

On Weighted Shapley Values

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Abstract: Nonsymmetric Shapley values for coalitional form games with transferable utility are studied. The nonsymmetries are modeled through nonsymmetric weight systems defined on the players of the games. It is shown axiomatically that two families of solutions of this type are possible. These families are strongly related to each other through the duality relationship on games. While the first family lends itself to applications of nonsymmetric revenue sharing problems the second family is suitable for applications of cost allocation problems. The intersection of these two families consists essentially of the symmetric Shapley value. These families are also characterized by a probabilistic arrival time to the game approach. It is also demonstrated that lack of symmetries may arise naturally when players in a game represent nonequal size constituencies.

1 Introduction

The Shapley (1953b) value is considered by many game theorists and economists as the main solution concept to cooperative games with transferable utility. These games and this solution concept have been applied to problems of revenue sharing and cost allocations.

One of the main axioms that characterize the Shapley value is one of symmetry. The underlying motivation for using this axiom is the assumption that except for the parameters of the games, the players are completely symmetric. However, in many applications this assumption of symmetry seems unrealistic for the situation that is being modeled and the use of nonsymmetric generalizations of the Shapley value was proposed in such cases.

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Consider, for example, a situation involving two players. If the two players cooperate in a joint project they can generate a unit profit which is to be divided between them. On their own they can generate no profit. The Shapley value views this situation as being symmetric and would allocate the profit from cooperative equally between the two players. However, in some applications lack of symmetry may be present in the underlying situation. It may be, for example, that a greater effort is needed on the part of player one than on the part of player two in order for the project to succeed. Another example arises in situations where player one represents a large constituency with many individuals and player two's constituency is composed of a small number of individuals. Other examples where lack of symmetry is present can easily be constructed for problems of cost allocations. Also, lack of symmetry may arise when different bargaining abilities for different players are modelled.

The family of weighted Shapley values was introduced by Shapley (1953a). Each weighted Shapley value associates a positive weight with each player. These weights are the proportions in which the players share in unanimity games. The symmetric Shapley value is the special case where all the weights are the same. In this paper we extend the notion of "weights" to "weight systems" enabling a weight of zero for some players. We then define in Section 2 the notion of the weighted Shapley value with a given weight system and relate it to a procedure of dividend allocation that was proposed by Harsanyi (1959) (see also Owen 1982) for games without sidepayments. In Section 3 we give an equivalent definition of the weighted Shapley value by random orders which generalize the random order approach to the symmetric Shapley value. In Section 4 we give an axiomatic characterization of the family of weighted Shapley values — that is, we provide a list of properties of a solution which is satisfied by and only by weighted Shapley values.

Shapley (1981) proposed also a family of weighted cost allocations schemes and axiomatically characterized, for exogenously given weights, the schemes associated with these weights. This family of solutions is related to the weighted Shapley values by duality. We explore further the relationship between these two families, provide an axiomatization of the latter family (which does not use the weights explicitly in the axioms as Shapley's axioms do) and get as a result an axiomatization of the symmetric Shapley value which does not use the symmetry axiom.

Owen (1968 and 1972) showed that weighted Shapley values can be computed by a "diagonal formula" providing another interpretation of the weights associated with the players. In Section 6 we extend the "diagonal formula" for weight systems and allocation schemes.

Finally, we note that if one accepts the axioms in Section 4, one is obliged to use a weighted Shapley value but no recommendation of the weights is implied by the axioms. The weights should be determined by considering such factors as bargaining ability, patience rates, or past experience. In Section 7 we examine cases in which the "size" of the players (where the players themselves are groups of individuals) are appropriate weights for the players.

2 Weighted Shapley Values

Let N be a finite set, the members of which will be called *players*. Subsets of N are called *coalitions* and N is called the *grand coalition*. Set $|N| = n$. For each coalition S we denote by E^S the $|S|$ -dimensional Euclidian space indexed by the players of S . A *game* v is a function which assigns to each coalition a real number and in particular $v(\emptyset) = 0$. The set of all games is denoted by Γ . Addition of two games v and w in Γ is defined by $(v + w)(S) = v(S) + w(S)$ for each S and multiplication of the game v by a scalar α is defined by $(\alpha v)(S) = \alpha v(S)$ for each coalition S . Thus Γ is a vector space. For each coalition S the *unanimity game of the coalition* S , u_S , is defined by $u_S(T) = 1$ if $T \supseteq S$ and $u_S(T) = 0$ otherwise. It is well known that the family of games $\{u_S\}_{S \subseteq N}$ is a basis for Γ .

The Shapley value ϕ is the linear function $\phi: \Gamma \rightarrow E^N$, which for each unanimity game u_S is defined by $\phi_i(u_S) = \frac{1}{|S|}$ if $i \in S$ and $\phi_i(u_S) = 0$ otherwise. Intuitively, in the game u_S any coalition which contains S can split one unit between its members and therefore players outside S do not contribute anything to the coalition they join. Hence, $\phi_i(u_S) = 0$ for $i \notin S$. The members of S on the other hand split equally the one unit between themselves. Since $\{u_S\}_{S \subseteq N}$ is a basis to Γ and ϕ is linear, ϕ is defined for all the games. A weighted Shapley value generalizes the Shapley value by allowing different ways to split one unit between the members of S in u_S . We prescribe a vector of positive weights $\lambda = (\lambda_i)_{i \in N}$ and in each u_S players split proportionally to their weights. We want to allow some players to have weight zero. This means that if they split one unit with players who have positive weights, they get zero. But then we have to specify how these zero-weight players split a unit when no positive-weight player is with them. This brings us to the following lexicographic definition of a weight system.

