by a utility function and a set of prior probabilities, such that the chosen act maximizes the minimal expected utility, where the minimum is taken over the priors in the set. Gilboa and Schmeidler axiomatized the maxmin decision rule in an environment where acts map states of nature into simple lotteries over a set of consequences. This approach presumes that objective probabilities exist, and, furthermore, that the decision maker is an expected utility maximizer when faced with risky choices (involving only objective probabilities).

This paper presents axioms for a derivation of the maxmin decision rule in a purely subjective setting, where acts map states to points in a connected topological space. This derivation does not rely on a pre-existing notion of probabilities, and, importantly, does not assume the von Neuman & Morgenstern (vNM) expected utility models for decision under risk. The axioms employed are simple and each refers to a bounded (and small) number of variables.

1 Introduction
There is a respectable body of literature dealing with axiomatic foundations of decision theory, and specifically with the non-Bayesian (or extended Bayesian) branch of it. We will mention a part of this literature to put our paper in context.

Building on the works of Ramsey (1931), de Finetti (1937), and von Neumann and Morgenstern (1944, 1953), Savage (1952, 1954) provided an axiomatic model of purely subjective expected utility maximization. The descriptive validity of this model was put in doubt long ago. Today many think that Savage’s postulates do not constitute a sufficient condition for rationality, and some doubt that they are all necessary conditions for it. However, almost all agree that his work is by far the most beautiful and important axiomatization ever written in the social or behavioral
sciences. "The crowning glory", as Kreps (1998) put it. Savage's work has had a tremendous influence on economic modeling, convincing many theorists that the only rational way to make decisions is to maximize expected utility with respect to a subjective probability. Importantly, due to Savage's axioms, many believe, that any uncertainty can and should be reduced to risk, and that this is the only reasonable model of decision making on which economic applications should be based. About a decade after Savage's seminal work Anscombe and Aumann (1963) (AA for short) suggested another axiomatic derivation of subjective probability, coupled with expected utility maximization. As in Savage's model, acts in AA's model map states of nature to a set of consequences. However, in Savage's model the set of consequences has no structure, and it may consist of merely two elements, whereas AA assume that the consequences are lotteries as in vNM's model, namely, distributions over a set of outcomes, whose support is finite. Moreover, AA impose the axioms of vNM-'S'-theory (including the independence axiom) on preferences over acts, which imply that the decision maker maximizes expected utility in the domain of risk. On the other hand, AA's model can deal with a finite state space, whereas Savage's axioms imply that there are infinitely many states, and, moreover, that none of them is an "atom". Since in many economic applications there are only finitely many states, one cannot invoke Savage's theorem to justify the expected utility hypothesis in such models.

A viable alternative to the approaches of Savage and of AA is the assumption that the set of consequences is a connected topological space. (See Fishburn (1970) and Kranz, Luce, Suppes, and Tversky (1971) and the references therein.) These spaces are "rich" and therefore more restrictive than Savage's abstract set of consequences. On the other hand, such spaces are natural in many applications. In particular, considering a consumer problem under uncertainty the consequences are commodity bundles which, in the tradition of neoclassical consumer theory, constitute a convex subset of an n-dimensional Euclidean space, and thus a connected topological space. As opposed to AA's model, the richness of the space does not necessarily derive from mixture operations on a space of lotteries. Thus, no notion of probability is presupposed, and no restrictions are imposed on the decision maker's behavior under risk.

Despite the appeal of Savage's axioms, the Bayesian approach has come under attack on descriptive and normative grounds alike. Echoing the view held by Keynes (1921), Knight (1921), and others, Ellsberg (1961) showed that Savage's axioms are not necessarily a good description of how people behave, because people tend to
prefer known to unknown probabilities. Moreover, some researchers argue that such preferences are not irrational. This was also the view of Schmeidler (1989), who suggested the first axiomatically based, general-purpose model of decision making under uncertainty, allowing for a non-Bayesian approach and a not necessarily neutral attitude to uncertainty. Schmeidler axiomatized expected utility maximization with a non-additive probability measure (also known as capacity), where the operation of integration is done as suggested by Choquet (1953-4). Schmeidler (1989) employed the AA model, thereby using objective probabilities and restricting attention to expected utility maximization under risk. Following his work, Gilboa (1987) and Wakker (1989) axiomatized Choquet Expected Utility theory (CEU) in purely subjective models; the former used Savage’s framework, whereas the latter employed connected topological spaces. Thus, when applying CEU, one can have a rather clear idea of what the model implies even if the state space is finite, as long as the consequence space is rich (or vice versa).

Gilboa and Schmeidler (1989) suggested the theory of Maxmin Expected Utility (MEU), according to which beliefs are given by a set of probabilities, and decisions are made as it were to maximize the minimal expected utility of the act chosen. This model has an overlap with CEU: when the non-additive probability is convex, CEU can be described as MEU with the set of probabilities being the core of the non-additive probability. More generally, CEU can capture modes of behavior that are incompatible with MEU, including uncertainty-liking behavior. On the other hand, there are many MEU models that are not CEU. Indeed, even with finitely many states, where the dimensionality of the non-additive measures is finite, the dimensionality of closed and convex sets of measures is infinite (if there are at least three states). Moreover, there are many applications in which a set of priors can be easily specified even if the state space is not explicitly given. As a result, there is an interest in MEU models that are not necessarily CEU. Gilboa and Schmeidler (1989) axiomatized MEU in AA’s framework, paralleling Schmeidler’s original derivation of CEU.

There are several reasons to axiomatize the MEU model without using objective probabilities. The raison d’être of the CEU and MEU models (and of several more recent models) is the assumption that, contrary to Savage’s claim, Knightian uncertainty cannot be reduced to risk. Thus to require the existence of exogenously given additive probabilities while modelling Knightian uncertainty seems odd at best.
Most of the applications in economics assume consequences lie in a convex subset of, say, Euclidean space, but don’t assume linearity of the utility function. It is therefore desirable to have an axiomatic derivation of the MEU decision rule in this framework that is applicable to a finite state space and that does not rely on objective probabilities. Such a derivation can help us see more clearly what the exact implication of the rule is in many applications, without restricting the model in terms of decision making under risk or basing it on vNM utility theory.

Our goal here is to suggest a set of axioms guaranteeing MEU representation\(^1\): i.e., the existence of a utility function \(u\) on \(X\) and a non empty, compact and convex set \(C\) of finitely additive probability measures on \((S, \Sigma)\) such that for all \(f\) and \(g\) in \(X^S\):

\[
f \succ g \iff \min_{P \in C} \int_S u(f(\cdot))dP \geq \min_{P \in C} \int_S u(g(\cdot))dP.
\]

The first axiomatic derivation of MEU of this type is by Casadesus-Masanell, Klibanoff and Ozdenoren (2000). Another one is by Ghirardato, Maccheroni, Marinacci and Siniscalchi (2002). They are discussed in section 4. To compare these works with ours, we conclude the introduction with a comment on axiomatization of the individual decision making under (Knightian) uncertainty. Such an axiomatization consists of a set of restrictions on preferences. This set usually constitutes a necessary and sufficient condition for the numerical representation of the preferences.

Axiomatizations may have different goals and uses, be distinguished across more than one normative or positive interpretations and be evaluated according to different tests. Here we emphasize and delineate simplicity and transparency as a necessary condition for an axiom. Let us start with a normative interpretation of the axioms.

In many decision problems, the decision maker does not have well-defined preferences that are accessible to her by introspection, and she has to invest time and effort to evaluate her options. In accordance with the models discussed in this paper we assume that the decision maker constructs a states space and conceives her options as maps from states to consequences.\(^2\) The role of axioms in this situation is twofold. First, the axioms may help the decision maker to construct her mostly unknown preferences: the axioms may be used as "inference rules", using some known instances of pairwise

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\(^1\)As usual \(S, \Sigma = 2^S\), \(X, X^S = \{ f : S \to X \}\), and \(\succ \subset X^S \times X^S\) denote the sets of states, events, consequences, acts, and the decision maker’s preferences over acts, respectively.

\(^2\)The decision maker may be aware that her construction is coarse and she may not be certain that the states are mutually exclusive and exhaustive.
preferences to derive others. Secondly, the set of axioms can be used as a general rule that justifies a certain decision procedure. To the extent that the decision maker can understand the axioms and finds them agreeable, the representation theorem might help her choose a decision procedure, thereby reducing her problem to the evaluation of some parameters. It is often the case that the mathematical representation of the decision rule also makes the evaluation of the parameters a simple task. For example, in the case of EU theory, one can use simple trade-off questions to evaluate one’s utility function and one’s subjective probability, and then use the theory to put these together in a unique way that satisfies the axioms.

Both normative interpretations of the axioms call for clarity and simplicity. In the first, instance-by-instance interpretation, the axioms are used by the decision maker herself to derive conclusions such as \( f \succsim h \) from premises such as \( f \succsim g \) and \( g \succsim h \). Only simple and transparent axioms can be used by actual decision makers to construct their preferences. In the second interpretation, the axioms should be accepted by the decision maker as universal statements to which she is willing to commit. If they fail to have a clear meaning, a critical decision maker will hesitate to accept the axioms and the conclusions that follow from them.

Finally, we briefly mention that descriptive interpretations of the axioms also favor simple over complex ones. One descriptive interpretation is the literal one, suggesting testing the axioms experimentally. The simpler the axioms, the easier it is to test them. Another descriptive interpretation is rhetorical: the axioms are employed to convince a listener that a certain mode of behavior may be more prevalent than it might appear at first sight. Here gain, simplicity of the axioms is crucial for such a rhetorical task.

But how does one measure the simplicity of an axiom? One can define a very coarse measure of opaqueness of an axiom by counting the number of variables in its formulation: acts, consequences, states or events, instances of preference relation, and logical quantifiers. For example, transitivity requires three variables, three preference instances, and one implication – seven in all. We suggest that a low level of opaqueness is a test, but not a criterion, for simplicity and transparency of an axiom. By a low level we mean first of all, a bounded number.\(^4\)

\(^3\)On this see Gilboa et al. (2009).
\(^4\)Of course, an axiom that requires half a page to write it down can hardly be considered simple and transparent.
A remark about the opaqueness of axiom of continuity is required. All the main representation theorems require such an axiom in one form or another. But any version of this axiom has an unbounded or infinite opaqueness as specified above. Ours is akin to continuity in the neoclassical consumer theory. As such it cannot be tested. Moreover, the decision maker should be agnostic toward it. Because of its opaqueness, it is as difficult for the decision maker to accept it as to reject it. Our purpose in this work is to use only axioms of small opaqueness, continuity excluded.

We conclude the Introduction with a short outline. In the next section we present the basic, almost necessary axioms. We say almost because we impose a restriction, "the set of states is finite", which is not implied by the representation. Its aim is to highlight and elucidate our approach. In Section 3 we add the axioms that lead to a cardinal, numeric representation of the preference \( \succeq \) on acts and a cardinal utility on consequences. We borrow two approaches from the literature: the tradeoffs (Koberling and Wakker (2003)), and the biseparable preferences (Ghirardato and Marinacci (2001)). In Section 4 we introduce our new axioms, in two versions, and state our main result, also in two versions: tradeoffs and biseparable preferences. This section contains an additional purely subjective representation theorem for the case where the MEU and CEU (Choquet expected utility, i.e., a non-additive prior) coincide. Proofs are relegated to the appendices.

2 Notation and basic axioms

Recall that \( S, \Sigma = 2^S, X, X^S = \{ f : S \to X \} \), and \( \succcurlyeq \subset X^S \times X^S \) denote the sets of states, events, consequences, acts, and the decision maker’s preferences over acts, respectively. As usual, \( \sim \) and \( \succcurlyeq \) denote the symmetric and asymmetric components of \( \succcurlyeq \). For \( x, y \in X \) and an event \( E \), \( xEy \) stands for the act which assigns \( x \) to the states in \( E \) and \( y \) otherwise. Let \( \overline{x} \) denote the constant act \( xEx \) (for any \( E \in \Sigma \)). Without causing too much of a confusion, we also use the symbol \( \succeq \) for a binary relation on \( X \), defined by: \( x \succeq y \) iff \( \overline{x} \succeq \overline{y} \). Another shortcut we use is \( xsy \) for \( x\{s\}y \), or more generally, we may write \( s \) instead of \( \{s\} \).

We start with the restrictions on the set of states and the set of consequences.

**A0. Structural assumption:**
(a) \( S \) is finite and \(|S| \geq 2\).

(b) \( X \) is a connected, topological space, and \( X^S \) is endowed with the product topology.

Part (a) of A0 is introduced to simplify the presentation in two ways. Although the condition "\( S \) is finite" restricts the applicability of our result, we avoid functional analysis formulations and concentrate on the essence of the axioms. We believe that the results here can and should be extended to infinite set of states. The restriction, \(|S| \geq 2\), together with a version of nondegeneracy below avoids uninteresting cases and awkward formalization.

