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AN APPLICATION OF THE AUMANN–SHAPLEY PRICES FOR COST ALLOCATION IN TRANSPORTATION PROBLEMS*

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The Aumann–Shapley (A-S) prices are axiomatically determined on certain classes of piecewise continuously differentiable cost functions. One of these classes consists of all cost functions derived from the transportation problems and some of their generalizations. These prices are used here to allocate costs among destinations in a way that each destination will pay its "real part" in the total transportation costs. An economic transportation model is presented in which the A-S prices are compatible with consumer demands. Finally an algorithm is provided to calculate both the optimal solution and the associated A-S prices for transportation problems.

1. Introduction. Allocating cost among users becomes a challenge when the cost of producing a list of commodities is not the sum of the costs of producing each of them separately. This occurs usually in cases where the cost function is defined by the solution of a certain mathematical programming problem for which the cost of producing a certain vector of commodities $\alpha = (\alpha_1, \ldots, \alpha_M)$ is minimized under production constraints and a given vector of input prices $c = (c_1, \ldots, c_N)$. If $y = (y_1, \ldots, y_N)$ is an input vector which provides the minimal cost, then the total cost is $c \cdot y$. However, it is not clear how to allocate this cost between the M types of goods and what is the "part" of each unit of mth good in the total cost $c \cdot y$.

Throughout this paper we restrict ourselves mostly to the specific class of cost functions derived from the optimal solution of the *transportation problem* (hereafter TP). This well-known optimization problem is a linear programming problem in which a certain good, available at several origins, is transported, under fixed transportation costs, to several destinations according to their needs, in a way that minimizes the *total* transportation cost. Given the optimal solution, it is sometimes natural to ask what is the contribution of a unit cargo in each destination to the total transportation cost. This is important, for instance in cost benefit analysis, where real costs vs. benefit are considered. For example, assume that the origins are army depot bases of ammunition and the destinations are field units. Given the transportation costs (which may include the price of the transported ammunition) it may be of interest for the army to determine the part of each field unit in the total cost, or, in other words, to determine the "real" expenses of each field unit. Another example deals with origins which are public monopoly (like government) regulated by a cost sharing rule. They produce, say, one good and transport it to several destinations in an optimal way. Their problem is to determine the price per unit of the good that each destination should be charged.

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We propose to use Aumann–Shapley (A-S) prices which reflect, in a sense, the contribution of a unit of cargo, at each destination, to the total transportation cost. These prices were first used by Billera, Heath and Raanan [5] who applied the value concept of nonatomic games as studied by Aumann and Shapley [1] to determine telephone billing rates at Cornell University.

Recently Billera and Heath [6] and independently Mirman and Tauman [8], [10], showed that the A-S price mechanism, as a price mechanism on the continuously differentiable cost functions, is uniquely determined by a set of natural and, in a sense, equitable axioms. These axioms, which do not depend on any notion of game theory, involve only cost functions and quantities consumed. Moreover, it is shown in [8] that the A-S price mechanism is compatible with demand and thus is justified from an economic point of view. It is worth mentioning that Samet and Tauman [13] discuss the relations between A-S prices and marginal cost prices. They show that by omitting the cost sharing assumption, the marginal cost prices can be uniquely determined by a set of axioms very similar to the one defining A-S prices.

It should be pointed out that while our analysis suits models involving public monopoly regulated through a cost sharing rule, it may be irrelevant for other situations. For example if each origin consists of many small producers and each destination consists of many small consumers then competitive prices will prevail in this market, and generally these prices will not coincide with A-S prices.

In §2 of this paper we formally introduce the problem of allocating transportation cost, and demonstrate using a simple example, the difficulties with charging each destination that part of the cost arising from the optimal TP solution and the given transportation prices.

In §3 we introduce the A-S price mechanism and show that it is uniquely determined, by the above mentioned axioms on three classes of cost functions. The first class consists of all piecewise continuously differentiable functions which satisfy certain requirements on the structure of their kinks. The second class consists of the piecewise linear functions in the first class and the third one, which is a subset of the former two, is spanned by those costs functions which emanate from generalized transportation problems.