A *weight system* ω is a pair (λ, Σ) where $\lambda \in E_{++}^N$ and $\Sigma = (S_1, \dots, S_m)$ is an ordered partition of N . A weight system $\omega = (\lambda, \Sigma)$ is called *simple* if $\Sigma = (N)$. The *weighted Shapley value with weight system* ω is the linear map $\phi_\omega: \Gamma \rightarrow E^N$ which is defined for each unanimity game u_S as follows.

Let $k = \max \{j | S_j \cap S \neq \emptyset\}$ and denote $\bar{S} = S \cup S_k$. Then

$$(\phi_\omega)_i(u_S) = \frac{\lambda_i}{\sum_{j \in \bar{S}} \lambda_j}$$

for $i \in \bar{S}$ and $(\phi_\omega)_i(u_S) = 0$ otherwise.

In other words, the weights of players in S_i are 0 with respect to players in S_j with $j > i$. The positive weights of players in S_i are used only for games u_S such that no player from S_j with $j > i$ is in S . Observe that ϕ_ω is the (symmetric) Shapley value

if and only if $\omega = (\lambda, (N))$ and λ is proportional to the vector $(1, 1, \dots, 1)$. Another computation procedure of $\phi_\omega(w)$ is along the lines proposed by Harsanyi (1959). In this procedure each coalition S allocates dividends to its members after all the proper subcoalitions of S have done it. The dividend allocation proceeds as follows. We first allocate to each player i his worth $v(\{i\})$. Suppose that all the coalitions of size k or less have already allocated dividends and let S be a coalition of size $k + 1$. Denote by $z(S)$ the sum of the dividends that members of S were paid by proper subcoalitions of S . Then $v(S) - z(S)$ (which is possibly 0) is the amount that S will allocate to its members. To determine how the amount is divided, we define the coalition \bar{S} (which is a subset of S) as above. The members of \bar{S} will divide $v(S) - z(S)$ in proportion to their weights while the rest of the players in S get nothing. The total amount that each player accumulated at the end of the procedure (i.e., after N allocated its dividends) is exactly $(\phi_\omega)_i(v)$. To see this one can easily prove by induction that if $v = \sum_{S \subseteq N} \alpha_S u_S$ then for each coalition S , $v(S) - z(S) = \alpha_S$ and the dividend allocation is therefore the allocation of the coefficients α_S in accordance with the definition of ϕ_ω .

A generalization of this procedure for the computation of the Shapley value was proposed by Maschler (1982). The same generalization applies also for ϕ_ω . We start by choosing any coalition S with $v(S) \neq 0$ and allocating $v(S)$ according to ω . In later steps of the computation we choose for dividend allocation any S for which $v(S) - z(S) \neq 0$ where $z(S)$ is the sum of the dividends paid for the players in S by subcoalitions of S which already allocated dividends (notice that a coalition may be chosen several times in this procedure). The procedure ends when $v(S) - z(S) = 0$ for all the coalitions. The proof that such procedure always terminates and gives indeed ϕ_ω is the same as in Maschler (1982). Harsanyi (1959) defined also a procedure of weighted dividend allocation for games without sidepayments. A family of solutions obtained by these procedures was axiomatized by Kalai and Samet (1985). They refer to these solutions as egalitarian, and it is shown there that the restriction of each egalitarian solution to games with sidepayments is a weighted Shapley value.

In the next section we provide a probabilistic approach to the weighted Shapley values, one which generalizes the probabilistic formula of the (symmetric) Shapley value.

3 Probabilistic Definition of Weighted Shapley Values

Let $\mathbb{R}(S)$ denote the set of all orders R of players in the coalition S . For an order R in $\mathbb{R}(N)$ we denote by $B^{R,i}$ the set of players preceding i in the order R . For an ordered partition $\Sigma = (S_1, \dots, S_m)$ of N , \mathbb{R}_Σ is the set of orders for N in which all

the players of S_i precede those of S_{i+1} for $i = 1, \dots, m - 1$. Each R in \mathbb{R}_Σ can be described as $R = (R_1, \dots, R_m)$ where $R_i \in \mathbb{R}(S_i)$, $i = 1, \dots, m$.

Let $|S| = s$ and let $\lambda \in E_{++}^S$. We associate with λ a probability distribution P_λ over $\mathbb{R}(S)$. For $R = (i_1, \dots, i_s)$ in $\mathbb{R}(S)$, we define

$$P_\lambda(R) = \prod_{j=1}^s \frac{\lambda_{i_j}}{\sum_{k=1}^j \lambda_{i_k}}$$

One way to obtain this probability distribution is by arranging the players of S in an order, starting from the *end*, such that the probability of adding a player to the beginning of a partially created line is the ratio between his weight and the total weight of the players of S that are not yet in the line.

With each weight system $\omega = (\lambda, \Sigma)$ where $\Sigma = (S_1, \dots, S_m)$ we associate a probability distribution P_ω over $\mathbb{R}(N)$ as follows. The distribution P_ω vanishes outside \mathbb{R}_Σ , and for $R = (R_1, \dots, R_m)$ in \mathbb{R}_Σ , $P_\omega(R) = \prod_{i=1}^m P_{\lambda_{S_i}}(R_i)$, where λ_{S_i} is the projection of λ on E^{S_i} .