A0(b) is an essential restriction of our approach. It follows the neoclassical economic model and the literature on separability.\(^5\) For its introduction into decision theory proper see Fishburn (1970) and the references there and Kranz, Luce, Suppes, and Tversky (1971) and the references there. We note that the topology does not have to be assumed. One can use the order topology on \( X \) whose base consists of open intervals of the form \( \{ z \in X \mid x \succ z \succ y \} \), for some \( x, y \in X \). The restriction then is the connectedness of the space. A0(b) can be further weakened by imposing the order topology and its connectedness on \( X/\sim \), i.e., on the \( \sim \)-equivalence classes. The connectedness restriction seems to exclude decision problems where there are just a few deterministic consequences like in medical decisions. On the other hand many medical consequences are measured by QALY, quality-adjusted life years, that is, in time units. These situations can be embedded in our model.

We state now the four basic axioms.

**A1. Weak Order:**

(a) For all \( f \) and \( g \) in \( X^S \), \( f \succcurlyeq g \) or \( g \succcurlyeq f \) (completeness).

(b) For all \( f, g, \) and \( h \) in \( X^S \), if \( f \succcurlyeq g \) and \( g \succcurlyeq h \) then \( f \succcurlyeq h \) (transitivity).

The weak order axiom is a necessary condition for MEU representation, and its opaqueness is low. The latter property of A1 does not hold for the continuity axiom below.

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\(^5\)It is still an open problem to obtain a purely subjective representation of MEU in a Savage like model, where \( X \) may be finite but \( S \) is rich enough.
A2. Continuity:
The sets \( \{ f \in X^S | f \succ g \} \) and \( \{ f \in X^S | f \prec g \} \) are open for all \( g \) in \( X^S \).

We delay elaboration on A2, and introduce the other basic axioms first. Recall that a state \( s \) is said to be null if for all \( x, y, z \in X \), \( xsy \sim zsy \).

A3. Substantiality:
All states in \( S \) are non null, and for some \( s \in S : x \succ y \Rightarrow xsy \succ y \).

This axiom together with A0(a) simplifies and shortens the formal presentation. One can easily extend the result by replacing the first part of A3 with, "there are at least two non null states in \( S \)”, and deleting from A0(a) the, now superfluous, restriction, \( |S| \geq 2 \). But this extension makes the statement of the axioms more cumbersome. If there are no two non null states, we are in a deterministic setting.

The last basic axiom is the usual monotonicity condition:

A4. Monotonicity:
For any two acts \( f \) and \( g \), \( f \succsim g \) holds whenever \( f(s) \succsim g(s) \) for all states \( s \) in \( S \).

Note that like A1, A4 is simple and necessary for MEU representation.

Let us consider once again the spacial case where \( X = \mathbb{R}^l_+ \), the standard neoclassical consumption set. Then \( X^S \) consists of state contingent consumption bundles, like in Chapter 7 of the classic 'Theory of value', Debreu (1959). Our A4 is "orthogonal" to the neoclassical monotonicity of preferences on \( \mathbb{R}^l_+ \). The latter says that increase in quantity of any commodity is desirable. A4 says that preferences between consequences do not depend on the state of nature that occurred. However if one assumes neoclassical monotonicity on \( (\mathbb{R}^l_+)^S = X^S \), it does not imply A4, but can accommodate it. Chapter 7 preferences on \( (\mathbb{R}^l_+)^S \) can be represented by a continuous utility function, say, \( J : (\mathbb{R}^l_+)^S \rightarrow \mathbb{R} \). One then can define a utility where for \( x \in X = \mathbb{R}^l_+ \), \( u(x) = J(\bar{x}) \). If also A4 is imposed, \( u \) represents \( \succeq \) on \( X \) and is continuous.\(^6\) To get such result under A0(b) assumption, a separability condition has to be added. However we are interested in a stronger representation where \( u \), and hence \( J \), are cardinal. The 'why' and 'how' are discussed in the next section.

Finally we comment on the opaqueness of A2. When a specific decision problem is posed, one gets a concrete set of consequences. If it happens to be of a finite linear

\(^6\)More precisely: the way \( \succeq \) was defined \( u \) represents it on \( X \) with or without A4. However without A4 this representation does not make sense.
dimension, say $X \subset \mathbb{R}^I$, then for anyone who has some education in economics or
statistics, even at MBA level, continuity is an apparent assumption.

3 Cardinal representations

In our quest for axioms which result in MEU representation we have to consider
a strategy for proving this result. An obvious possibility is to follow Gilboa and
Schmeidler (1989). They accomplished it in several steps. First they constructed a
vNM utility over consequences, and a numerical representation of the preferences over
acts, where the latter coincides with the utility on constant acts. Next they translated
acts to functions from states to utiles, i.e., to real numbers, and they translated the
axioms on preferences to properties of the functional on $\mathbb{R}^S$. Finally they showed that
this functional can be represented via a set of priors. One way to carry out the first
step is to take an “off the shelf” result based on a purely subjective axiomatization
which agrees with our structural assumption and basic axioms. There are two obvious
candidates. The first one is the biseparable preferences representation by Ghirardato
and Marinacci (2001). It is presented in Subsection 3.2. Another result is a special
case of a theorem in Kobberling and Wakker (2003). We start with the latter.

3.1 Tradeoffs and tradeoff consistency axiom

First we have to recall and restate several definitions.

**Definition 1.** A set of acts is **comonotonic** if there are no two acts $f$ and $g$ in the
set and states $s$ and $t$, such that, $f(s) \succ f(t)$ and $g(t) \succ g(s)$. Acts in a comonotonic
set are said to be comonotonic.

Comonotonic acts induce essentially the same ranking of states according to the
desirability of their consequences. Given any numeration of the states, say
$\nu : S \rightarrow \{1, ..., |S|\}$, the set \{ $f \in X^S \mid f(\nu(1)) \succeq \ldots \succeq f(\nu(|S|))$ \} is comonotonic. It
is a largest-by-inclusion comonotonic set of acts.

**Definition 2.** Given a comonotonic set of acts $A$, and an event $E$, $E$ is said to be
comonotonically nonnull on $A$ if there are acts $f$ and $g$ such that:
$f \succ g, \quad f(s) = g(s)$ for all $s \in S \setminus E$, and the set $A \cup \{f, g\}$ is comonotonic.

The definition of tradeoff indifference we use restricts attention to binary comonotonic
acts, that is, comonotonic acts which obtain at most two consequences.
Definition 3. Let $a, b, c, d$ be consequences. We write $< a; b > \sim^* < c; d >$ if there exist consequences $x, y$ and a state $s$ such that

$$asx \sim bsy \quad \text{and} \quad csx \sim dsy$$

with all four acts comonotonic, and $\{s\}$ comonotonically nonnull on this set of acts.

Given this notation we can express the cardinality of a utility function within our framework.

Definition 4. We say that $u : X \rightarrow R$ cardinally represents $\succ$ on $X$ (or for short, $u$ is cardinal), if it represents $\succ$ ordinally, and

$$< a; b > \sim^* < c; d > \Rightarrow u(a) - u(b) = u(c) - u(d)$$

It will be shown in the sequel that given our axioms so far, and A5 below, such $u$ exists, and that $u$ and $v$ cardinally represent $\succ$ iff $v(x) = \alpha u(x) + \beta$ for some positive number $\alpha$, and any number $\beta$.

A special case of Definition 3 is of the form, $< a; b > \sim^* < b; c >$. It essentially says that $b$ is half the way between $a$ and $c$. It will be used in the statement of axioms six and seven below. To make the definition of tradeoffs indifference useful we need the following axiom.

A5. Binary Comonotonic Tradeoff Consistency (BCTC):

For any eight consequences $a, b, c, d$, $x, y, v, w$, and events, $E, F \in \{ \{s\} \mid s \in S \} \cup \{ S \backslash \{s\} \mid s \in S \}$:

$$aEx \sim bEy, \quad cEx \sim dEy, \quad aFv \sim bFw \Rightarrow cFv \sim dFw$$

whenever the sets of acts $\{ aEx, bEy, cEx, dEy \}$ and $\{ aFv, bFw, cFv, dFw \}$ are comonotonic, $E$ is comonotonically nonnull on the first set, and $F$ is comonotonically nonnull on the second.

This axiom guarantees that the relations $asx \sim bsy$ and $csx \sim dsy$ defining the $\sim^*$ equivalence relation above do not depend on $s$, $x$ and $y$. It is somewhat weaker than the axiom used in Kobberling and Wakker (2003), in that it applies only to binary acts obtaining one consequence on a single state, and another consequence on all other states, and not to all binary acts.

How complex is this axiom. Simple counting yields ten variables, four instances of preferences and one implication. However the conditions about comonotonicity have
also nontrivial opaqueness. The condition, "the set of acts \( \{ aEx, bEy, cEx, dEy \} \) is comonotonic" is equivalent to the condition, "\{ \( a \gtrsim x, b \gtrsim y, c \gtrsim x, d \gtrsim y \) \} or \{ \( x \gtrsim a, y \gtrsim b, x \gtrsim c, y \gtrsim d \) \}. So we have here six variables, eight instances of preferences, and an ‘or’. Similarly the other comonotonicity condition. Since the variables are the same eight consequences, we have to add the eight instances of preferences\(^7\). Next we have to add the opaqueness of \( E \) being comonotonically non null w.r. to \{ \( a \gtrsim x, b \gtrsim y, c \gtrsim x, d \gtrsim y \) \} (and similarly if the preferences go the other way). This restriction says that there are consequences, \( e, e', e'' \) such that: \( e \succ e' \succ e'' \), and \( eEe'' \succ e'Ee'' \) (and similarly if the preferences go the other way). So the opaqueness of this axiom is expressed in the decimal system by two and not one numeral.

There is however another way to evaluate the opaqueness of axioms. We deal here not with one axiom at a time but with a collection of eight behavioral axioms. Adding several definitions like non mull, comonotonic, comonotonically nonnull, and tradeoff indifference, is not cognitively overtaxing. After internalizing these definitions, it becomes simpler not only to present the axioms, but also to perceive them. This indeed is the way how technical and non technical material is presented and taught.

So far five axioms and a few definitions were presented. Allowing the use of the definitions in the formulation of the axioms (and other definitions) leaves us with relatively simple axioms, involving a bounded number of variables, continuity excluded (Or connectedness excluded in another formulation). The first five axioms imply the existence of a continuous, cardinal representation of the preferences \( \succcurlyeq \) on \( X \), and a numerical representation of the preferences \( \succcurlyeq \) on \( X^S \). Denoting the latter representation by \( J \) and former by \( u \) as above, we also have for all \( x \in X : J(\bar{x}) = u(x) \). Hence \( J \) is cardinal too. This is Proposition 12 at the beginning of the proofs in the Appendix.\(^8\)

### 3.2 Biseparable preferences

We recall the definition from Girardato and Marinacci (2001) (henceforth called GM).

**Definition 5.** A preferences relation over acts, \( \succcurlyeq \), is said to be biseparable if there exist a real valued function \( J \) on acts that represents the relation, and a monotonic

\[^7\] One can decrease the complexity of the definition of comonotonicity by considering its counterpositive. For any pair of acts, \((f, g)\), and any two states, \(s\), and \(t\): if \( f(s) \succeq f(t) \), then \( g(s) \succeq g(t) \). It is smaller than what we computed above, and can be used instead.

\[^8\] A result from Kobberling and Wakker (2003) is used in the proof.
set function \( \eta \) on events such that for binary acts with \( x \succ y \),

\[
J(xEy) = u(x)\eta(E) + u(y)(1 - \eta(E)).
\]

The function \( u \) is defined on \( X \) by \( u(x) = J(x) \), and it represents \( \succ \) on \( X \). The set function \( \eta \), when normalized s.t. \( \eta(S) = 1 \), is unique, and \( u \) and \( J \) are unique up to a positive multiplicative constant and an additive constant.

GM showed that this representation generalizes CEU and MEU. They characterized biseparable preferences using several axioms. We will restate their axioms within our restrictions, the main restriction being that \( S \) is finite. We will also follow our simplification strategy of assuming that there are no null states and \( |S| \geq 2 \). With these constrains in mind GM use A0, A1, A2, and A4. We will denote with * the new axioms required for biseparability of the preferences. First of all an additional structural assumption is needed:

**A0** Separability:
The topology on \( X \) is separable.

Instead of A3 they use another axiom, which we term essentiality.

**A3**: Essentiality
All states in \( S \) are non null, and for some \( x \succ y \), and an event \( E \), \( x \succ xEy \succ y \).

An event \( E \) from the second part of A3* is termed **essential**. The axiom above guarantees the existence of an essential event.