In §4 we discuss an economic model in which there are several types of commodities each having different transportation prices (e.g., wheat and oil) and several consumers who consume bundles of commodities at the various destinations (e.g., chains and department stores). Each consumer endowed with a given budget has a utility function over these bundles. Given any consumption vector one cannot expect the total demand (dictated by utility maximization) under A-S prices to be equal to this consumption vector. However, we show in this section that an equilibrium in this context can be achieved, i.e., there exist A-S prices for a vector of outputs which is chosen by consumers who maximize their utilities subject to their budget constraints. This result is a special case of a more general theorem proved in [8] which deals with partial equilibrium.

Finally, in §5 we outline an algorithm which computes, for a given transportation problem and an overall consumption vector, the associated A-S prices and the optimal solution. This algorithm is an application of a general parametric programming approach by Srinivasan and Thompson [15], [16] for the TP.

2. Cost allocation for TP cost functions. In this section we consider a transportation model where a certain commodity, available at N origins $B_1, \ldots, B_N$, is shipped to M different destinations $A_1, \ldots, A_M$. The $N \times M$ nonnegative matrix $C = (c_{nm})$ represents the cost of shipping a unit from each origin to each destination. Let $b = (b_1, \ldots, b_N)$ be the vector of the available resources in the N origins (b_n in B_n).

The vector b is assumed to be fixed. Consider the vector of quantities $x = (x_1, \ldots, x_M)$ where x_m is the demand at A_m, satisfying the condition $\sum_{m=1}^M x_m \leqslant \sum_{n=1}^N b_n$, and let $F(x_1, \ldots, x_M)$ be the minimal cost of x. Formally

$$F(x_1, \ldots, x_M) = \min \sum_{n,m} c_{nm} y_{nm}$$

$$\text{subject to} \quad \sum_{n=1}^{N} y_{nm} = x_m, \quad m = 1, \ldots, M, \qquad \sum_{m=1}^{M} y_{nm} \leqslant b_n, \quad n = 1, \ldots, N, \qquad y_{nm} \geqslant 0. \tag{TP}$$

It is well known that the cost function F, which is defined on the simplex $\{x \in E_+^M \mid \sum_{m=1}^M x_m \leqslant \sum_{n=1}^N b_n\}$, is piecewise linear and convex.

The question is how should the total cost be allocated by prices per load at each destination A_m, $m = 1, \ldots, M$. A straightforward mechanism can be suggested: Suppose that the minimal cost of transporting $x = (x_1, \ldots, x_M)$ under the above constraints is obtained by shipping y_{nm} units from B_n to A_m. Then the cost of transporting $y_m = (y_{1m}, \ldots, y_{Nm})$ to A_m is $\sum_{n=1}^N c_{nm} y_{nm}$. The price $\sum c_{nm} y_{nm} / x_m$ per unit in the mth destination will cover this cost. In general, this price does not reflect the "real" effect on the total cost of supplying the destination A_m. The following example shows that since the amount is an outcome of an overall minimization which takes into account the whole system, it may yield distortions.

EXAMPLE. Let $M = N = 2$, and suppose that the available resources are 20 units at each origin, i.e., $b = (20, 20)$ and the demand at the destinations is the vector $x = (20, 20)$. The following table represents the costs of transportation:

$c_{11} = 10$	$c_{12} = 15$	B_1 $b_1 = 20$
$c_{21} = 1000$	$c_{22} = 1500$	B_2 $b_2 = 20$
A_1 $x_1 = 20$	A_2 $x_2 = 20$	

It is easy to verify that the optimal solution is obtained by transporting 20 units from B_2 to A_1 with associated cost of $20 \cdot 1000 = 20{,}000$, and by transporting 20 units from B_1 to A_2 with costs $20 \cdot 15 = 300$. However, destination A_1 should not be allocated 20,000 out of the total cost of 20,300.