For a given game v and order R in $\mathbb{R}(N)$ the contribution of player i is $C_i(v, R) = v(B^{R,i} \cup \{i\}) - v(B^{R,i})$. We prove now:

Theorem 1: For each player $i \in N$, weight system ω , and game v ,

$$(\phi_\omega)_i(v) = E_{P_\omega}(C_i(v, \cdot))$$

where the right hand side is the expected contribution of player i with respect to the probability distribution P_ω .

Proof: We say that i is last for S in the order R if $i \in S$ and $S \subseteq B^{R,i} \cup \{i\}$. For a given order R and player i the coalition $N \setminus (B^{R,i} \cup \{i\})$ is called the *tail of i in R* . A coalition T is said to be a *tail for R* if for some i , T is a tail of i in R .

Let $\omega = (\lambda, (S_1, \dots, S_m))$ be a weight system and let S be a coalition. Denote $k = \max \{j | S \cap S_j \neq \emptyset\}$ and $\bar{S} = S \cap S_k$. We show that for each $i \in S \setminus \bar{S}$, $P_\omega(i \text{ is last for } S) = 0$, for each $i \in \bar{S}$, $P_\omega(i \text{ is last for } S) > 0$, and for each $j, i \in \bar{S}$:

$$\frac{P_\omega(i \text{ is last for } S)}{P_\omega(j \text{ is last for } S)} = \frac{\lambda_i}{\lambda_j} \tag{*}$$

Indeed, if $i \in S \setminus \bar{S}$ then in order to be last for S , i must be preceded by players from S_k which occurs with probability 0. Now suppose $i, j \in \bar{S}$. Let $A = (\bigcup_{t \geq k} S_t) \setminus S$ then we have

$$\begin{aligned} P_\omega(i \text{ is last for } S) &= \sum_{T \subseteq A} P_\omega(T \text{ is a tail of } i) \\ &= \sum_{T \subseteq A} P_\omega(T \text{ is a tail of } i | T \text{ is a tail}) P_\omega(T \text{ is a tail}) \\ &= \sum_{T \subseteq A} \lambda_i (1 / \sum_{r \in (N \setminus T) \cap S_k} \lambda_r) P_\omega(T \text{ is a tail}) \\ &= \lambda_i h \text{ where } h \text{ is positive.} \end{aligned}$$

Similarly, $P_\omega(j \text{ is last for } S) = \lambda_j h$ and (*) follows.

Now consider the game u_S . The contribution of $i \notin S$ is 0 in each order and thus $E_{P_\omega}(C_i(u_S, \cdot)) = 0 = (\phi_\omega)_i(u_S)$. The contribution of $i \in S$ in the order R is 1 if i is last for S in R , and is 0 otherwise. If $i \in S \setminus \bar{S}$ then

$$E_{P_\omega}(C_i(u_S, \cdot)) = P_\omega(i \text{ is last for } S) = 0 = (\phi_\omega)_i(u_S).$$

If $i, j \in \bar{S}$, then

$$\frac{E_{P_\omega}(C_i(u_S, \cdot))}{E_{P_\omega}(C_j(u_S, \cdot))} = \frac{P_\omega(i \text{ is last for } S)}{P_\omega(j \text{ is last for } S)} = \frac{\lambda_i}{\lambda_j}.$$

But

$$\sum_{i \in N} E_{P_\omega}(C_i(u_S, \cdot)) = E_{P_\omega}(\sum_{i \in N} C_i(u_S, \cdot)) = E_{P_\omega}(1) = 1.$$

On the other hand, as we have shown:

$$\sum_{i \in N} E_{P_\omega}(C_i(u_S, \cdot)) = \sum_{i \in \bar{S}} E_{P_\omega}(C_i(u_S, \cdot))$$

and therefore for each $i \in \bar{S}$

$$E_{P_\omega}(C_i(u_S, \cdot)) = \frac{\lambda_i}{\sum_{i \in \bar{S}} \lambda_j} = (\phi_\omega)_i(u_S).$$

Clearly $E_{P\omega}(C_i(v, \cdot))$ is a linear map from Γ to E and so is $(\phi_\omega)_i(v)$ and therefore, since they coincide on the basis consisting of the unanimity games, they coincide on Γ . Q.E.D.

4 An Axiomatic Characterization of the Weighted Shapley Values

A solution for Γ is a function ϕ from Γ to E^N . For a coalition S we denote by $\phi(v)(S)$ the sum $\sum_{i \in S} \phi_i(v)$. A coalition S is said to be a *coalition of partners* or a *p-type coalition*, in the game v , if for each $T \subsetneq S$ and each $R \subseteq N \setminus S$, $v(R \cup T) = v(R)$.

Consider now the following axioms imposed on ϕ . For all games $v, w \in \Gamma$:

1. *Efficiency.* $\phi(v)(N) = v(N)$
2. *Additivity.* $\phi(v + w) = \phi(v) + \phi(w)$
3. *Positivity.* If v is monotonic (i.e., $v(T) \geq v(S)$ for each T and S such that $T \supseteq S$) then $\phi(v) \geq 0$.
4. *Dummy Player.* If i is a dummy player in the game v (i.e., for each S , $v(S \cup \{i\}) = v(S)$) then $\phi_i(v) = 0$.
5. *Partnership.* If S is a *p-type coalition* in v then $\phi_i(v) = \phi_i(\phi(v)(S)u_S)$, for each $i \in S$.

Axioms 1–4 are standard in various axiomatizations of the Shapley value. In order to examine axiom 5 consider first the character of a *p-type coalition*. A *p-type coalition* S in the game v behaves in a certain sense like one individual in the game v since all its subcoalitions are completely powerless. In this sense S behaves internally the same in V as in u_S . One can expect therefore that S will take its share in the game v as one individual and then bargain over this share. This is the content of axiom 5. $\phi(v)(S)u_S$ is a unanimity game in which the members of S bargain over $\phi(v)(S)$ which is what they received together in $\phi(v)$. $\phi_i(\phi(v)(S)u_S)$ is what i receives as a result of this bargaining. This should be exactly what he received in v .