**A3**: Extension of Partial Essentiality
For an event \( E \), if for some \( x \succ y \), \( xEy \succ y \), (resp. \( x \succ xEy \)), then for all \( a \succ b \succ c \), \( aEc \succ bEc \) (resp. for all \( c \succ a \succ b \), \( cEa \succ cEb \)).

To state the last axiom required for biseparability of preferences we first recall that for an act \( f \), \( x \in X \) is its **certainty equivalent** if \( f \sim \tilde{x} \). In the following definition and in the axiom the concept of certainty equivalence is used.

**Definition 6.** Let there be given two acts, \( f \) and \( g \), and an essential event \( G \). An act \( h \) is termed a \( G \)-mixture of \( f \) and \( g \), if for all \( s \in S \), \( h(s) \sim f(s)Gg(s) \).

Given our axioms so far it is obvious that certainty equivalent of each act, and as a result event mixtures, can easily be proved to exist. However when stating the
next axiom these proofs are not assumed.

A5* Binary Comonotonic Act Independence:
Let two essential events, \( D \) and \( E \), and three pairwise comonotonic, binary acts, \( aEb \), \( cEd \), and \( xEy \) be given. Suppose also that either \( xEy \) weakly dominates \( aEb \) and \( cEd \), or is weakly dominated by them. Then \( aEb \preceq cEd \) implies that a \( D \)-mixture of \( aEb \) and \( xEy \) is weakly preferred to a \( D \)-mixture of \( cEd \) and \( xEy \), provided that both mixtures exist.

GM (2001) Theorem 11 says that assuming \( A0 \) and \( A0^* \), preferences are biseparable iff they satisfy \( A1, A2, A3^*, A3^{**}, A4, \) and \( A5^* \).

The theorem implies that if \( x \succ y \succ z \), and \( u(y) = u(x)/2 + u(z)/2 \), then \( y \) is half way between \( x \) and \( z \). However we would like to express that \( y \) is half way between \( x \) and \( z \) in the language of preferences, and not by using an artificial construct like utility. Ghirardato et al. (2002) did it in their Proposition 1. They showed that if the preferences over acts are biseparable, and \( x \succ y \succ z \) in \( X \), then:

\[
\exists a, b \in X, \text{ and an essential event } E \text{ s.t. } \bar{a} \sim x Ey, \bar{b} \sim y Ez, \text{ and } x Ez \sim aEb. \quad (4)
\]

The line above is a behavioral definition of "\( y \) is half way between \( x \) and \( z \)." We denote it by \( y \in \mathcal{F}(x, z) \) \(^9\). We will use this notation to state our new axioms.

How complex are \( A0^*, A3^*, \) and \( A3^{**} \)? Separability is as complex as continuity. But it comes for free if \( X \) is in a Euclidean space with a topology induced by the Euclidean distance. \( A3^* \) has very low opaqueness, and \( A3^{**} \) is about the opaqueness of \( A5 \). Note also that the opaqueness of the notation, \( y \in \mathcal{F}(x, z) \) is not unlike that of \( <x, y > \sim^* <y, z> \).

We conclude this subsection with a comment about comparisons of the two approaches: biseparability and tradeoffs. The axioms \( A5 \) and \( A5^* \) seem unrelated. On the other hand, axioms \( A3 \) and \( A3^* \) cum \( A3^{**} \) seem related. Nevertheless, neither \( A3 \) implies \( A3^* \) cum \( A3^{**} \), nor vice versa. At this point it seems that the two axiomatizations so far: \( A0, \ldots, A5 \), and \( A0, A0^*, A1, A2, A3^*, A3^{**}, A4, A5^* \) are independent. Neither of them implies the other. We will return to this point at the end of the next section.

\(^9\)This is Definition 4. in Ghirardato et al. (2002).
4 New axioms and main results

As mentioned in the Introduction, in an AA type model the set of consequences \( X \) consists of an exogenously given set of all lotteries over some set of deterministic consequences, say \( Z \). As a result, for any two acts \( f \) and \( g \) and \( \theta \in [0,1] \), the act \( h = \theta f + (1 - \theta)g \) is well defined where \( h(s) = \theta f(s) + (1 - \theta)g(s) \) for all \( s \in S \). This mixture is used in the statement of two axioms central in the derivation of the MEU decision rule. One axiom, uncertainty aversion, states that \( f \succ g \) implies \( \theta f + (1 - \theta)g \succ g \) for any \( \theta \in [0,1] \). The other axiom, that of certainty independence, uses a mixture of acts with constant acts. Without lotteries the sets \( X \) and \( X^S \) are not linear spaces, and the two axioms have to be restated without availability of the mixture operation. We assume, however, that \( X \) is a connected topological space.

Given acts \( f, g, \) and \( h \), we introduced two ways to express the intuition that \( g \) is half way between \( f \) and \( h \). One is by tradeoffs, and the \( \sim^* \)-notation, and the other by certainty equivalence and the \( H(\cdot, \cdot) \) notation. So one way to formalize certainty independence goes as follows: Let \( f, g, \) and \( h \) be acts with \( f \succ g \), and \( h \) a constant act. Suppose that for some \( k \geq 1 \) and acts \( f = f_0, f_1, f_2, \ldots, f_{k-1}, f_k = h \), and \( g = g_0, g_1, g_2, \ldots, g_{k-1}, g_k = h \), \( f_i \) is half way between \( f_{i-1} \) and \( f_{i+1} \), and \( g_i \) is half way between \( g_{i-1} \) and \( g_{i+1} \), for \( i = 1, \ldots, k - 1 \). Then \( f_i \succ g_i \) for \( i = 1, \ldots, k - 1 \).

This essentially is the way that Casadesus-Masanell et al. (2000) stated the axiom. Such sequences are called standard sequences\(^{10}\) (Casadesus-Masanell et Al. also use a standard sequence to state uncertainty aversion). Ghirardato et al. (2003), on the other hand, introduce an artificial construct, \( \oplus \), coupled with a mixture set \( (M, \dot{\oplus}, \dot{+}) \). Their C-independence axiom states that for all \( \alpha \in [0,1] \), \( f \succ g \) implies \( \alpha f \oplus (1 - \alpha)\bar{x} \succ \alpha g \oplus (1 - \alpha)\bar{x} \). The \( \oplus \) notation stands for a consecutive application of our \( H(\cdot, \cdot) \) notation (for all \( s \in S \)). Both formulations (the one using standard sequences and that applying the \( \oplus \) notation) require an unbounded number of variables, thus have unbounded opaqueness.

As explained we would like to avoid, whenever possible, axioms of the form: “for any positive integer \( n \), and for any two \( n \)-lists, etc...”. Our uncertainty aversion axiom states that if \( f \succ h \) and \( g \) is half way between \( f \) and \( h \), then \( g \succ h \). Our certainty independence axiom is: Suppose that \( g \) is half way between \( f \) and a constant act \( \bar{w} \), and for some constant acts \( \bar{x} \) and \( \bar{y} \), \( \bar{y} \) is half way between \( \bar{x} \) and \( \bar{w} \). Then \( f \sim \bar{x} \) iff

\(^{10}\)see Krantz et al. (1971) and the references there.
However our version of the certainty independence axiom does not suffice to obtain the required representation. An axiom named certainty covariance is added. The certainty covariance axiom says that given acts \( f \) and \( g \), and consequences \( x \) and \( y \): If, for all states \( s \), the strength of preference of \( f(s) \) over \( g(s) \) is the same as the strength of preference of \( x \) over \( y \), then \( [f \sim x \iff g \sim y] \). In other words: when consequences are translated into utiles, the conditions of the axiom imply that the change from the vector of utiles \( (u(f(s)))_{s \in S} \), to the vector of utiles \( (u(g(s)))_{s \in S} \), is parallel to the move on the diagonal from the constant vector with coordinates \( u(x) \) to the constant vector with coordinates \( u(y) \). The axiom requires that indi¤erence (between \( x \) and \( f \)) be preserved by movements parallel to the diagonal in the utiles space.

4.1 The tradeoffs version

We now formally present the three axioms introduced above, using the \( \sim^* \) relation.

A6. Uncertainty Aversion:
For any three acts \( f, g, \) and \( h \) with \( f \succeq g \): if \( \forall s \in S, < f(s); h(s) > \sim^* < h(s); g(s) > \), then \( h \succeq g \).

A7. Certainty Independence:
Suppose that two acts, \( f \) and \( g \), and three consequences, \( x, y, \) and \( w \), satisfy
\( < x; y > \sim^* < y; w > \), and \( \forall s \in S, < f(s); g(s) > \sim^* < g(s); w > \). Then \( g \sim \overline{y} \iff f \sim \overline{x} \).

A8. Certainty Covariance:
Let \( f, g \) be acts and \( x, y \) consequences such that \( \forall s \in S, < f(s); g(s) > \sim^* < x; y > \). Then, \( f \sim \overline{x} \iff g \sim \overline{y} \).

As mentioned earlier, the three axioms are stated in the language of tradeoffs indifference \( \sim^* \) and not in the language of the preferences on \( X \) (or \( X^S \)). Assuming that the decision maker internalized the \( \sim^* \) relation, the opaqueness of each of these axioms is quite small.

Keeping this in mind, the axiom of uncertainty aversion, A6, is a tradeoffs version of the uncertainty aversion axiom introduced by Schmeidler (1989, preprint 1984).
Essentially, it replaces utility mixtures with the use of tradeoffs terminology. It is analogous to the uncertainty aversion axiom in a model with exogenous lotteries for the case of $\theta = 1/2$. Applying the axiom consecutively, and then using continuity, A2, guarantees the conclusion for any mixture. A6 expresses uncertainty aversion in that it describes the decision maker’s will to reduce the impact of not knowing which state will occur. The reduction is achieved by averaging the consequences of $f$ and $g$ in every state.

In a similar manner to A6, axiom A7 produces a utilities analogue of the certainty independence axiom phrased by Gilboa and Schmeidler (1989) in their axiomatization of MEU, for the case $\theta = 1/2$. Consecutive application of A7 will yield the analogue for $\theta = 1/2^m$, where $\theta$ is the coefficient of the non-constant act. To get the analogue for all dyadic mixtures we have to supplement it with A8. When acts are represented in the utiles space, one can see that axiom A8 is some version of a parallelogram where one side is an interval on the diagonal. This is the order interval $[x, y]$. The parallel side is $[f, g] \approx \{f(s), g(s)\}_{s \in S}$, which should be thought of as an off diagonal “order” interval in $X^S$. The other two sides of the parallelogram are delineated by equivalences: $f \sim \pi$ iff $g \sim \gamma$.

Having stated all axioms, we can formulate our main result,

**Theorem 7.** Suppose that a binary relation $\succsim$ on $X^S$ is given, where $X$ and $S$ satisfy the structural assumptions, A0(a) and A0(b). Then the following two statements are equivalent:

(1) $\succsim$ satisfies

(A1) Weak Order

(A2) Continuity

(A3) Substantiality

(A4) Monotonicity

(A5) Binary Comonotonic Tradeoff Consistency

(A6) Uncertainty Aversion

(A7) Certainty Independence

(A8) Certainty Covariance
There exist a continuous utility function $u : X \to \mathbb{R}$ and a non-empty, closed and convex set $C$ of additive probability measures on $\Sigma$, such that, for all $f, g \in X^S$,

$$f \succ g \iff \min_{P \in C} \int u \circ f dP \geq \min_{P \in C} \int u \circ g dP$$

Furthermore, the utility function $u$ is unique up to an increasing linear transformation, and the set $C$ is unique, $C \neq \Delta(S)$, and for all $s$ in $S$, $\max\{P(s) | P \in C\} > 0$.

Note that if $C = \Delta(S)$, each act is evaluated according to its worst consequence and there is no meaning to cardinal utility. The requirement that for all $s$ in $S$, $\max\{P(s) | P \in C\} > 0$, reflects the assumption that all states are nonnull. As mentioned earlier, this assumption is made merely for the sake of ease of presentation and may be dropped (at a cost). All proofs appear in Appendix A.

### 4.2 The biseparable version

For formulation of the theorem using biseparable preferences we simply repeat the new axioms with the $\mathcal{H}(\cdot, \cdot)$ notation.

**A6*. Uncertainty Aversion:**
For any three acts $f, g$, and $h$ with $f \succ g$: if $\forall s \in S, h(s) \in \mathcal{H}(f(s), g(s))$, then $h \succ g$.

**A7*. Certainty Independence:**
Suppose that two acts, $f$ and $g$, and three consequences, $x, y,$ and $w$, satisfy $y \in \mathcal{H}(x, w)$, and $\forall s \in S, g(s) \in \mathcal{H}(f(s), w)$. Then $g \sim y$ iff $f \sim x$.