The above solution turns to be optimal since the penalty paid, once A_2 is not supplied from its cheapest origin, is much higher than the penalty paid in case A_1 is not supplied from its cheapest origin. Therefore A_1 should be supplied from the more expensive origin for him, namely from B_2, in order to achieve the minimal cost. Hence it seems that the destination A_2 subsidizes destination A_1. But this cannot be a reason to charge A_1 a price higher than A_2. On the contrary, since from each of the origins transportation to A_1 is cheaper than to A_2 it is expected that the price per unit charged at A_1 should be less than the one charged at A_2.

It should be mentioned that shadow prices are not applicable in our study since we are interested in cost sharing prices. Because the cost function $F(x_1, \ldots, x_M)$ in our problem is convex, the shadow prices yield a profit. For instance, in the above example the shadow prices are 1000 and 1005 for A_1 and A_2, respectively. Thus, the total revenue under these prices is $20 \cdot 1000 + 20 \cdot 1005 = 40{,}100$ and the profit is 40,100 −

20,300 = 19,800. Shadow prices are cost sharing prices for any $x = (x_1, \ldots, x_M)$ if and only if F is homogeneous of degree 1. This happens, for example, when resources at the origins are unlimited and then each destination is totally supplied by its cheapest origin. In these cases A-S prices and shadow prices coincide (see §3).

3. The Aumann–Shapley prices. Billera, Heath and Raanan [5] proposed equitable telephone billing rates which share the cost of service. Their approach is based on the theory of nonatomic games as studied in [1]. One can use their idea to allocate costs through prices when a finite number of infinitely divisible commodities is produced. Their idea can be described roughly as follows. Suppose that $F(x_1, \ldots, x_M)$ is a cost function satisfying $F(0) = 0$. The variables x_m denote nonnegative quantities of the commodities produced. Let $\alpha = (\alpha_1, \ldots, \alpha_M)$ be a vector of these commodities. Suppose that the commodities are various types of corn, and they are piled together into one heap. Identify this heap with a continuum of players and associate with it a cooperative game v_α defined as follows: For each subset S of the heap let $v_\alpha(S)$ be the cost of producing S. The Aumann–Shapley value for this nonatomic game is a measure defined on the space of players (the heap) which assigns to each coalition its contribution to the total cost of the heap, i.e., to the cost of producing α. The A-S price of the mth commodity is the value of a unit of this commodity, i.e., the contribution of this unit to the total cost. The existence of a value for the game v_α described above is guaranteed whenever v_α belongs to a well-known class of games called pNAD. Moreover, on this class of games, there is only one continuous value (this follows from Proposition 43.13 and 44.22 of [1] together with Neyman's result stated in [12]). Using the formula for the value on pNAD, the value of the game v_α assigns to a unit of the mth commodity the magnitude,

$$P_m(F, \alpha) = \int_0^1 \frac{\partial F}{\partial x_m}(t\alpha)\, dt,$$

which is defined to be the price of the mth commodity. The components of the vector

$$P(F, \alpha) = (P_1(F, \alpha), \ldots, P_M(F, \alpha)),$$

are the Aumann–Shapley prices. Corollary 5 below guarantees that any game v_α derived from a TP cost function is in pNAD and therefore A-S prices are applicable for this class of functions. For an intuitive interpretation of the A-S prices, assume that the vector α is produced in an homogeneous way, starting from 0 and ending at α. Suppose also that along the above production process each time a "small" proportion (an "infinitesimal" one) of α is produced, the mth commodity is charged its current marginal production cost. Then the average cost per unit of the mth commodity once α has been produced will be its A-S price.

A different derivation of A-S prices is introduced and discussed in [6], [8] and [13]. It has been shown in these works that A-S prices are uniquely determined by a set of neutral, and in a sense equitable, axioms imposed on price mechanisms. Rather than using game theoretic notions, these axioms are stated in purely economic terms, hence providing an economic justification for using the A-S prices. However, the above papers deal only with continuously differentiable cost functions and obviously, in general, TP cost functions are not of that type. In this section we prove that the above mentioned axioms uniquely determine A-S prices on some natural classes of cost functions that include the TP cost functions.