Theorem 2: A solution ϕ satisfies axioms 1–5 if and only if there exists a weight system ω such that ϕ is the weighted Shapley value ϕ_ω .

Proof: We first show that for $\omega = (\lambda, (S_1, \dots, S_m))$, ϕ_ω satisfies axioms 1–5. To prove efficiency we observe that for each v and R , $\sum_{i \in N} C_i(v, R) = v(N)$ and therefore

$$\phi_\omega(v)(N) = \sum_{i \in N} (\phi_\omega)_i(v) = \sum_{i \in N} E_{P_\omega}(C_i(v, \cdot)) = E_{P_\omega}(\sum_{i \in N} C_i(v, \cdot)) = v(N).$$

The additivity of ϕ_ω follows from the additivity of E_{P_ω} and C_i . The positivity and the dummy player axioms follow also immediately. To check the partnership axiom assume that S is a p -type coalition in a game v . Observe first that since S is of p -type a player i in S makes a nonzero contribution in an order R only if i is last for S in R . Now let $k = \max \{j | S_j \cap S \neq \emptyset\}$ and let $\bar{S} = S \cap S_k$. For $i \in S \setminus \bar{S}$ the orders in which i is last for S have probability zero and therefore $(\phi_\omega)_i(v) = 0$. For $i \in \bar{S}$ we have:

$$\begin{aligned} (\phi_\omega)_i(v) &= E_{P_\omega}(C_i(v, \cdot)) \\ &= \sum_{T \subseteq N \setminus S} E_{P_\omega}(C_i(v, \cdot) | T \text{ is a tail of } i) P_\omega(T \text{ is a tail of } i) \end{aligned}$$

But

$$E_{P_\omega}(C_i(v, \cdot) | T \text{ is a tail of } i)$$

is the same for every $i \in \bar{S}$ since S is of p -type. Moreover $P_\omega(T \text{ is a tail of } i)$ is of the form $\lambda_i h(T)$ where $h(T)$ is the same for each $i \in S$.

Thus, there exists a constant K such that for every $i \in S$, $(\phi_\omega)_i(v) = \lambda_i K$ which shows that ϕ_ω satisfies the partnership property.

Now let ϕ be a solution which satisfies axioms 1–5 and we will show that for some weight system ω , $\phi = \phi_\omega$. We define first a weight system $\bar{\omega} = (\lambda, (\bar{S}_1, \dots, \bar{S}_m))$ as follows. The coalition \bar{S}_1 contains all players i for which $\phi_i(u_N) \neq 0$, ($\bar{S}_1 \neq \emptyset$ because of the efficiency axiom).³ We define $\lambda_i = \phi_i(u_N)$ for each $i \in \bar{S}_1$. Assuming that the coalitions $\bar{S}_1, \dots, \bar{S}_k$ are already defined then denote $T = N \setminus (\bar{S}_1 \cup \dots \cup \bar{S}_k)$ and let \bar{S}_{k+1} contain all the players i for which $\phi_i(u_T) \neq 0$ and define $\lambda_i = \phi_i(u_T)$ for all $i \in \bar{S}_{k+1}$. (\bar{S}_{k+1} is not empty because of the efficiency and dummy player axioms.) By the positivity axiom, $\lambda > 0$. Now for $i = 1, \dots, m$ we define $S_i = \bar{S}_{m-i+1}$ and $\omega = (\lambda, (S_1, S_2, \dots, S_m))$.

³ This is the only place where we use the efficiency axiom. Therefore we could use a much weaker axiom, namely that for each S , $\phi(u_S) \neq 0$. It is easy to see that such an axiom plus axiom 5 imply efficiency.

Next we show that ϕ is homogeneous, i.e. $\phi(tv) = t\phi(v)$ for each game v and scalar t . Since every game is the difference of two monotonic games it is enough, by the additivity axiom, to consider only monotonic games. Again by additivity, homogeneity follows for rational scalars. Let v be a monotonic game. Choose sequences of rationals $\{r_k\}$ and $\{s_k\}$ which converge to t from above and below, correspondingly. By the additivity and positivity axioms, $\phi(r_k v) - \phi(tv) = \phi((r_k - t)v) \geq 0$ and similarly $\phi(tv) - \phi(s_k v) \geq 0$. But $\phi(r_k v) - \phi(s_k v) = (r_k - s_k)\phi(v) \rightarrow 0$ as $k \rightarrow \infty$ and therefore $\phi(r_k v) \rightarrow \phi(tv)$ and $\phi(s_k v) \rightarrow \phi(tv)$ which proves the homogeneity of ϕ . Since both ϕ and ϕ_ω are linear maps on Γ , it suffices to show as we do next that ϕ and ϕ_ω coincide on each unanimity game.