**A8*. Certainty Covariance:**
Let $f, g$ be acts and $x, y$ consequences such that $\forall s \in S$:

$$[z \in \mathcal{H}(f(s), y) \iff z \in \mathcal{H}(x, g(s))]$$

Then, $f \sim x$ iff $g \sim y$.

Note that the relation between A8 and A8* is based on the Euclidean geometry theorem which says: A (convex) quadrangle is a parallelogram iff its diagonals bisect each other. The relation between A6 and A7, and their starred counterparts, A6* and A7*, is derived from the trivial observation: $y \in \mathcal{H}(x, z)$ iff $x < y < z$. The corresponding representation theorem below follows.

**Theorem 8.** Suppose that a binary relation $\succeq$ on $X^S$ is given, where $X$ and $S$ satisfy the structural assumptions, $A0(a), A0(b)$, and $A0*$. Then the following two statements are equivalent:
(1) \( \succeq \) satisfies

(A1) Weak Order
(A2) Continuity
(A3*) Essentiality
(A3**) Extension of Partial Essentiality
(A4) Monotonicity
(A5*) Binary Comonotonic Act Independence
(A6*) Uncertainty Aversion
(A7*) Certainty Independence
(A8*) Certainty Covariance

(2) There exist a continuous utility function \( u : X \rightarrow \mathbb{R} \) and a non-empty, closed and convex set \( C \) of additive probability measures on \( \Sigma \), such that, for all \( f, g \in X^S \),
\[
 f \succeq g \iff \min_{P \in C} \int u \circ f dP \geq \min_{P \in C} \int u \circ g dP
\]

Furthermore, the utility function \( u \) is unique up to an increasing linear transformation, and the set \( C \) is unique, \( C \neq \Delta(S) \), and for all \( s \) in \( S \), \( \max\{P(s) | P \in C\} > 0 \).

In the two theorems, 7 and 8, the parts (2) are identical. Hence, given a binary relation \( \succeq \) on \( X^S \), axioms A0,..., A8 hold iff axioms, A0, A0*, A1, A2, A3*, A3**, A4, A5*,...,A8* hold. Still, A5 and A5* are quite different axioms.

4.3 Preference relations admitting CEU and MEU representations

Theorem 9 below is a purely subjective counterpart of a proposition in Schmeidler (1989), characterizing preference relations representable by both CEU and MEU rules. Axiom A5 is replaced by a more restrictive axiom, A5.1, below. The latter axiom, together with Axioms A1-A4, yield a purely subjective CEU representation (Definition 2). Addition of uncertainty aversion (A6) implies that the preference relation can equivalently be represented by an MEU functional, where the set of additive probabilities is the core of the nonadditive probability.
A5.1 Comonotonic Tradeoff Consistency:
For any four consequences \(a, b, c, d\), states \(s, t\) and acts \(f, g, f', g'\),
\[
asf \sim bsg, \ csf \sim dsg, \ atf' \sim btg' \Rightarrow cf' \sim dtg' \tag{5}
\]
whenever the sets of acts \(\{asf, bsg, csf, dsg\}\) and \(\{atf', btg', cf', dtg'\}\) are comonotonic, 
\(s\) is comonotonically nonnull on the first set, and \(t\) on the second.

This definition originates in Wakker (1986,1989).

Theorem 9. Assume that the structural assumptions, A0(a) and A0(b), hold, and 
let \(\succsim\) be a binary relation on \(X^S\). Then the following two statements are equivalent:

(1) \(\succsim\) satisfies:

(A1) Weak Order
(A2) Continuity
(A3) Substantiality
(A4) Monotonicity
(A5.1) Comonotonic Tradeoff Consistency
(A6) Uncertainty Aversion

(2) There exist a continuous, non-constant, cardinal utility function \(u : X \to \mathbb{R}\) 
and a unique, convex, and nonadditive probability \(\eta\) on \(\Sigma\) such that, for all 
\(f, g \in X^S\),
\[
f \succsim g \iff \int_S u \circ f d\eta \geq \int_S u \circ g d\eta
\]
where the nonadditive probability \(\eta\) satisfies
\[
\int_S u \circ f d\eta = \min\{\int_S u \circ f dP \mid P \in \text{core}(\eta)\}\tag{6}
\]
Furthermore, \(\text{core}(\eta) \neq \Delta(S)\), and for all \(s \in S\), \(\eta(S \setminus \{s\}) < 1\).

The proof appears in Appendix A, subsection 5.5.

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Peter Wakker.
Appendix A. Proof of the theorems

We begin by listing two observations, which are standard in neoclassical consumer theory. These observations will be used in the sequel, sometimes without explicit reference.

Observation 10. Weak order, Substantiality and Monotonicity imply that there are two consequences \( x^*, x_* \in X \) such that \( x^* \succ x_* \).

Observation 11. Weak order, Continuity and Monotonicity imply that each act \( f \) has a certainty equivalent, i.e., a constant act \( \bar{x} \) such that \( f \sim \bar{x} \).

5.1 Proof of the implication (1)⇒(2) of Theorem 7

Our first step is to use A0 through A5 to derive a utility function, which represents \( \succcurlyeq \) on constant acts and respects the tradeoff indifference relation, in the sense that \( < a; b > \sim^* < c; d > \) implies \( u(a) - u(b) = u(c) - u(d) \). An implied representation on \( X^S \) is defined through certainty equivalents.

Proposition 12. Under axioms A0 to A5 the following is satisfied:

1) there exists a continuous function \( u : X \rightarrow \mathbb{R} \) such that for all \( x, y \in X \),
\[ x \succcurlyeq y \iff u(x) \geq u(y) \text{ and } < a; b > \sim^* < c; d > \text{ implies } u(a) - u(b) = u(c) - u(d) \]. Furthermore, \( u \) is unique up to a positive linear transformation.

2) Given a function \( u \) as in (1) there exists a unique continuous \( J : X^S \rightarrow \mathbb{R} \), such that
\[ f \succcurlyeq g \iff J(f) \geq J(g) \text{ for all } f, g \in X^S \text{, and } J(\bar{x}) = u(x) \text{ for all } x \in X \].

5.1.1 Proof of Proposition 12

We define a new decision problem. Choose some state \( s \in S \), and let \( \{\{s\}, S \setminus \{s\}\} \) be the new state space. Let \( X^{\{s\}, S \setminus \{s\}} \) be the set of acts, functions from the new state space to the set of consequences \( X \). For consequences \( a \) and \( b \), \( (a, b) \) denotes an act in \( X^{\{s\}, S \setminus \{s\}} \), assigning \( a \) to \( \{s\} \) and \( b \) to \( S \setminus \{s\} \). To avoid confusion, we use the notation \( (a, b) \) and \( (a, a) \) for acts in \( X^{\{s\}, S \setminus \{s\}} \), and reserve the notation \( asb \) and \( \bar{a} \) for acts in \( X^S \). Define a binary relation \( \succcurlyeq_s \) on \( X^{\{s\}, S \setminus \{s\}} \) by:
\[ \text{for all } (a, b), (c, d) \in X^{\{s\}, S \setminus \{s\}} \text{, } (a, b) \succcurlyeq_s (c, d) \iff asb \succ csd, \text{ asb, csd } \in X^S. \]
There exists a one-to-one correspondence between the sets $X^{\{s\}, S \setminus \{s\}}$ and \{ asb $|$ a, b $\in$ X \}. Thus, the product topology on $X^{\{s\}, S \setminus \{s\}}$ is equivalent to the original topology, restricted to the set \{ asb $|$ a, b $\in$ X \}. We summarize the attributes of $\succeq_s$, required to obtain a CEU representation of it, in the next lemma.

**Lemma 13.** The binary relation $\succeq_s$ on $X^{\{s\}, S \setminus \{s\}}$ satisfies Weak order, Continuity, Monotonicity, Non-nullity of the states \{s\} and $S \setminus \{s\}$ and (Binary) Comonotonic Tradeoff Consistency.

Proof. Weak Order, Continuity, Monotonicity and Non-nullity follow from the definition of $\succeq_s$ and the attributes of the original relation $\succeq$, and from the equivalence between the topology on $X^{\{s\}, S \setminus \{s\}}$ and the topology on $X^S$ restricted to \{ asb $|$ a, b $\in$ X \}.

Note that when there are only two states, axioms A5 (Binary Comonotonic Tradeoff Consistency) and A5.1 (Comonotonic Tradeoff Consistency) coincide. It follows that $\succeq_s$ satisfies Comonotonic Tradeoff Consistency.

Additive representation of $\succeq_s$ on $X^{\{s\}, S \setminus \{s\}}$ will be obtained using Corollary 10 and Observation 9 from Kobberling and Wakker (2003), applied to the case of two states. We state it in Lemma 14 below. KW use a slightly different tradeoff consistency condition than ours. Assuming the rest of our axioms, their condition follows (see proof in Appendix B).

**Lemma 14.** (Corollary 10 and Observation 9 of KW 2003:) Assume the conclusions of Lemma 13. Then there exists a nonadditive probability $\rho$ on $2^{\{s\}, S \setminus \{s\}}$ and a continuous utility function $u : X \rightarrow \mathbb{R}$, such that $\succeq_s$ is represented on $X^{\{s\}, S \setminus \{s\}}$ by the following CEU functional $U$:

$$U((a, b)) = \begin{cases} u(a)\rho(s) + u(b)[1 - \rho(s)] & a \succeq b \\ u(b)\rho(S \setminus s) + u(a)[1 - \rho(S \setminus s)] & \text{otherwise} \end{cases}$$

(7)

Furthermore, if both \{s\} and $S \setminus \{s\}$ are comonotonically nonnull on one of the sets \{asb $|$ a $\succeq$ b \} or \{asb $|$ a $\not\succeq$ b \}, then $u$ is unique up to an increasing linear transformation and $\rho$ is unique. Otherwise, $u$ is unique up to a continuous, strictly increasing transformation, and $\rho$ if unique, obtaining zero on the null state and one on the nonnull state.

Let $s' \in S$ be a state such that $x \succ y \Rightarrow \pi \succ xs' y \succ \overline{y}$ (such a state exists by Substantiality). Then \{s'\} and $S \setminus \{s'\}$ are comonotonically nonnull on \{as'b $|$ a $\succeq$ b \}.
Apply Lemma 14 to the corresponding relation $\succeq_{s'}$, and denote by $U$ the resulting CEU functional, with a utility function $u$ and nonadditive probability $\rho$. By definition of $\succeq_{s'}$, $U$ represents $\succeq$ on acts of the form $as'b$ and the utility function $u$ represents $\succeq$ on $X$, and is unique up to an increasing linear transformation.

We proceed to show that $u$ respects tradeoff indifference. Let $t \neq s'$. A CEU functional, representing $\succeq$ on acts of the form $atb$, may be obtained, as described in Lemma 14. The link between this functional and $U$ is elucidated in the following lemma.

**Lemma 15.** Let $t \neq s'$, $W$ a CEU representation on $X^{\{|t\}, S \setminus \{t\}}$, and $w$ the corresponding utility function. Suppose that $\{t\}$ and $S \setminus \{t\}$ are comonotonically nonnull on some comonotonic set of the form $atb$. Then $w = \sigma u + \tau$, $\sigma > 0$.

Proof. Denote by $\varphi$ the nonadditive probability corresponding to $W$. Denote, for $x \in X$, $V_1(x) = u(x)\rho(s')$, and recall that $\rho(s') > 0$ because $s'$ is comonotonically nonnull on $\{as'b \mid a \succcurlyeq b\}$. Suppose that $\{t\}$ and $S \setminus \{t\}$ are comonotonically nonnull on $\{atb \mid a \succcurlyeq b\}$, and let $W_1(x) = w(x)\varphi(t)$. Binary Comonotonic Tradeoff Consistency allows us to follow the steps of Lemma VI.8.2 in Wakker 1989, to obtain $W_1 = \eta V_1 + \lambda$, $\eta > 0$, thus $w(x) = \sigma u + \tau$ (set $\sigma = \eta \rho(s')/\varphi(t) > 0$, where $\varphi(t) > 0$ follows from $t$ being comonotonically nonnull as assumed).

Otherwise, if $\{t\}$ and $S \setminus \{t\}$ are comonotonically nonnull on $\{atb \mid a \not{\succcurlyeq} b\}$, the result may be obtained using a functional $W_2(x) = w(x)\varphi(S \setminus t)$ (here again $\varphi(S \setminus t) > 0$ because $S \setminus \{t\}$ is comonotonically nonnull as assumed).