Let $\mathscr{F}$ be a family of functions F, such that each F in $\mathscr{F}$ is defined, for some M, on a full dimensional comprehensive subset, C^F of E_+^M (i.e., $\alpha \in C^F$ implies $C_\alpha \subseteq C^F$ where $C_\alpha = \{x \in E_+^M \mid x \leqslant \alpha\}$). By a price mechanism on $\mathscr{F}$ we mean a function $P(\cdot, \cdot)$ that assigns to each cost function F in $\mathscr{F}$ and each vector α in C^F with $\alpha \gg 0$ a vector of prices,

$$P(F,\alpha) = (P_1(F,\alpha), \ldots, P_M(F,\alpha)).$$

The set of axioms that will be imposed on price mechanisms on $\mathscr{F}$ as stated in [8], is given by,

AXIOM 1 (Cost Sharing). For every $F \in \mathscr{F}$ and every $\alpha \in C^F$,

$$\alpha \cdot P(F,\alpha) = F(\alpha),$$

i.e., total cost equals total revenue.

AXIOM 2 (Additivity). If F and G are in $\mathscr{F}$ and $\alpha \in C_F \cap C_G$, then

$$P(F+G,\alpha) = P(F,\alpha) + P(G,\alpha)$$

where $C_{F+G} = C_F \cap C_G$.

AXIOM 3 (Positivity). If $F \in \mathscr{F}$ is nondecreasing on C_α for some $\alpha \in C^F$ then,

$$P(F,\alpha) \geqslant 0.$$

AXIOM 4 (Consistency). Let F be in $\mathscr{F}$ and assume that $C^F \subseteq E_+^M$. Let C be the subset of E_+^1 defined by $C = \{y \in E^1 \mid y = \sum_{m=1}^M x_m,\ x \in C^F\}$, and let G be a function on $C^G = C$ such that,

$$F(x_1, x_2, \ldots, x_M) = G\left(\sum_{m=1}^M x_m\right).$$

Then for each m, $1 \leqslant m \leqslant M$, and for each $\alpha \in C^F$,

$$P_m(F,\alpha) = P\left(G, \sum_{m=1}^M \alpha_m\right).$$

This axiom is implied by the requirement that commodities which have the same effect on the cost have the same prices.

AXIOM 5 (Rescaling). Let F be in $\mathscr{F}$ with $C^F \subseteq E_+^M$. Let $\lambda_1, \ldots, \lambda_M$ be M positive real numbers. Define $C = \{(x_1, \ldots, x_M) \mid (\lambda_1 x_1, \ldots, \lambda_M x_M) \in C^F\}$ and let G be the function defined on $C^G = C$ by

$$G(x_1, \ldots, x_M) = F(\lambda_1 x_1, \ldots, \lambda_M x_M).$$

Then for each $\alpha \in C^G$, and each m, $1 \leqslant m \leqslant M$,

$$P_m(G,\alpha) = \lambda_m P_m(F, (\lambda_1 \alpha_1, \ldots, \lambda_M \alpha_M)).$$

(A change in the scale of the commodities should yield an equivalent change in the prices.)

DEFINITION. Let $\mathscr{F}_0$ be the family of all functions F such that

(a) F is defined for some M on a full dimensional comprehensive subset C^F of E_+^M.

(b) $F(0) = 0$, i.e., F does not contain a fixed cost component.

(c) F is continuously differentiable (c.d.) on C_α for each $\alpha \in C^F$.

THEOREM 1. *There exists one and only one price mechanism $P(\cdot,\cdot)$ on $\mathscr{F}_0$ which obeys the above five axioms. This is the A-S price mechanism, i.e.,*

$$P_m(F,\alpha) = \int_0^1 \frac{\partial F}{\partial x_m}(t\alpha)\,dt, \qquad m = 1, \ldots, M,$$

for each $F \in \mathscr{F}_0$ and $\alpha \in C^F$ ($\subseteq E_+^M$).