For a unanimity game u_S define $k = \max \{j | S \cap S_j \neq \emptyset\}$ and let $\bar{S} = S \cap S_k$. Let $T = \bigcup_{j=1}^k S_j$. The coalition S is of p -type in u_T (as each subset of T is) and by the partnership axiom for each $i \in S$

$$\phi_i(u_T) = \phi_i(\phi(u_T)(S)u_S) = \phi(u_T)(S)\phi_i(u_S).$$

By the definition of T the only members of T who have nonzero payoffs in u_T are those of S_k , thus $\phi(u_T)(S) = \sum_{j \in \bar{S}} \lambda_j > 0$ and therefore

$$\phi_i(u_S) = \frac{\phi_i(u_T)}{\sum_{j \in \bar{S}} \lambda_j}.$$

It follows that for $i \in \bar{S}$,

$$\phi_i(u_S) = \frac{\lambda_i}{\sum_{i \in \bar{S}} \lambda_j}$$

and for $i \notin \bar{S}$, $\phi_i(u_S) = 0$, i.e., $\phi(u_S) = (\phi_\omega)(u_S)$. Q.E.D.

The family of all weighted Shapley values ϕ_ω for simple weight systems ω , can also be characterized by slightly changing the positivity axiom. We replace now axiom 3 by the following one.

(3') *Positivity.* If v is monotonic and there are no dummy players in v then $\phi(v) > 0$.

Theorem 3: A solution ϕ satisfies axioms 1, 2, 3', 4, and 5 if and only if there exists a simple weight system $\omega = (\lambda, (N))$ such that $\phi = \phi_\omega$.

Proof: If ω is a simple weight system then for each order R in $\mathbb{R}(N)$, $P_\omega(R) > 0$. If v satisfies the condition of axiom 3' then for each player i , $C_i(v, \cdot) \geq 0$ and for some R , $C_i(v, R) > 0$ which shows that $\phi_\omega(v) > 0$.

The proof of the other direction is along the same line of the proof of Theorem 2. The only difference is that because of axiom 3', $\phi(u_N) > 0$ and therefore the partition built in the proof of Theorem 2 contains only N .

In the next theorem we show that weighted Shapley values can be approximated by simple weighted Shapley values.

Theorem 4: For each weight system $\omega = (\lambda, (S_1, \dots, S_m))$ there exists a sequence of simple weight systems $\omega^t = (\lambda^t, (N))$ such that for each game v , $\phi_{\omega^t}(v) \rightarrow \phi_\omega(v)$ when $t \rightarrow \infty$.

Proof: Let $0 < \epsilon < 1$ and define for each t , $1 \leq l \leq m$ and $i \in S_l$, $\lambda_i^t = \epsilon^{t(m-l+1)}\lambda_i$, and define $\omega^t = (\lambda^t, (N))$. It is easy to see that for each S , $\phi_{\omega^t}(u_S) \rightarrow \phi(u_S)$ and since ϕ_ω and ϕ_{ω^t} are linear, $\phi_{\omega^t}(v) \rightarrow \phi_\omega(v)$ for each v . Q.E.D.

5 Duality

The dual game of a game v is denoted by v^* and is defined by

$$v^*(S) = v(N) - v(N \setminus S) \quad \text{for each } S \subseteq N.$$

The transformation $v \rightarrow v^*$ is a one-to-one linear map from Γ onto itself. In particular the set $\{u_S^*\}_{S \subseteq N}$ is a basis for Γ . Observe that $u_S^*(T) = 0$ for each T with $T \cap S = \emptyset$ and $u_S^*(T) = 1$ if $T \cap S \neq \emptyset$. We call the game u_S^* the *representation game for the coalition* S . The game u_S^* has a natural interpretation as a cost-game where $u_S^*(T)$ is the cost incurred by T . The presence of any number of members of S in T incurs a unit cost (compare Shapley 1981). For a weight system $\omega = (\lambda, (S_1, \dots, S_m))$ we define a linear map $\phi_\omega^* : \Gamma \rightarrow R^N$ by defining ϕ_ω^* on the basis $\{u_S^*\}_{S \subseteq N}$ as follows. For a given S denote $k = \max \{j | S_j \cap S \neq \emptyset\}$ and let $\bar{S} = S \cap S_k$. Then for $i \in \bar{S}$,

$$(\phi_\omega^*)_i(u_S^*) = \frac{\lambda_i}{\sum_{j \in \bar{S}} \lambda_j}$$

and $(\phi_\omega^*)_i(u_S^*) = 0$ if $i \notin \bar{S}$.

An equivalent random order approach is defined for ϕ_ω^* . For an order R we denote by R^* the reverse order. For a given probability distribution P over $\mathbb{R}(N)$ we define P^* by $P^*(R) = P(R^*)$. We now have the following equivalence.

*Theorem 1**: For each player i , weight system ω and game v ,

$$(\phi_\omega^*)_i(v) = E_{P_\omega^*}(C_i(v, \cdot)).$$

The proof is analogous to the proof of Theorem 1, where the notion “ i is last for S in R ” is replaced by “ i is first for S in R ” which means $S \cap B^{R,i} = \emptyset$. The solutions ϕ_ω and ϕ_ω^* can be related in a simple way.

Theorem 5: For each game v and weight system ω ,

$$\phi_\omega^*(v) = \phi_\omega(v^*).$$

Proof: Consider the game $v = u_S^*$. Then $v^* = (u_S^*)^* = u_S$, and by the definition of ϕ_ω and ϕ_ω^* , $\phi_\omega^*(v) = \phi_\omega(v^*)$. Now let $v = \sum_{S \subseteq N} \alpha_S u_S^*$. Then

$$\phi_\omega^*(v) = \sum_{S \subseteq N} \alpha_S \phi_\omega^*(u_S^*) = \sum_{S \subseteq N} \alpha_S \phi_\omega(u_S) = \phi_\omega(\sum_{S \subseteq N} \alpha_S u_S) = \phi_\omega(v^*). \quad \text{Q.E.D.}$$