**Corollary 16.** For any four consequences $a, b, c, d$, $<a;b > \sim^* < c;d >$ implies $u(a) - u(b) = u(c) - u(d)$.

Proof. Let $<a;b > \sim^* < c;d >$. Then there exist consequences $x, y$ and a state $t$ such that $atx \sim bty$ and $ctx \sim dty$, with $\{atx, bty, ctx, dty\}$ comonotonic and $t$ comonotonically nonnull on this set. Let $W$ be a CEU representation on $X^{\{|t\}, S \setminus \{t\}}$, with a corresponding utility function $w$. Suppose first that $S \setminus \{t\}$ is comonotonically null on $\{atx, bty, ctx, dty\}$, and let $\alpha t \beta$ be an act comonotonic with the acts in this set. Then by Lemma 14, $W(\alpha t \beta) = w(\alpha)$, and the indifference relations above imply $w(a) = w(b)$ and $w(c) = w(d)$, thus $a \sim b$ and $c \sim d$, resulting trivially $u(a) - u(b) = 0 = u(c) - u(d)$.

Otherwise $S \setminus \{t\}$ is comonotonically nonnull on the relevant binary comonotonic set. The indifference relations $atx \sim bty$ and $ctx \sim dty$ then imply $w(a) - w(b) =
$w(c) - w(d)$. If $t = s'$, $s'$ the state on which the representation $U$ was obtained, then by the uniqueness result in Lemma 14, $w = \sigma u + \tau$ with $\sigma > 0$. If $t$ is another state, then Lemma 15 yields this relation. In any case, $w(a) - w(b) = w(c) - w(d)$ implies $u(a) - u(b) = u(c) - u(d)$. 

Statement (1) of Proposition 12 is thus proved. Having a specific $u$, a unique representation $J$ on $X^S$ follows through the use of certainty equivalents. By Observation 11 there exists, for any $f \in X^S$, a constant act $CE(f)$ such that $f \sim CE(f)$. Set $J(f) = u(CE(f))$, then for all $f, g \in X^S$,

$$f \succeq g \iff CE(f) \succeq CE(g) \iff J(f) = u(CE(f)) \geq u(CE(g)) = J(g)$$

That is, $J$ represents $\succeq$ on $X^S$. $J$ is unique and continuous by its definition, and by continuity of $u$. The proof of the proposition is completed.

5.2 Proof of the implication (1)$\Rightarrow$(2) of Theorem 7 - continued

Throughout this section $s'$ will denote a state that satisfies $x \succ s'y \succ y$ whenever $x \succ y$ (exists by Substantiality), and $U$ will denote the CEU representation on binary acts of the form $as'b$. $u$ and $\rho$ will notate the corresponding utility function on $X$ and nonadditive probability on $2^{|s'| \times S \setminus \{s'\}}$, respectively. By choice of $s'$, $0 < \rho(s')$ and $\rho(S \setminus \{s'\}) < 1$. As proved above, $u$ satisfies (1) of Proposition 12, and we denote by $J$ the representation of $\succeq$ over $X^S$, obtained according to (2) of the proposition. By connectedness of $X$ and continuity of $u$, $u(X)$ is an interval. Since $u$ is unique up to a positive linear transformation, we fix it for the rest of this proof such that there are $x^*, x_*$ for which $u(x_*) = -2$ and $u(x^*) = 2$. $\theta$ is a consequence such that $u(\theta) = 0$.

The proof is conducted in three logical steps. First, an MEU representation is obtained on a subset of $X^S$. This is done by moving to work in utiles space, and applying tools from Gilboa and Schmeidler (1989). Afterwards, the representation is extended to a 'stripe', in utiles space, around the main diagonal. The last step extends the representation to the entire space.

Claim 17. There exists a subset $Y \subset X$ such that $u(Y)$ is a non-degenerate interval, $\theta \in Y$, and for all $a, b, c, d \in Y$, $u(a) - u(b) = u(c) - u(d)$ implies $a \sim b < c \sim d$.
Proof. Employing the representation $U$ and Continuity of $u$, there exists an interval $[-\tau, \tau]$ $(0 < \tau)$ small enough such that if $u(a), u(b), u(c), u(d) \in [-\tau, \tau]$ satisfy $u(a) - u(b) = u(c) - u(d)$, there are $x, y \in X$ for which $u(x), u(y) \leq -\tau$ and
\[
u(a) - u(b) = \frac{1 - \rho(s')}{\rho(s')} [u(y) - u(x)] = u(c) - u(d).
\]

It follows that $as'x \sim bs'y$ and $cs'x \sim ds'y$. By choice of the consequences, the set $\{asx, bsy, csx, dsy\}$ is comonotonic, and $s'$ is comonotonically nonnull on this set. Thus $<a; b > \sim^* < c; d >$.

Let $Y = \{y \in X \mid u(y) \in [-\tau, +\tau]\}$. $Y$ is a subset of $X$, $u(Y)$ is an interval, $\theta \in Y$, and for all $a, b, c, d \in Y$, $u(a) - u(b) = u(c) - u(d)$ implies $<a; b > \sim^* < c; d >$.

Throughout this subsection $Y$ will denote a subset of $X$ as specified in the above claim. Note that utility mixtures of acts in $Y^S$ are also in $Y^S$ (this fact is employed in the proof without further mention).

The following is a utilities analogue to Schmeidler’s (1989) uncertainty aversion axiom.

A6u. Uncertainty Aversion in Utilities:
Let the preference relation $\succeq$ be represented on constant acts by a utility function $u$. If $f, g, h$ are acts such that $f \succeq g$, and for all states $s$, $u(h(s)) = \frac{1}{2}u(f(s)) + \frac{1}{2}u(g(s))$, then $h \succeq g$.

Lemma 18. If (1) of the theorem holds then uncertainty aversion in utilities (A6u) is satisfied on $Y^S$.

Proof. Let $f, g, h \in Y^S$ be acts satisfying $f \succeq g$, $h$ such that in all states $s$,
\[u(h(s)) = \frac{1}{2}u(f(s)) + \frac{1}{2}u(g(s)).\]
Then for all states $s$,
\[u(f(s)) - u(h(s)) = u(h(s)) - u(g(s)),\]
therefore $<f(s), h(s) > \sim^* < h(s), g(s) >$. By A6 it follows that $h \succeq g$. ■

Comment 19. Applying A6u consecutively implies that if $f, g, h \in Y^S$ are acts satisfying $f \succeq g$, $h$ an act such that for all states $s$,
\[u(h(s)) = \frac{k}{2m}u(f(s)) + (1 - \frac{k}{2m})u(g(s)), \ (\frac{k}{2m} \in [0, 1])\]
then $h \succeq g$. Continuity then implies that the same is true for any $\alpha : 1 - \alpha$ utilities mixture ($\alpha \in [0, 1]$).
Lemma 20. Let \( t \) be a state such that \( t \neq s' \). Then both \( \{t\} \) and \( S \setminus \{t\} \) are nonnull on the set \( \{atb \mid a \preceq b\} \).

Proof. Let \( t \) be some state, and \( W \) the CEU representation on acts of the form \( atb \) (exists by lemma 14). Denote by \( \varphi \) the nonadditive probability corresponding to \( W \).

Schmeidler (1989) gives a definition of Uncertainty Aversion, which is identical to Uncertainty Aversion in Utilities if mixtures are interpreted as utility mixtures. Thus, according to Comment 19, \( \succeq \) on \( Y^S \) satisfies Schmeidler’s Uncertainty Aversion. Consider the set of binary acts \( Y^{\{t\}, S \setminus \{t\}} \), on which \( \succeq \) is represented by a CEU functional. A proposition of Schmeidler (1989), stated below in Lemma 33, may be applied to \( \succeq \) on this set, to obtain that the nonadditive probability \( \varphi \) is convex.

Suppose, contrary to the assumption above, that \( \{t\} \) is null on \( \{atb \mid a \preceq b\} \). This is equivalent to assuming that \( \varphi(S \setminus \{t\}) = 1 \). By convexity of \( \varphi \) it must then be that \( \varphi(\{t\}) = 0 \). But such a nonadditive probability means that for all \( x, y, z \in X \), \( xty \sim \overline{y} \sim zty \), hence \( t \) is null, contradicting the axiom of Substantiality. We conclude that \( \{t\} \) is nonnull on \( \{atb \mid a \preceq b\} \).

Choose \( t \neq s' \). Monotonicity entails \( ytx \succeq xs'y \) for \( x > y \), and therefore, since \( xs'y \succ \overline{y} \), also \( ytx \succ \overline{y} \). It follows that \( S \setminus \{t\} \) is nonnull on \( \{atb \mid a \preceq b\} \). 

Let \( t' \neq s' \) be some state. We denote by \( W \) the CEU representation on acts of the form \( at'b \), with corresponding utility function \( w \) and nonadditive probability \( \varphi \). By the claim above it follows that \( 0 < \varphi(S \setminus \{t'\}) < 1 \), and, according to Lemma 14, \( w \) is unique up to an increasing linear transformation. Lemma 15 further implies that \( w = \sigma u + \tau, \sigma > 0 \), so we fix \( w = u \) by setting \( u(x_s) = -2, u(x^*) = 2 \).

Claim 21. Let \( f \in Y^S \) be an act, \( \overline{\pi} \in Y^S \) a constant act such that \( f \sim \overline{\pi} \), and \( \alpha \in (0,1) \). If \( g \) is an act such that for all states \( s \), \( u(g(s)) = \alpha u(f(s)) + (1-\alpha)u(x) \), then \( g \sim \overline{\pi} \).

Proof. We first prove the claim for \( \alpha = \frac{1}{2m} \), that is, when \( g \) satisfies, for all states \( s \), \( u(g(s)) = \frac{1}{2m} u(f(s)) + (1-\frac{1}{2m}) u(x) \). The proof is by induction on \( m \). For \( m = 1 \), in all states \( s \), \( f(s) > g(s) > x \), which by certainty independence (A7) (with \( \overline{w} = \overline{\pi} \)) implies \( g \sim \overline{\pi} \). Assume that for \( g \) that satisfies \( u(g(s)) = \frac{1}{2m} u(f(s)) + (1-\frac{1}{2m}) u(x) \), \( g \sim \overline{\pi} \). Let \( h \) be an act such that \( u(h(s)) = \frac{1}{2m+1} u(f(s)) + (1-\frac{1}{2m+1}) u(x) \). \( h \) satisfies, for all states \( s \),
Claim 22. Let $f \in Y^S$ be an act, $\overline{w}, \overline{x}, \overline{y} \in Y^S$ constant acts and $\alpha \in (0, 1)$, such that $f \sim \overline{x}$ and $u(y) = \alpha u(x) + (1 - \alpha) u(w)$. If $h$ is an act such that in all states $s$, $u(h(s)) = \alpha u(f(s)) + (1 - \alpha) u(w)$, then $h \sim \overline{y}$.

Proof. Let $g$ be an act such that for all states $s$, $u(g(s)) = \alpha u(f(s)) + (1 - \alpha) u(x)$, then $g \in Y^S$. For all states $s$,
\[
\begin{align*}
u(h(s)) &= \alpha u(f(s)) + (1 - \alpha) u(w) = u(g(s)) + u(y) - u(x) \
u(h(s)) - u(g(s)) &= u(y) - u(x) \\< h(s); g > &\sim^* < y; x >.
\end{align*}
\]

According to Claim 21, $g \sim \overline{x}$, then by Certainty Covariance it follows that $h \sim \overline{y}$.

The following is an analogue to Lemma 3.3 from Gilboa and Schmeidler (1989), and follows the lines of the proof given there. Denote by $B$ the set of all real-valued functions on $S$. For any $\gamma \in \mathbb{R}$, $\overline{\gamma} \in B$ is the constant function $\gamma^S$.

Lemma 23. There exists a functional $I : B \to \mathbb{R}$ such that:

(i) For $f \in Y^S$, $I(u \circ f) = J(f)$. 

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(ii) \( I \) is monotonic (i.e., for \( a, b \in B \), if \( a \geq b \) then \( I(a) \geq I(b) \)).

(iii) \( I \) is superadditive and homogeneous of degree 1 (therefore \( I(\overline{1}) = 1 \)).

(iv) \( I \) satisfies certainty independence: for any \( a \in B \) and \( \overline{\gamma} \) a constant function, \( I(a + \overline{\gamma}) = I(a) + I(\overline{\gamma}) \).

Proof. Define \( I(a) \) on \( (u(Y))^S \) by (i). By monotonicity, if in all states \( s \), \( f(s) \sim g(s) \), then \( f \sim g \). \( I \) is therefore well defined.

We show that \( I \) is homogeneous on \( (u(Y))^S \). Let \( a, b \in (u(Y))^S \), \( b = \alpha a \). Let \( f, g \) be acts such that \( u \circ f = a \), \( u \circ g = b \), and \( x, y \in X \) such that \( f \sim \overline{x} \), \( u(y) = \alpha u(x) \). Monotonicity and the fact that \( Y \) contains \( \theta \) imply that \( \overline{x}, \overline{y} \in Y^S \). Applying Claim 22 with \( \overline{\theta} = \overline{x} \) implies that \( g \sim \overline{y} \). Thus \( I(b) = J(g) = u(y) = \alpha u(x) = \alpha J(f) = \alpha I(a) \), and homogeneity is established. Note also that if \( y \in Y \) and \( u(y) = \gamma \), then \( I(\overline{\gamma}) = u(y) = \gamma \), and by homogeneity it follows that \( I(\overline{1}) = 1 \).