Theorem 1 above is Theorem 1.2 of [8]. Its proof appears in [10] and [13].

REMARK. Observe that the price mechanism which assigns to each pair (F, α), $F \in \mathscr{F}_0$, $\alpha \in C^F$, the shadow prices of F at α, obeys all the axioms but the cost-sharing one. A similar characterization (as of Theorem 1 above) for shadow prices is given in [13].

Cost functions which are derived from the solution of linear programming (LP) problems are piecewise linear. In many cases however, one may have an LP cost function F together with a vector α in its domain such that the line segment $[0, \alpha]$ contains a continuum of kinks (of F). For example let

$$F(x_1, x_2) = \min y \quad \text{s.t.} \quad y \geqslant x_1, \quad y \geqslant x_2,$$

and let $\alpha = (1, 1)$. Then $F(x_1, x_2) = \max(x_1, x_2)$, and F is not differentiable along the line segment $[0, \alpha]$. Thus, one cannot apply the A-S formula to obtain prices for these F and α. Fortunately, this is not the case for TP cost functions or for the generalized TP cost functions (as shown in Corollary 4, below). The latter are cost functions of the form:

$$\begin{aligned} F(x) = \min & \sum_{n=1}^{N} \sum_{m=1}^{M} c_{nm} y_{nm} \\ \text{subject to} \quad & \sum_{n=1}^{N} a_{nm} y_{nm} = x_m \qquad m \in \mathscr{M}, \\ & \sum_{m=1}^{M} d_{nm} y_{nm} \leqslant b_n, \quad n \in \mathscr{N}, \\ & y_{nm} \geqslant 0, \qquad n \in \mathscr{N}, \quad m \in \mathscr{M}, \end{aligned} \tag{WDP}$$

where $\mathscr{N} = \{1, \ldots, N\}$, $\mathscr{M} = \{1, \ldots, M\}$ and a_{nm}, d_{nm} and c_{nm} are nonnegative for all $n \in \mathscr{N}$, $m \in \mathscr{M}$. In addition, it is required that each y_{nm} has a nonzero coefficient in at least one equation. We also assume, without loss of generality, that the rank of the constraints matrix is $M + N$. This linear programming problem or some of its variations is known in the literature as the *weighted distribution problem* (WDP) or as the *generalized transportation problem* (see Dantzig [7, Chapter 21]). Our next lemma and theorem apply to WDP problems for which all constraints $n \in \mathscr{N}$ are equalities. We denote this problem by WDPE. Note that a WDP problem can be transformed into an equivalent WDPE problem by the addition of slack variables to the constraints in $\mathscr{N}$. Thus, Theorem 3 below applies to WDP cost functions as a special case. To prove this theorem we use Theorem 1 of [7, p. 42] (Lemma 2, below) which makes use of the linear graph associated with the constraints of WDPE. There are $N + M$ nodes in this graph, each corresponds to one of the constraints of WDPE. Two nodes are joined by an arc if and only if y_{nm} appears in both constraints with nonzero coefficients. When a variable appears in only one equation with a nonzero coefficient, the node corresponding to this equation is joined by an arc to itself. Note that it is impossible for two nodes in $\mathscr{N}$ (or in $\mathscr{M}$) to be joined by an arc. A basic graph for the above system will be the subgraph corresponding to a specific linear programming basis. This graph is obtained by deleting the arcs corresponding to the nonbasic variables. We have ([7]):

LEMMA 2. *Each maximal connected subgraph of a basic graph for the system of equations in WDPE has precisely one loop.*

We now prove:

THEOREM 3. *Let F be a WDPE cost function. Then F is a piecewise linear function and there is a finite number of hyperplanes $H_1, \ldots, H_l$ of the form*

$$H_j = \left\{ x \in E^M \,\middle|\, \sum_{m \in \mathcal{M}} \lambda_m^j x_m = \beta^j \right\}, \tag{1}$$

where $\lambda_m^j \geq 0$ and $\beta^j \geq 0$ such that F is continuously differentiable off $\bigcup_{j=1}^{l} H_j$.