An axiomatic characterization of the family $\{\phi_\omega^*\}$ is obtained by changing axiom 5. We say that a coalition S is of p^* -type in the game v if for each $R \supseteq S$ and $T \subsetneq S$, $v(R \setminus T) = v(R)$. Here again, as in the case of p -type coalitions, a p^* -type coalition can be considered as one individual represented by several agents. But in the p^* -type case any nonempty subcoalition of agents has the same effect on the cost as the coalition of all agents, while in the p -type case all the proper subcoalitions of agents are powerless. We call a coalition of a p^* -type a *coalition of representatives*. Common to both p -type and p^* -type coalitions is the fact that the inner coalitional structure of such coalitions is trivial. Axiom 5* is analogous to Axiom 5; it requires that if S is a p^* -type coalition in the game v , then the cost shared by each one of its members can be computed by letting the players in S bargain over the splitting of the total cost shared by S in $\phi_\omega^*(v)$. Clearly by the nature of the p^* -type coalition this bargaining is represented by the game u_S^* .

*Axiom 5**: If S is of p^* -type in v , then $\phi_i(v) = \phi_i(\phi(v)(S)u_S^*)$, for each $i \in S$.

*Theorem 2**: A solution ϕ satisfies axioms 1, 2, 3, 4, and 5* if and only if there exists a weight system ω such that $\phi = \phi_\omega^*$.

The proof is analogous to that of Theorem 2.

One might expect that ϕ_ω^* can be obtained from ϕ_ω by an appropriate transformation of the weight system. To see that this is not the case we examine first simple weight systems.

Theorem 6: Let $|N| \geq 3$. If $\omega = (\mu, (N))$ and $\omega' = (\lambda, (N))$ are two simple weight systems and $\phi_\omega^*(v) = \phi_{\omega'}(v)$ for each game v then both λ and μ are multiples of the vector $(1, 1, \dots, 1)$ and thus both ϕ_ω^* and $\phi_{\omega'}$ are the Shapley value.

Proof: Assume $\phi_{\omega'} = \phi_\omega^*$. Then for any coalition $\{i, j\} \subseteq N$, $(\phi_{\omega'})_i(u_{\{i,j\}}) = (\phi_\omega^*)_i(u_{\{i,j\}})$. But

$$(\phi_{\omega'})_i(u_{\{i,j\}}) = \frac{\lambda_i}{\lambda_i + \lambda_j}$$

and

$$(\phi_\omega^*)_i(u_{\{i,j\}}) = (\phi_\omega)_i(u_{\{i,j\}}^*) = (\phi_\omega)_i(u_{\{i\}} + u_{\{j\}} - u_{\{i,j\}}) = \frac{\mu_j}{\mu_i + \mu_j}.$$

Therefore, for each i and j in N

$$\frac{\lambda_i}{\lambda_i + \lambda_j} = \frac{\mu_j}{\mu_i + \mu_j}$$

from which we conclude that $\lambda_i \mu_i = \lambda_j \mu_j$ for each $i, j \in N$. It follows that there exists a positive number C for which $\lambda_i = \frac{C}{\mu_i}$ for each $i \in N$. Consider now a coalition $\{i, j, k\}$. We find that

$$\begin{aligned} (\phi_{\omega'})_i(u_{\{i,j,k\}}) &= \frac{\lambda_i}{\lambda_i + \lambda_j + \lambda_k} = \frac{C/\mu_i}{C/\mu_i + C/\mu_j + C/\mu_k} \\ &= \frac{\mu_j \mu_k}{\mu_j \mu_k + \mu_i \mu_j + \mu_i \mu_k}. \end{aligned} \tag{1}$$

Using the probabilistic definition of ϕ_{ω}^* we can compute

$$\begin{aligned}
 (\phi_{\omega}^*)_i(u_{\{i,j,k\}}) &= \frac{\mu_k}{\mu_i + \mu_j + \mu_k} \cdot \frac{\mu_j}{\mu_i + \mu_j} \\
 &+ \frac{\mu_j}{\mu_i + \mu_j + \mu_k} \cdot \frac{\mu_k}{\mu_i + \mu_k}.
 \end{aligned}
 \tag{2}$$

Equating the two expressions (1) and (2), dividing by $\mu_j\mu_k$, and multiplying by $\mu_i + \mu_j + \mu_k$ we find that:

$$\frac{1}{\mu_i + \mu_j} + \frac{1}{\mu_i + \mu_k} = \frac{\mu_i + \mu_j + \mu_k}{\mu_j\mu_k + \mu_i\mu_j + \mu_i\mu_k}.
 \tag{3}$$

We can obtain an equation similar to (3) for $(\phi_{\omega})_j$ and $(\phi_{\omega}^*)_j$ applied to the game $u_{\{i,j,k\}}$. By symmetry the right hand side of this equation will be the same as in (3) and therefore equating the left hand sides we get:

$$\frac{1}{\mu_i + \mu_j} + \frac{1}{\mu_i + \mu_k} = \frac{1}{\mu_j + \mu_i} + \frac{1}{\mu_j + \mu_k}.$$

From that we conclude $\mu_i = \mu_j$ and therefore $\lambda_i = \lambda_j$. Since this true for any $i, j \in N$, the proof is completed. Q.E.D.

Corollary 1: For $|N| \geq 3$, the only distribution which is common to the family of distributions $\{P_{\omega}\}$ and the family $\{P_{\omega}^*\}$ where ω ranges over all simple weight systems is P_{ω_0} where $\omega_0 = ((1, \dots, 1), (N))$.