The functional \( I \) may now be extended from \( (u(Y))^S \) to \( B \) by homogeneity. By its definition, \( I \) is homogeneous on \( B \), and by monotonicity of the preference relation \( I \) is also monotonic. By homogeneity it suffices to show that certainty independence and superadditivity of \( I \) hold on \( (u(Y))^S \).

For certainty independence, let \( a, \overline{\gamma} \in (u(Y))^S \). Let \( f \in Y^S \) be such that \( u \circ f = a \), and denote \( I(a) = \beta \). Suppose \( \overline{x}, \overline{y}, \overline{\theta} \in Y^S \) are constant acts such that \( u(x) = \beta \), \( u(w) = \gamma \), \( u(y) = (u(x) + u(w))/2 \) and \( h \) an act such that in all states \( s \), \( u(h(s)) = \frac{1}{2}(u(f(s)) + u(w)) \). Then \( h \in Y^S \), \( f \sim \overline{x} \), and by Claim 22 it follows that \( h \sim \overline{y} \), which implies
\[
J(h) = I(u \circ h) = I(\frac{1}{2}(u \circ f + \overline{\gamma})) = u(y) = \frac{1}{2}(\beta + \gamma) = \frac{1}{2}(I(a) + \gamma).
\]
Hence, using homogeneity to extend the result to \( B \), for all \( a, \overline{\gamma} \in B \),
\[
I(a + \overline{\gamma}) = I(a) + I(\overline{\gamma}).
\]

Finally we show that \( I \) is superadditive on \( (u(Y))^S \). Let \( a, b \in (u(Y))^S \) and \( f, g \in Y^S \) such that \( u \circ f = a \), \( u \circ g = b \). Assume first that \( I(a) = I(b) \). By Lemma 18, if \( h \) is an act such that \( u \circ h = \frac{1}{2}(u \circ f + u \circ g) \), then \( h \succsim g \). Thus,
\[
J(h) = I(u \circ h) = I(\frac{1}{2}(u \circ f + u \circ g)) \geq J(g) = \frac{1}{2}(I(a) + I(b)).
\]
so by homogeneity, if \( a, b \in B \) satisfy \( I(a) = I(b) \) then \( I(a + b) \geq I(a) + I(b) \).
Now assume that \( I(a) > I(b) \) and let \( \gamma = I(a) - I(b), \ c = b + \gamma \). Certainty independence of \( I \) implies \( I(c) = I(b) + \gamma = I(a) \). Again by certainty independence and by superadditivity for the case \( I(a) = I(b) \), it follows that
\[
I\left( \frac{1}{2}(a + b) \right) + \frac{1}{2}\gamma = I\left( \frac{1}{2}(a + c) \right) \geq \frac{1}{2}(I(a) + I(c)) = \frac{1}{2}(I(a) + I(b)) + \frac{1}{2}\gamma.
\]
and superadditivity is proved for all \( a, b \in (u(Y))S \), and may be extended to \( B \) by homogeneity.

Lemma 3.5 from Gilboa and Schmeidler (1989) may now be applied to obtain an MEU representation of \( \succcurlyeq \) on \( Y^S \).

**Lemma 24.** (Lemma 3.5 and an implied uniqueness result from Gilboa and Schmeidler (1989)):
If \( I \) is a monotonic superlinear certainty-independent functional on \( B \), satisfying \( I(1) = 1 \), there exists a closed and convex set \( C \) of finitely additive probability measures on \( \Sigma \) such that for all \( b \in B \), \( I(b) = \min\{ \int b dP \mid P \in C \} \). The set \( C \) is unique.

**Corollary 25.** \( \succcurlyeq \) is represented on \( Y^S \) by an MEU functional.

Let \( C \) denote the set of additive probability measures on \( \Sigma \), involved in the MEU representation of \( \succcurlyeq \) on \( Y^S \). The functional \( I \) is as in Lemma 23: for all \( f \in Y^S \), \( I(u \circ f) = J(f) \), which by the above claim implies that for all \( f \in Y^S \), \( I(u \circ f) = \min\{ \int u \circ f dP \mid P \in C \} \). Outside of \((u(Y))^S\), \( I \) is extended by homogeneity. By Lemma 23 \( I \) is monotonic, superadditive, homogeneous of degree 1 and satisfies certainty independence.

It is now required to show that the homogeneous extension of \( I \) outside of \((u(Y))^S\) is consistent with the preference relation. That is, that for all \( f \in X^S \),
\[
J(f) = I(u \circ f) = \min\{ \int u \circ f dP \mid P \in C \}.
\]
The proof consists of two steps. First, we extend the representation to a stripe (in utilities space) parallel to the main diagonal, obtained by sliding \((u(Y))^S\) along the diagonal. Then we extend the representation from that stripe to the entire space.

For an act \( g \), let \( \text{diag}(g) = \{ f \in X^S \mid u \circ f = u \circ g + \gamma, \ \gamma \in \mathbb{R} \} \). That is, \( \text{diag}(g) \) contains all acts that can be obtained from \( g \) by constant shifts (translations parallel to the main diagonal, in utilities space). Note that \( \text{diag}(g) \) is convex w.r.t. utility mixtures.

**Claim 26.** Let \( g \in Y^S \). Then for all acts \( f \in \text{diag}(g) \), \( J(f) = I(u \circ f) \).
Proof. Let \( f, g \) be acts that satisfy \( g \in Y^S \) and \( u \circ f = u \circ g + \bar{\tau} \), \( \varepsilon > 0 \), so \( f \in \text{diag}(g) \). Suppose that \( x, y \) are consequences such that \( g \sim \bar{\tau} \) and \( u(x) - u(y) = \varepsilon \). It follows that for all states \( t \), \( u(f(t)) - u(g(t)) = u(x) - u(y) \).

Similarly to the proof of Claim 17, if \( \varepsilon \) is small enough then for all states \( s \) there are \( z_1, z_2 \) such that \( u(z_1), u(z_2) < \min_{s \in S} u(g(s)) \), and
\[
\varepsilon = u(f(s)) - u(g(s)) = \frac{1}{p(s)}[u(z_2) - u(z_1)] = u(x) - u(y).
\]
It follows that for all states \( s \), \( f(s)z_1 \sim g(s)z_2 \) and \( xs'z_1 \sim yys'z_2 \), with \( \{f(s)s'z_1, g(s)s'z_2, xs'z_1, ys'z_2\} \) comonotonic, and \( s' \) comonotonically nonnull on this set. Thus
\[
\langle f(s); g(s) > \sim s < x; y > \text{ for all states } s \)]

Certainty Covariance axiom implies that \( f \sim \bar{\tau} \). Applying certainty independence of \( I \),
\[
I(u \circ f + u \circ \bar{\tau}) = I(u \circ f) + u(y) = I(u \circ g + u \circ \bar{\tau}) = J(g) + u(x),
\]
and \( I(u \circ f) = u(x) = J(f) \). The same procedure may be repeated (in small 'tradeoff steps') for all \( f \in \text{diag}(g) \) that satisfy \( u \circ f = u \circ g + \bar{\tau} \), \( \gamma \geq 0 \), to obtain \( J(f) = I(u \circ f) \).

In order to extend the representation to the lower part of the diagonal, we employ the representation \( W \) on states of the form \( at'b \) (\( t' \) as above, i.e. \( t' \neq s' \)). Let \( f, g \) be acts that satisfy \( g \in Y^S \) and \( u \circ f = u \circ g + \bar{\tau} \), \( \varepsilon > 0 \). Let \( x, y \) be consequences such that \( g \sim \bar{\tau} \) and \( u(y) - u(x) = \varepsilon \). Similarly to the previous case, if \( \varepsilon \) is small enough, then for all states \( s \) there are \( z_3, z_4 \) such that \( u(z_3), u(z_4) > \max_{s \in S} u(g(s)) \), and
\[
\varepsilon = u(g(s)) - u(f(s)) = \frac{\varphi(S \setminus \{t'\})}{1 - \varphi(S \setminus \{t'\})} |u(z_3) - u(z_4)| = u(y) - u(x) \text{ (recall that} \)
\[
0 < \varphi(S \setminus \{t'\}) < 1. \]
It follows that for all states \( s \), \( f(s)t'z_3 \sim g(s)t'z_4 \) and \( xt'z_3 \sim yut'z_4 \), with the sets \( \{f(s)t'z_3, g(s)t'z_4, xt'z_3, yut'z_4\} \) comonotonic, and \( t' \) comonotonically nonnull on these sets. Therefore
\[
< f(s); g(s) > \sim s < x; y > \text{ for all states } s \]

By Certainty Covariance it follows that \( f \sim \bar{\tau} \), and certainty independence of \( I \) again implies that \( J(f) = I(u \circ f) \). Subsequent applications yield the result for all acts \( f \) such that \( u \circ f = u \circ g - \bar{\tau} \), \( \gamma \geq 0 \), and the proof is completed.

Claim 25 implies that \( J(f) = \min\{\int u \circ f dP \mid P \in C\} \) for all \( f \in X^S \) that satisfy \( u \circ f = u \circ g + \bar{\tau} \) for some \( g \in Y^S \) and \( \gamma \in \mathbb{R} \). That is, \( \bar{\zeta} \) is represented by an MEU functional in a 'stripe' containing all acts with utilities in \([\bar{\tau}, \tau]\), and those obtained from them by translations parallel to the main diagonal (in utiles space).

We next show that for any two consequences \( M \succ m \) there exists a consequence \( \xi \) that has a tradeoff-midpoint with every intermediate consequence (including the extreme ones).
Claim 27. Let $M, m \in X$ be consequences such that $M \succ m$. Then there exists a consequence $\xi \in X$, $M \succ \xi \succ m$, such that for all $x \in X$ with $M \succeq x \succeq m$, there is $y \in X$ that satisfies

\[ < x; y > \sim y; \xi > \]

Proof. We first prove the claim for the two extreme consequences, $M$ and $m$. To show that a consequence $\xi$ exists with tradeoff-midpoints with both $m$ and $M$, it suffices to show that there exist consequences $a, b$ such that

\[ Ms'a \sim ys'b, \quad ys'a \sim \xi s'b \]

with $M \succ y \succ \xi \succeq b \succ a \succeq m$, and consequences $c, d$ for which

\[ \xi t'c \sim zt'd, \quad zt'c \sim mt'd \]

with $M \succeq d \succ c \succeq \xi \succ z \succ m$.

Using the CEU representations $U$ and $W$, the above indifference relations translate to

\[ u(M)\rho(s') + u(a)[1 - \rho(s')] = \frac{u(M) + u(\xi)}{2}\rho(s') + u(b)[1 - \rho(s')] \]

for $M \succ \xi \succeq b \succ a \succeq m$, and

\[ u(c)\varphi(S \setminus \{t'\}) + u(\xi)[1 - \varphi(S \setminus \{t'\})] = u(d)\varphi(S \setminus \{t'\}) + \frac{u(\xi) + u(m)}{2}[1 - \varphi(S \setminus \{t'\})] \]

for $M \succeq d \succ c \succeq \xi \succ z \succ m$.

Since $u$ is continuous, then in the first equation $u(b) - u(a)$ may range from zero to $u(\xi) - u(m)$, and in the second equation $u(d) - u(c)$ may range from zero to $u(M) - u(\xi)$. Hence, for the desired $a, b, c, d$ to exist it is sufficient if the following two conditions are satisfied:

\[ \frac{\rho(s')}{2(1 - \rho(s'))}[u(M) - u(\xi)] \leq u(\xi) - u(m) \]

or, equivalently,

\[ \rho(s')u(M) + 2[1 - \rho(s')]u(m) \leq \rho(s')u(\xi) + 2[1 - \rho(s')]u(\xi) \]

\[ 2\varphi(S \setminus \{t'\})u(\xi) + [1 - \varphi(S \setminus \{t'\})]u(\xi) \leq 2\varphi(S \setminus \{t'\})u(M) + [1 - \varphi(S \setminus \{t'\})]u(m) \]
If \( M \succ \xi \succ m \), then \( u(M) > u(\xi) > u(m) \), and for (8) to hold it suffices that
\[
\rho(s')u(M) + [1 - \rho(s')]u(m) \leq \rho(s')u(\xi) + [1 - \rho(s')]u(\xi) \\
\varphi(S \setminus \{t'\})u(\xi) + [1 - \varphi(S \setminus \{t'\})]u(\xi) \leq \varphi(S \setminus \{t'\})u(M) + [1 - \varphi(S \setminus \{t'\})]u(m)
\]

Translating back from the representation to the implied \( \succeq \) relations, we need to show that there exists a consequence \( \xi, M \succ \xi \succ m \), such that \( mt'M \succ \xi \succeq Ms'm \). Monotonicity entails \( mt'M \succ Ms'm \). Since every act has a certainty equivalent, there are constant acts \( \bar{x} \) and \( x^0 \) that satisfy \( \bar{x} \sim mt'M \) and \( x^0 \sim Ms'm \), and we can choose, for instance, \( \xi = \bar{x} \).