PROOF. Observe that if the cost function F has a kink at x then the optimal solution must be degenerate. This follows since, in case the solution to WDP at x is nondegenerate, then, in a neighborhood of x, the same basis remains optimal. Hence F is linear in this neighborhood. To prove the theorem, we derive expressions for the basic variables y_{nm} in terms of x, and show that degeneracy implies that x lies in one of a finite number of hyperplanes of the form given in (1). Assume now that the optimal solution is degenerate at x and consider a maximal connected subgraph of the basic graph which contains one of the arcs that corresponds to a vanishing basic variable. Let us denote by P the set of nodes contained in the unique loop of this subgraph, the existence of which is implied by Lemma 2. To calculate the values of the basic variables, consider first all chains of this subgraph beginning with nodes having exactly one arc and containing exactly one node of P (necessarily the last one in the chain). In case no such chain exists then we proceed to calculate the variables in P as described below.

In each of the above chains, a node $n \in \mathcal{N}$ must be followed by a node $m \in \mathcal{M}$ and vice versa. Thus, it is easy to verify that for the arcs joining these nodes we must have

$$y_{nm} = \sum_{\tilde{\mathcal{M}}} \lambda_m x_m - \sum_{\tilde{\mathcal{N}}} \lambda_n b_n, \tag{2}$$

in case this arc joins a node $n \in \mathcal{N}$ to a node $m \in \mathcal{M}$, or

$$y_{nm} = \sum_{\tilde{\mathcal{N}}} \lambda_n b_n - \sum_{\tilde{\mathcal{M}}} \lambda_m x_m, \tag{3}$$

in case $m \in \mathcal{M}$ is joined to $n \in \mathcal{N}$, where $\tilde{\mathcal{N}} \subseteq \mathcal{N}$, $\tilde{\mathcal{M}} \subseteq \mathcal{M}$. Note that the coefficients λ_m and λ_n are positive, since they are products and ratios of numbers a_{nm} and d_{nm} which are positive. Certainly, if one (or more) of these variables is zero, then by (2) or (3), x lies in a hyperplane of the form (1). Moreover, because the number of elements a_{nm}, d_{nm} and b_n is finite then there is a finite number of such possible hyperplanes. Suppose now that all the above chains are evaluated, none of the variables in these chains is degenerate, and that some $y_{\bar{n}\bar{m}}$ corresponding to $\bar{n}, \bar{m} \in P$ is degenerate; that is $y_{\bar{n}\bar{m}} = 0$. Then it is left to evaluate the variables y_{nm} where both n and m are in P. In case P contains exactly one node then the corresponding y_{nm} is given by (2) or (3) above. In any other case we have a loop with at least four nodes. Then, it is possible to solve for all the other variables y_{nm}, $n, m \in P$ starting either from node $\bar{n}$ or from node $\bar{m}$. Let $n', m' \in P$, then accordingly, the following equivalent expressions for $y_{n'm'}$ are obtained:

$$y_{n'm'} = \sum_{\tilde{\mathcal{M}}} \tilde{\lambda}_m x_m - \sum_{\tilde{\mathcal{N}}} \tilde{\lambda}_n b_n, \quad \text{and}$$

$$y_{n'm'} = \sum_{\bar{\mathcal{N}}} \bar{\lambda}_n b_n - \sum_{\bar{\mathcal{M}}} \bar{\lambda}_m x_m,$$

where $\tilde{\mathcal{N}} \subseteq \mathcal{N}$, $\bar{\mathcal{N}} \subseteq \mathcal{N}$, $\tilde{\mathcal{M}} \subseteq \mathcal{M}$ and $\bar{\mathcal{M}} \subseteq \mathcal{M}$ satisfying $\tilde{\mathcal{N}} \cap \bar{\mathcal{N}} = \emptyset$ and $\tilde{\mathcal{M}} \cap \bar{\mathcal{M}} = \emptyset$ and where again $\tilde{\lambda}_m$, $\tilde{\lambda}_n$, $\bar{\lambda}_m$ and $\bar{\lambda}_n$ are positive. Equating the right-hand sides of the above expressions we see that x lies in an hyperplane of the form (1). This completes the proof.