We can also obtain a characterization of the (symmetric) Shapley value, one which does not use the symmetry axiom.

Theorem 7: For $|N| \geq 3$, a solution ϕ satisfies axioms 1, 2, 3', 4, 5, and 5* if and only if it is the Shapley value.

For $N = 2$ there exists a transformation $\omega \rightarrow \omega^*$ of simple weight systems such that $\phi_{\omega}^* = \phi_{\omega^*}$. Indeed, it is easy to see that if for $\omega = (\lambda, (N))$ we set $\omega^* = (\lambda^*, (N))$ where $\lambda^* = (\lambda_2, \lambda_1)$ then $\phi_{\omega}^* = \phi_{\omega^*}$. We state now the extension of Theorem 6 to general weight systems and omit the proof.

Theorem 8: If $\omega = (\mu, (S_1, \dots, S_m))$ and $\omega' = (\lambda, (T_1, \dots, T_k))$ are weight systems for which $\phi_\omega^* = \phi_{\omega'}$, then:

- (1) $m = k$.
- (2) $S_i = T_{m+1-i}$ for $i = 1, \dots, m$
- (3) If $|S_i| > 3$ then μ_{S_i} and $\lambda_{T_{m+1-i}}$ are proportional to $(1, 1, \dots, 1)$.
- (4) If $|S_i| = 2$ then μ_{S_i} is proportional to $\lambda_{T_{m+1-i}}^*$.

6 Other Formulas for ϕ_ω and ϕ_ω^*

Owen (1972) has shown that $\phi_\omega(v)$ for $\omega = (\lambda, (N))$ can be computed as an integral of the gradient of the multilinear extension over some path. We now generalize this result for general weight systems and develop an integration formula for $\phi_\omega^*(v)$. The multilinear extension for a game v is the function F_v defined on the unit cube $[0, 1]^n$ as follows:

$$F_v(x_1, \dots, x_n) = \sum_{S \subseteq N} \prod_{i \in S} x_i \prod_{j \notin S} (1 - x_j) v(S).$$

The coordinate x_i can be interpreted as the probability that player i will join the game to form a coalition and $F_v(x_1, \dots, x_n)$ is the expected payoff made. For a given $\omega = (\lambda, (S_1, \dots, S_m))$ define for $i \in S_k$

$$\xi_i(t) = \begin{cases} 0 & \text{if } t \leq \frac{k-1}{m}, \\ \left[m \left(t - \frac{k-1}{m} \right) \right]^{\lambda_i} & \text{if } \frac{k-1}{m} \leq t \leq \frac{k}{m}, \text{ and} \\ 1 & \text{if } \frac{k}{m} \leq t \end{cases}$$

Intuitively $\xi_i(t)$ is the probability that player i will join the game until time t . One can prove that

$$(\phi_\omega)_i(v) = \int_0^1 \frac{\partial F_v}{\partial x_i} \Big|_{\xi(t)} \frac{d\xi_i(t)}{dt} dt$$

just by checking the equality for $v = u_S$, since the right hand side is linear in v (observe that $F_{u_S}(x) = \prod_{i \in S} x_i$). It is easy to see that if the players' arrival time is distributed according to the ξ_i 's, then the probability that they arrive in a certain order R is $P_\omega(R)$.

Now define F_v^* by

$$F_v^*(x_1, \dots, x_n) = v(N) - F_v(1 - x_1, \dots, 1 - x_n)$$

It is easy to check that $F_{v^*}(x) = F_v^*(x)$. Therefore

$$(\phi_\omega^*)_i(v) = (\phi_\omega)_i(v^*) = \int_0^1 \frac{\partial F_v^*}{\partial x_i} \Big|_{\xi(t)} \frac{d\xi_i(t)}{dt} dt$$

Denote $\eta_i(t) = 1 - \xi_i(t)$ and observe that $\frac{\partial F_v^*}{\partial x_i} \Big|_{(\xi(t))} = \frac{\partial F_v}{\partial x_i} \Big|_{(\eta(t))}$ and $\frac{d\xi_i}{dt} = -\frac{d\eta_i}{dt}$.

It follows that

$$(\phi_\omega^*)_i(v) = \int_1^0 \frac{\partial F_v}{\partial x_i} \Big|_{(\eta(t))} \frac{d\eta_i(t)}{dt} dt$$

$\eta_i(t)$ can be interpreted as the probability that player i arrives after time t .

7 Reduction of p -type and p^* -type Coalitions

Part of the reasoning of the partnership axiom is that a coalition of partners can be treated in a certain sense as one individual. In this section we show how a p -type coalition can be practically defined as one player, thereby reducing the size of the game. Let us fix a coalition S_0 with more than one player. Consider the set \bar{N} which consists of all the players of N except that all the players in S_0 are replaced by a single player denoted by s , i.e., $\bar{N} = (N \setminus S_0) \cup \{s\}$. For any game v on N we define a game \bar{v} on \bar{N} by $\bar{v}(S) = v(S)$ if $s \notin S$ and $\bar{v}(S) = v((S \setminus \{s\}) \cup S_0)$ if $s \in S$. Let $\omega = (\lambda, (S_1, \dots, S_m))$ be a weight system for N , and let k be the highest index for which $S_k \cap S_0 \neq \emptyset$. The weight system $\bar{\omega} = (\bar{\lambda}, (\bar{S}_1, \dots, \bar{S}_m))$ for \bar{N} is defined as follows. For each $i \neq s$, $\bar{\lambda}_i = \lambda_i$ and $\bar{\lambda}_s = \sum_{i \in S_0} \lambda_i$. For each $j \neq k$, $\bar{S}_j = S_j \setminus S_0$ and $\bar{S}_k = (S_k \setminus S_0) \cup \{s\}$. We can state now

Theorem 9: If S_0 is a p -type coalition in v then for each $i \neq s$, $(\phi_{\bar{\omega}})_i(\bar{v}) = (\phi_{\omega})_i(v)$ and $(\phi_{\bar{\omega}})_s(\bar{v}) = \sum_{i \in S_0} (\phi_{\omega})_i(v)$.