Next it is proved that the consequence \( \xi \) found, that has tradeoff midpoints with the extreme consequences \( m \) and \( M \), has a tradeoff midpoint with every consequence \( x \) such that \( M \succ x \succ m \).

First suppose that \( M \succ x \succ \xi \). Then for a tradeoff midpoint \( y \), \( < x; y > \sim_s < y; \xi > \), to exist, it is sufficient that \( xs'a \sim ys'b \) and \( ys'a \sim \xi s'b \) for consequences \( a, b \) that satisfy \( \xi \succeq b \succeq a \). Then, applying the same arguments as above, for a tradeoff midpoint \( y \) to exist it is sufficient to show that
\[
\frac{\rho(s')}{2(1 - \rho(s'))}(u(x) - u(\xi)) \leq u(\xi) - u(m) \quad \text{for every consequence } x \text{ it holds that } \\
u(x) \leq u(M), \text{ and we already know that } \frac{\rho(s')}{2(1 - \rho(s'))}(u(M) - u(\xi)) \leq u(\xi) - u(m), \text{ hence a tradeoff midpoint of } x \text{ and } \xi \text{ exists.}
\]

Analogous arguments may be used to prove that a tradeoff midpoint of \( \xi \) and a consequence \( x, \xi \succ x \succ m \), exists (in this case the representation \( W \) over acts \( at'b \) is applied).

**Corollary 28.** For all \( f \in X^S \), \( J(f) = I(u \circ f) \).

Proof. As stated above, by claims 25 and 26, \( \succeq \) is represented by an MEU functional on a 'stripe' containing all acts with utilities in \([-\tau, \tau]\), and those obtained from them, in utiles space, by translations parallel to the main diagonal. It is left to show that \( J(f) = I(u \circ f) \) for all acts outside the stripe as well.

Let \( f, g \) be acts and \( x, y, \xi \) consequences such that: \( < f(s); g(s) > \sim_s < g(s); \xi > \) for all states \( s \), \( J(g) = I(u \circ g) \), \( g \sim \bar{y} \) and \( < x; y > \sim_s < y; \xi > \). The existence
of the required $\xi$ is guaranteed by applying Lemma 27 to consequences $M, m$ such that $M \succeq f(s) \succeq m$ for all $s \in S$ (x such that $u(y) = (u(x) + u(\xi))/2$ must satisfy $M \succeq x \succeq m$). Employing the axiom of Certainty Independence (A7) it may be asserted that $f \sim \pi$. Thus, by homogeneity and certainty independence of $I$,

$$u(y) = I(u \circ g) = \frac{1}{2} (I(u \circ f) + u(\xi)) = \frac{1}{2} (I(u \circ f) + 2u(y) - u(x))$$

and $I(u \circ f) = u(x) = J(f)$.

In that manner we may extend the MEU representation that applies in the stripe to any act which has a tradeoff-midpoint inside the stripe. By repeating this procedure, the MEU representation is seen to apply to the entire acts space. ●

We conclude that for all $f \in X^S$, $J(f) = \min\{\int u \circ f dP \mid P \in C\}$, that is, $\succeq$ on $X^S$ is represented by an MEU functional.

The fact that for all $s$ in $S$, $\max\{P(s)\mid P \in C\} > 0$, follows from Substantiality: Suppose that for some state $s$, $\max\{P(s)\mid P \in C\} = 0$. Then for all $x, y, z \in X$, $\int u \circ (x sy) dP = u(y) = \int u \circ (z sy) dP$ for every $P \in C$, yielding $x sy \sim z sy$, and thus nullity of state $s$. Contradiction. By Substantiality it also follows that $C \neq \Delta(S)$: Let $s'$ be a state for which $xs'y \succ y$ whenever $x \succ y$. The MEU representation on $\{xs'y \mid x \succeq y\}$ takes on the form

$$J(xs'y) = u(x) \min\{P(s')\mid P \in C\} + u(y)[1 - \min\{P(s')\mid P \in C\}]$$

and it follows that $\min\{P(s')\mid P \in C\} > 0$, therefore $C \neq \Delta(S)$.

The proof of the direction $(1) \Rightarrow (2)$ of the main theorem is completed.

### 5.3 Proof of the implication $(2) \Rightarrow (1)$ of Theorem 7

By definition of the representation on $X^S$, $\succeq$ is a weak order, satisfying Monotonicity. Define a functional $I : B \to \mathbb{R}$ by $I(b) = \min\{\int_S bdP \mid P \in C\}$. Hence the preference relation is represented by $J(f) = I(u \circ f)$. By its definition, $I$ is continuous and superlinear, therefore $\succeq$ is continuous and satisfies Uncertainty Aversion in Utilities (A6u). Uncertainty aversion (A6) results by observing that

$$<f(s), h(s)> \sim <h(s), g(s)> \text{ implies } u(h(s)) = \frac{1}{2}[u(f(s)) + u(g(s))].$$
To see that Substantiality holds, let \( x \succ y \). For any state \( t \), \( \max_{P \in C} P(t) > 0 \), so
\[
J(ytx) = u(y) \max_{P \in C} P(t) + u(x)[1 - \max_{P \in C} P(t)] < u(x),
\]
implying that \( \pi \succ ytx \) and \( t \) is nonnull. It follows also that \( C \neq \{\delta_i\} \) for any \( t \), thus \( \min_{P \in C} P(t) < 1 \) for all states \( t \). By the fact that \( C \neq \Delta(S) \) we conclude that there must be a state \( s \) for which \( \min_{P \in C} P(s) > 0 \), hence
\[
J(xsy) = u(x) \min_{P \in C} P(s) + u(y)[1 - \min_{P \in C} P(s)]
\]
is strictly between \( u(y) \) and \( u(x) \), implying \( \pi \succ xsy \succ \overline{y} \).

Next it is proved that Certainty Independence is satisfied. If \( \gamma \in \mathbb{R} \) and \( \overline{\gamma} \in B \) is the constant function returning \( \gamma \) in every state, then for all \( b \in B \),
\[
I(b + \gamma) = I(b) + I(\gamma) = I(b) + \gamma.
\] If \( \alpha > 0 \) then \( I(ab) = \alpha I(b) \). Let \( f, g \) be acts and \( \xi, x, y \) consequences that satisfy: for all states \( s \), \( < f(s), g(s) > \sim^s < g(s), \xi > \) and \( < x; y > \sim^s < y; \xi > \). Then \( u(g(s)) = \frac{1}{2}[u(f(s)) + u(\xi)] \) and \( u(y) = \frac{1}{2}[u(x) + u(\xi)] \).

Therefore,
\[
J(g) = I(u \circ g) = \frac{1}{2}I(u \circ f) + \frac{1}{2}u(\xi) = \frac{1}{2}J(f) + \frac{1}{2}u(\xi)
\]
and \( J(f) = u(x) \iff J(g) = u(y) \), that is, \( f \sim \pi \iff g \sim \overline{\gamma} \), as required.

To see that certainty covariance axiom \((A8)\) holds let \( f, g \) be acts and \( x, y \) consequences such that for all states \( s \), \( < f(s); g(s) > \sim^s < x; y > \). Then for all states \( s \),
\[
u(f(s)) - u(g(s)) = u(x) - u(y),
\]
which implies \( I(u \circ f) + u(y) = I(u \circ g) + u(x) \) and thus \( f \sim \pi \iff g \sim \overline{\gamma} \).

Finally, under MEU, sets of binary comonotonic acts are EU-sets, hence binary comonotonic tradeoff consistency is satisfied.

### 5.4 Proof of Theorem 8

By Theorem 11 in GM (2001), assuming A0 and A0*, the binary relation \( \succeq \) satisfies axioms A1,A2,A3*,A3**,A4 and A5* if and only if there exist a real valued, continuous, monotonic and nontrivial representation \( J : X^S \rightarrow \mathbb{R} \) of \( \succeq \), and a monotonic set function \( \eta \) on events, such that for binary acts with \( x \succ y \),
\[
J(xEy) = u(x)\eta(E) + u(y)(1 - \eta(E)).
\] (9)
The function \( u \) is defined on \( X \) by, \( u(x) = J(\pi) \), and it represents \( \succeq \) on \( X \). The set function \( \eta \), when normalized s.t. \( \eta(S) = 1 \), is unique, and \( u \) and \( J \) are unique up to a positive multiplicative constant and an additive constant.

It remains to show that addition of axioms A6*,A7* and A8* yields an MEU representation as in (2) of the theorem. This is done simply by showing that when
these axioms are translated into utiles space, they imply the exact same attributes as their counterpart axioms A6, A7 and A8. The key result is proposition 1 of Ghirardato, Maccheroni, Marinacci and Siniscalchi (2002), which states that for biseparable preference relations, and \( x \succ y \succ z \) \( x, y \in \mathcal{H}(x, z) \) if and only if \( u(y) = u(x)/2 + u(z)/2 \). Applying that result, we obtain equivalence between A6*, A7* and A8*, and their utilities-analogue axioms. First, axiom A6* is equivalent to Uncertainty Aversion in Utilities (A6u), as phrased above. Second, A7* is equivalent to the following utilities-based axiom:

**A7u. Certainty Independence in Utilities:**

Let the preference relation \( \succ \) be represented on constant acts by a utility function \( u \). Suppose that two acts, \( f \) and \( g \), and three consequences, \( x, y \) and \( w \), satisfy: 
\[
u(y) = \frac{1}{2}u(x) + \frac{1}{2}u(w), \quad \text{and for all states } s, \ u(g(s)) = \frac{1}{2}u(f(s)) + \frac{1}{2}u(w).
\]
Then \( f \sim x \iff g \sim y \).

Finally, A8* is equivalent to the utilities axiom:

**A8u. Certainty Covariance in Utilities:**

Let \( f, g \) be acts and \( x, y \) consequences such that \( \forall s \in S : \)
\[
\frac{1}{2}u(f(s)) + \frac{1}{2}u(y) = \frac{1}{2}u(g(s)) + \frac{1}{2}u(x).
\]
Then \( f \sim x \iff g \sim y \).

Having these utilities axioms (which immediately apply to the entire acts space, as opposed to the tradeoffs case), the rest of the proof (both directions) is identical to the proof of the tradeoffs theorem, Theorem 7.

### 5.5 Proof of Theorem 9

To obtain a CEU representation we apply Corollary 10 from Kobberling and Wakker (2003), stated below. As in the proof of Theorem 7, here as well KW use a slightly different tradeoff consistency condition than ours, however assuming the rest of our axioms, their condition follows (see Appendix B).

**Lemma 29. (Corollary 10 of KW 2003:)** Given a binary relation \( \succ \) on \( X^S \), the following two statements are equivalent:

1. \( \succ \) satisfies:

   (A1) Weak Order
(A2) *Continuity*

(A4) *Monotonicity*

(A5.1) *Comonotonic Tradeoff Consistency*

(2) There exist a continuous utility function \( u : X \rightarrow \mathbb{R} \) and a nonadditive probability \( v \) on \( \Sigma \) such that, for all \( f, g \in X^S \),

\[
 f \succeq g \iff \int_S u \circ f dv \geq \int_S u \circ g dv
\]

**Claim 30.** If, in addition to axioms A1, A2, A4 and A5.1, Substantiality is satisfied, then \( u \) is cardinal and \( v \) is unique and satisfies:

for all \( s \in S \), either \( v(s) > 0 \) or \( v(S \backslash \{s\}) < 1 \), and

\[
0 < v(s') < 1 \text{ for at least one } s' \in S.
\]

In the other direction, assuming that \( u \) is cardinal, and that \( v \) is unique and satisfies (10), Substantiality holds.

Proof. Suppose that Substantiality is satisfied. By Observation 9 of Kobberling and Wakker (2003), the utility function \( u \) is cardinal, and the nonadditive probability function \( v \) is unique. Non-nullity of the states imply that for all states \( s \) there are \( x, y, z \) such that \( xsz \succ ysz \ (x \succ y) \). If \( z \succeq x \) then the preference implies \( v(S \backslash \{s\}) < 1 \). If \( y \succeq z \), it follows that \( v(s) > 0 \). Otherwise, if \( x \succ z \succ y \), then \( xsz \succeq sz \succ zsy \) with at least one of the preferences strict, which again implies either \( v(S \backslash \{s\}) < 1 \) or \( v(s) > 0 \). By Substantiality, there exists a state \( s' \) for which \( \exists x \succ y \succ y \) whenever \( x \succ y \). Applying the CEU representation we obtain, for this state, \( 0 < v(s') < 1 \).