Obviously, TP is a special case of WDP. Moreover, since each λ^j_m is a ratio of products of the a_{nm} and d_{nm} and since for TP, $a_{nm} = d_{nm} = 1$ each λ^j_m in (1) equals one. Note that cost functions derived from assignment problems (see [7, Chapter 15]) are a special case of the transportation problem. Hence, they too belong to the WDP class. Also note that if WDP is feasible for $x = \alpha$ then it is also feasible for all $0 \leqslant x \leqslant \alpha$. Hence, the domain of F is a comprehensive set. Moreover $c_{nm} \geqslant 0$ implies $F(0) = 0$.

Next we have an immediate consequence of Theorem 3.

COROLLARY 4. *If $F(x_1, \ldots, x_m)$ is a WDP cost function defined on a comprehensive domain C^F, then for each α in C^F, F is continuously differentiable along the line segment $[0, \alpha]$, except perhaps for finitely many points.*

For the readers who are familiar with the theory of nonatomic games we state one more corollary.

COROLLARY 5. *If F is a WDP cost function with $F(0) = 0$ defined on a comprehensive domain C^F, then for each $\alpha \in C^F$ the game v_α, as described in the beginning of this section,[1] is in pNAD.*

PROOF. Follows from Corollary 4 and from the main theorem of [11].

Let C be a subset of E^M and let $H_1, \ldots, H_l$ be l hyperplanes in E^M. Each hyperplane H_j defines two closed halfspaces which we shall denote by H_j^+ and H_j^-. We call each nonempty subset of the form $C \cap H_1^{\epsilon_1} \cap \cdots \cap H_l^{\epsilon_l}$, where ϵ_j stands for $+$ or $-$, a *region*.

DEFINITION. A function F defined on a subset C of E^M is piecewise continuously differentiable (p.c.d.) if it is continuous and there are l hyperplanes $H_1, \ldots, H_l$ in E^M and r continuously differentiable function $F^1, \ldots, F^r$ on E^M such that F coincides on each of the regions of C (determined by $H_1, \ldots, H_l$) with some F^j, $1 \leqslant j \leqslant r$.

DEFINITION. Let $\mathcal{F}_1^M$ be the family of all functions F such that:

(a) F is defined, on a full dimensional comprehensive subset C^F of E_+^M;

(b) $F(0) = 0$;

(c) F is p.c.d.;

(d) the hyperplanes $H_1, \ldots, H_l$ involved in the definition of F are defined by positive functionals, i.e.,

$$H_j = \{x \mid \lambda^j x = a^j\} \qquad \text{where} \quad \lambda^j_m \geqslant 0, \quad \lambda^j \neq 0, \quad j = 1, \ldots, l, \quad m = 1, \ldots, M.$$

Notice that $\mathcal{F}_1^M$ is a linear space when $F + G$ is taken to be the function on $C_F \cap C_G$ defined by

$$(F + G)(x) = F(x) + G(x), \qquad x \in C_F \cap C_G.$$

Denote now $\mathcal{F}_1 = \bigcup_{M=1}^{\infty} \mathcal{F}_1^M$. Note that $\mathcal{F}_1$ contains, in view of Corollary 4 and the discussion which preceded it, all WDP cost functions. Moreover all TP cost functions are always in $\mathcal{F}_1$.

Let $\mathcal{F}_2^M$ be the linear space consisting of the piecewise linear functions in $\mathcal{F}_1^M$ and let $\mathcal{F}_2 = \bigcup_{M=1}^{\infty} \mathcal{F}_2^M$. By $\mathcal{F}_3^M$ we denote the linear space spanned by all WDP cost functions with M variables. Again let $\mathcal{F}_3 = \bigcup_{M=1}^{\infty} \mathcal{F}_3^M$. Notice that $\mathcal{F}_1 \supseteq \mathcal{F}_0$ and that $\mathcal{F}_1 \supseteq \mathcal{F}_2 \supseteq \mathcal{F}_3$.