Similarly, if S_0 is a p^* -type coalition then for each $i \neq s$, $(\phi_{\bar{\omega}^*})_i(\bar{v}) = (\phi_{\omega^*})_i(v)$ and $(\phi_{\bar{\omega}^*})_s(\bar{v}) = \sum_{i \in S_0} (\phi_{\omega^*})_i(v)$. To prove this theorem we use the following lemmas.

Lemma 1: If i is a dummy player in v and $v = \sum_{S \subseteq N} \alpha_S u_S$ then $\alpha_S = 0$ for each S which contains i .

Proof: By induction on the size of S . For $S = \{i\}$, $0 = v(\{i\}) = \alpha_{\{i\}}$. Suppose we proved for all coalitions of size k which contain i and let S be a coalition of size $k + 1$ and such that $i \in S$. Then $0 = v(S) - v(S \setminus \{i\}) = \sum_{T \subseteq S} \alpha_T - \sum_{T \subseteq S \setminus i} \alpha_T = \sum_{i \in T \subseteq S} \alpha_T$. But for $i \in T \subsetneq S$, $\alpha_T = 0$ and therefore $\alpha_S = 0$. Q.E.D.

Lemma 2: Let S_0 be a p -type coalition in v and let $v = \sum_{S \subseteq N} \alpha_S u_S$ then $\alpha_S = 0$ for each S which satisfies $S \cap S_0 \neq \emptyset$ and $S \cap S_0 \neq S_0$.

Proof: For a coalition T and a game v denote by v^T the restriction of the game u to the coalition T . Since $v \rightarrow v^T$ is a linear map from the space of games on N to the space of games on T and since $u_S^T = 0$ if $S \not\subseteq T$ it follows that

$$v^T = \sum_{S \subseteq N} \alpha_S u_S^T = \sum_{S \subseteq T} \alpha_S u_S^T. \tag{*}$$

Now if T satisfies $T \cap S_0 \neq \emptyset$ and $T \cap S_0 \neq S_0$ then all the players of $T \cap S_0$ are dummies in the game v^T . In particular, we conclude by Lemma 1 and (*) that $\alpha_T = 0$. Q.E.D.

Proof of Theorem 9: It can be easily shown by Lemma 2 that if $v = \sum \alpha_S u_S$ then $\bar{v} = \sum_{s \notin S} \alpha_S \bar{u}_S + \sum_{s \in S} \alpha_S \bar{u}_S$. Therefore for $i \neq s$

$$\begin{aligned} (\phi_{\bar{\omega}})_i(\bar{v}) &= \sum_{s \notin S} \frac{\bar{\lambda}_i}{\sum_{j \in S} \bar{\lambda}_j} \alpha_S + \sum_{s \in S} \frac{\bar{\lambda}_i}{\sum_{j \in S} \bar{\lambda}_j} \alpha_S \\ &= \sum_{S \subseteq N \setminus S_0} \frac{\lambda_i}{\sum_{j \in S} \lambda_j} \alpha_S + \sum_{S \supseteq S_0} \frac{\lambda_i}{\sum_{j \in S} \lambda_j} \alpha_S = (\phi_{\omega})_i(v). \end{aligned}$$

For $i = s$

$$(\phi_\omega)_s(\bar{v}) = \sum_{s \in S} \frac{\bar{\lambda}_s}{\sum_{j \in S} \bar{\lambda}_j} \alpha_S = \sum_{S \supseteq S_0} \frac{\sum_{i \in S_0} \lambda_i}{\sum_{j \in S} \lambda_j} \alpha_S = \sum_{i \in S_0} (\phi_\omega)_i(v).$$

Now if S_0 is of p^* -type in v , then S_0 is of p -type in v^* . To prove the second half of the theorem one has to observe only that $(\bar{v})^* = \bar{v}^*$ and use the equality of Theorem 5. Q.E.D.

The following corollary follows from Theorem 9. It is important for applications in which the players themselves are, or are representing, groups of individuals. Such is the case for example when the players are parties, cities, or management boards. The use of the symmetric Shapley value seems to be unjustified in certain cases of this type because the players represent constituencies of different sizes. A natural candidate for a solution is the weighted Shapley value where the players are weighted by the size of the constituencies they stand for. The following corollary shows that such a procedure is justified in the two special cases described below.

Corollary 2: Let v be a game on N ($|N| = n$) in which each player i is a set of individuals M_i with m_i members. Consider the set of individuals $\bar{N} = \bigcup_{i \in N} M_i$ and the games v_1 and v_2 defined on \bar{N} as follows. For each $S \subseteq \bar{N}$,

$$v_1(S) = v(\{i | M_i \subseteq S\})$$

$$v_2(S) = v(\{i | M_i \cap S \neq \emptyset\}).$$

Let ω be the simple weight system $((m_1, \dots, m_n), (\bar{N}))$. Then for each i

$$(\phi_\omega)_i(v) = \phi(v_1)(M_i)$$

and

$$(\phi_\omega^*)_i(v) = \phi(v_2)(M_i)$$

where ϕ is the symmetric Shapley value.

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