Now assume that \( u \) is cardinal and \( v \) unique, satisfying the attributes in (10). If for every state \( s \) either \( v(s) > 0 \) or \( v(S \backslash \{s\}) < 1 \), then for every state \( s \) and \( x \succ y \) either \( xsy \succ y \) or \( x \succ ysx \), yielding that all states are nonnull. If for some state \( s' \), \( 0 < v(s') < 1 \), then for this state \( \exists x \succ y \succ y \) whenever \( x \succ y \). We conclude that Substantiality is satisfied. ■

Let \( u \) be the utility function and \( v \) the non-additive probability obtained by Lemma 29. The proof of the theorem continues in subsections 5.5.1 and 5.5.2 below.
5.5.1 Proof of the implication (1)⇒(2) of Theorem 9.

It is left to prove that addition of Uncertainty Aversion yields the specific CEU form in (6) (along with the detailed attributes of $v$).

**Lemma 31.** Let $\succsim$ satisfy (1) of Lemma 29. Then Uncertainty Aversion (A6) implies Uncertainty Aversion in Utilities (A6u).

Proof. Let $f, g, h$ be such that $f \succsim g$, and for all states $t$,

$$u(h(t)) = \frac{1}{2}u(f(t)) + \frac{1}{2}u(g(t)).$$

It is proved that $h \succsim g$.

Let $x$ be an internal consequence, that is, $x^* \succ x \succ x_*$ for some $x^*, x_*$. Since $X$ is connected and $u$ continuous, there exist acts $f', g', h'$ such that, in all states $s$,

$$u(f'(t)) = \frac{1}{2^n}u(f(t)) + (1 - \frac{1}{2^n})u(x)$$
$$u(g'(t)) = \frac{1}{2^n}u(g(t)) + (1 - \frac{1}{2^n})u(x)$$
$$u(h'(t)) = \frac{1}{2^n}u(h(t)) + (1 - \frac{1}{2^n})u(x) = \frac{1}{2}u(f'(t)) + \frac{1}{2}u(g'(t))$$

By definition of the CEU functional, $f \succsim g$ $\iff f' \succsim g'$ and $h \succsim g$ $\iff h' \succsim g'$. Therefore it suffices to prove that $h' \succsim g'$.

In order to use Uncertainty Aversion we need to show that for every state $t$, $< f'(t); h'(t) > \sim^* < h'(t); g'(t) >$. That is, it should be proved that for every state $t$ there are consequences $y, z$ and a state $s$ such that $f'(t) sy \sim h'(t) sz$ and $h'(t) sy \sim g'(t) sz$. We next show that these indifference relations may be satisfied if $s$ is chosen such that $\bar{x} \succ xsy \succ \bar{y}$ whenever $x \succ y$ (exists by Substantiality), and $f', g', h'$ are made ’close enough’ to the internal consequence $x$.

Let $s$ be a state such that $\bar{x} \succ xsy \succ \bar{y}$ whenever $x \succ y$. By the CEU representation it follows that $0 < v(s) < 1$. Let $z_1 \succ z_2 \succ z_3$ be internal consequences that satisfy

$$z_1 \succ x \succ z_2 \quad \text{and} \quad u(z_1) - u(z_2) < \frac{2(1 - v(s))}{v(s)}(u(z_2) - u(z_3))$$
For any two consequences \( a \succ c \) between \( z_1 \) and \( z_2 \), \( u(a) - u(c) \leq u(z_1) - u(z_2) \). Thus, applying continuity of \( u \), there are consequences \( y, z \) such that

\[
z_2 \succ z \succ y \succ z_3 \quad \text{and} \quad u(a) - u(c) = \frac{2(1 - v(s))}{v(s)}(u(z) - u(y)).
\]

Rearranging the expression and letting \( b \) be a consequence with \( u(b) = \frac{u(a) + u(c)}{2} \), we obtain \( asy \sim bsz \) and \( bsy \sim csz \). Since all acts are comonotonic and \( s \) is comonotonically nonnull on the set containing them, we conclude that for every \( a, c \) such that \( z_1 \succ a \succ c \succ z_2 \) there is a consequence \( b \) that satisfies \( u(b) = \frac{u(a) + u(c)}{2} \) and \( < a; b > \sim * < b; c > \).

Taking a large enough \( n \), it may be obtained that \( z_1 \succ f'(t), h'(t), g'(t) \succ z_2 \). Having a finite state space, the same can be achieved, for a finite \( n \), for all states simultaneously. We conclude that \( < f'(t); h'(t) > \sim * < h'(t); g'(t) > \) for all states \( t \). By Uncertainty Aversion \( h' \succ g' \) and the proof is completed. \( \blacksquare \)

Define a functional \( I: \mathbb{R}^S \rightarrow \mathbb{R} \) by \( I(a) = \int_S adv \).

**Claim 32.** Let \( a, b \in \mathbb{R}^S \) be such that \( I(a) = I(b) \), then \( I(a + b) \geq I(a) + I(b) \).

Proof. By its definition, \( I \) is homogeneous, so it suffices to show that the claim is satisfied for all \( a, b \in (u(X))^S \). Let \( a, b \in (u(X))^S \) be such that \( I(a) = I(b) \). Let \( f, g \in X^S \) be such that \( u \circ f = a \), \( u \circ g = b \). Then \( J(f) = I(a) = I(b) = J(g) \), and \( f \sim g \). By A6u it follows that if \( h \) is an act such that in all states \( s \), \( u(h(s)) = \frac{1}{2} u(f(s)) + \frac{1}{2} u(g(s)) \), then \( h \succ g \). Thus, applying homogeneity once more,

\[
J(h) = I(u \circ h) = I\left( \frac{1}{2} u \circ f + \frac{1}{2} u \circ g \right) = \frac{1}{2} I(a + b) \geq J(g) = I(b) = \frac{1}{2} (I(a) + I(b)).
\]

\( \blacksquare \)

To obtain the specific CEU form as minimum expected utility over the core of \( v \), we state a result from Schmeidler (1989).

**Lemma 33.** (part of a proposition from Schmeidler 1989):

Suppose that \( \succ \) on \( X^S \) is represented by a CEU functional

\( J(f) = I(u \circ f) = \int_S u \circ f dv \). The following conditions are equivalent:

(i) For all \( a, b \in \mathbb{R}^S \), if \( I(a) = I(b) \) then \( I(a + b) \geq I(a) + I(b) \).

(ii) \( v \) is convex.
(ii) For all $a \in \mathbb{R}^S$, $I(a) = \min\{\int_S a dP \mid P \in \text{core}(v)\}$.

Thus, by Lemma 32, the representation in (2) obtains. If $\text{core}(v) = \Delta(S)$, then for all states $s$ and consequences $x \succ y$, $xsy \sim y$, contradicting Substantiality. Hence $\text{core}(v) \neq \Delta(S)$. By Lemma 29, non-nullity implies that for all states $s$ either $v(S \setminus \{s\}) < 1$ or $v(s) > 0$. Since $v$ is convex, $v(s) > 0$ itself implies $v(S \setminus \{s\}) < 1$, thus for all $s \in S$, $v(S \setminus \{s\}) < 1$, and the implication $(1) \Rightarrow (2)$ is proved.

5.5.2 Proof of the implication $(2) \Rightarrow (1)$ of Theorem 9.

By Lemma 29, axioms A1, A2, A4 and A5.1 are satisfied.

If for some state $s$, $v(S \setminus \{s\}) < 1$, then $\overline{x} \succ ysy$, therefore $v(S \setminus \{s\}) < 1$ for all states implies that all states are nonnull. Observe that $v(s) < 1$ for all states $s$: Suppose on the contrary that $v(s) = 1$ for some $s$. Then by monotonicity and convexity of the nonadditive probability $v$ (see Lemma 33), $v(A) = \mathbb{I}(s \in A)$ for all events $A$, yielding a non-cardinal $u$. Since $\text{core}(v) \neq \Delta(S)$, then there must be a state $s$ for which $v(s) > 0$. For this state, $v(s)$ being strictly between 0 and 1 implies $\overline{x} \succ xsy \succ \overline{y}$, and Substantiality is established.

It remains to show that when the integral w.r.t. $v$ takes on the special form of minimum expectation over a set of priors, Uncertainty Aversion (A6) is satisfied. But since $< f(s); h(s) > \sim^* < h(s); g(s) >$ implies $u(h(s)) = \frac{1}{2}[u(f(s)) + u(g(s))]$, and the representing functional consists of taking a minimum, the required inequality is easily seen to hold.

6 Appendix B. On the equivalence of definitions

Kobberling and Wakker (2003) use the following condition in eliciting a CEU representation:

**KW Comonotonic Tradeoff Consistency:**

Improving any outcome in the $\sim^*$ relation breaks that relation.

Our formulation of Comonotonic Tradeoff Consistency (A5.1), as well as Binary Comonotonic Tradeoff Consistency (BCTC, A5) applied to a state space with two
states, correspond to an axiom KW call *Comonotonic Strong Indifference-Tradeoff Consistency*:

**KW Comonotonic Strong Indifference-Tradeoff Consistency:**
For any four consequences \( a, b, c, d \), four acts \( f, g, f', g' \) and states \( s, t \in S \),

\[
as f \sim b sg, \ cs f \sim d sg, \ at f' \sim bt g' \Rightarrow ct f' \sim dt g'
\]

whenever the sets of acts \( \{ as f, b sg, cs f, d sg \} \) and \( \{ at f', bt g', ct f', dt g' \} \) are comonotonic, \( s \) is comonotonically nonnull on the first comonotonic set, and \( t \) is comonotonically nonnull on the second.

We show that assuming the rest of our axioms, Comonotonic Tradeoff Consistency (equivalently, BCTC for \( |S| = 2 \)) implies KW Comonotonic Tradeoff Consistency. Thus, we may use their results to derive a CEU representation, as is done in Appendix A. To show their axiom is implied, we require an additional condition of monotonicity.

**Comonotonic Strong Monotonicity:**
For any two comonotonic acts \( f \) and \( g \), if \( f(s) \succ g(s) \) for all states \( s \), and \( f(t) \succeq g(t) \) for a state \( t \) that is comonotonically nonnull on \( \{ f, g \} \), then \( f \succ g \).

**Lemma 34.** Assume that the binary relation \( \succ \) satisfies Weak Order, Continuity, Substantiality, Monotonicity and Comonotonic Tradeoff Consistency. Then \( \succ \) satisfies Comonotonic Strong Monotonicity.

Proof. Let \( f, g \) be comonotonic acts such that \( f(s) \succ g(s) \) for all states \( s \in S \), and \( f(t) \succ g(t) \) for some state \( t \in S \) which is comonotonically nonnull on \( \{ f, g \} \). Denote by \( s_1, s_2, \ldots, s_{|S|} \) the ordering of states such that \( f(s_1) \succ f(s_2) \succ \ldots f(s_{|S|}) \) (the same ordering holds for \( g \)). Define acts \( h_0, h_1, \ldots, h_{|S|} \) as follows: \( h_0 = g, \ h_2 = f\{s_1\}g, \ h_3 = f\{s_1, s_2\}g, \ldots, \ h_{|S|-1} = f\{s_1, \ldots, s_{|S|-1}\}g, \ h_{|S|} = f \). All these acts are comonotonic, and by Monotonicity \( h_{|S|} \succ \ldots \succ h_1 \succ h_0 \), and \( h_i, h_{i-1} \) differ by at most one consequence. Thus, it suffices to prove that for any two comonotonic acts \( ath, bth \), with \( a \succ b \) and \( t \) comonotonically nonnull on \( \{ ath, bth \} \), \( ath \succ bth \).

Denote by \( s' \) a state for which \( \alpha \succ \alpha s' \beta \succ \beta \) whenever \( \alpha \succ \beta \) (such a state exists by Substantiality). We assume that \( ath \sim bth \) and derive a contradiction. Obviously \( ath \sim ath \) and \( as'b \sim as'b \), therefore by Comonotonic Tradeoff Consistency (all comonotonicity and non-nullity conditions are satisfied) it must be that \( as'b \sim b \), contradicting the choice of \( s' \). We conclude that \( ath \succ bth \), hence Comonotonic...
Strong Monotonicity holds. ■

Having proved that \( \succeq \) satisfies Comonotonic Strong Monotonicity, Lemma 24 of KW (2003) implies that it also satisfies KW Comonotonic Tradeoff Consistency.

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