[1] For a precise definition of v_α see [8, p. 48].

THEOREM 6. *For each of the spaces $\mathscr{F}_1$, $\mathscr{F}_2$, and $\mathscr{F}_3$ there exists one and only one price mechanism $P(\cdot,\cdot)$ which obeys the five axioms. This is the A-S price mechanism, i.e., for each $F \in \mathscr{F}_i^M$ $(i = 1,2,3)$ and $\alpha \in C^F$, $\alpha \gg 0$,*

$$P_m(F,\alpha) = \int_0^1 \frac{\partial F}{\partial x_m}(t\alpha)\,dt, \qquad m = 1,\dots,M.$$

We shall first outline the proof of this theorem. It is easy to verify that the A-S formula indeed defines a price mechanism which obeys the five axioms on each of the three sets $\mathscr{F}_1$, $\mathscr{F}_2$, and $\mathscr{F}_3$. As for the uniqueness part, denote by $\mathscr{S}_i^M$ $(i = 1,2,3)$, the minimal linear space containing both $\mathscr{F}_i^M$ and PO^M where the latter is the space of all polynomials p in M variables s.t. $p(0) = 0$. Note that the sum $F + p$ for $F \in \mathscr{F}_i^M$ and $p \in \mathrm{PO}^M$ is defined on C^F. Let $\mathscr{S}_i = \bigcup_{M=1}^{\infty} \mathscr{S}_i^M$. Clearly $\mathscr{S}_1 = \mathscr{F}_1$. We first prove that $\mathscr{F}_i$ is dense in $\mathscr{S}_i$ when the latter is equipped with an appropriate topology (Lemma 7, below). Then we show (Lemma 8, below) that any price mechanism which obeys the five axioms is continuous w.r.t. this topology. These two lemmas enable us to prove that any price mechanism on $\mathscr{F}_i$ $(i = 1,2,3)$ can be extended to all of $\mathscr{S}_i$ (this is Lemma 10, below). Now on $\mathrm{PO} = \bigcup_{M=1}^{\infty} \mathrm{PO}^M$ which is a subset of $\mathscr{S}_i$ $(i = 1,2,3)$ the extended price mechanism is given by the A-S formula since the A-S price mechanism is the only one on PO which obeys the five axioms (see [8] and [13]). Finally we prove that any function F in $\mathscr{F}_i$, $(i = 1,2,3)$, can be approached by polynomials in such a way that their A-S prices approach the prices determined for F by the given mechanism on $\mathscr{F}_i$. Since this sequence of polynomials is independent of the price mechanism on $\mathscr{F}_i$ and depends only on F and α there is only one way to define the prices on $\mathscr{F}_i$ and this completes the proof of the theorem.

The rest of the section is devoted to the formal proof of Theorem 6.

DEFINITION. Let $F \in \mathscr{F}_i^M$ $(i = 1,2,3)$ and let $\alpha \in C^F$, $\alpha \gg 0$. The α-norm of F, $\|F\|_\alpha$ is defined by

$$\|F\|_\alpha = \sum_{m=1}^{M} \left\| \frac{\partial F}{\partial x_m} \right\|_\alpha^{L_\infty}$$

where $\|\partial F/\partial x_m\|_\alpha^{L_\infty}$ is the L_∞-norm of $\partial F/\partial x_m$ on C_α. Observe that $\|\cdot\|_\alpha$ is a norm on $\mathscr{F}_i^m$ $(i = 1,2,3)$.

LEMMA 7. *Let p be a polynomial in PO^M and let $\alpha \in E_{++}^M$. Then for each i $(i = 1,2,3)$ there exists a sequence $(F_n)_{n=1}^{\infty}$ of functions in $\mathscr{F}_i^M$ with $\alpha \in \bigcap_{n=1}^{\infty} C^{F_n}$ such that*

$$\|F_n - p\|_\alpha \to 0, \qquad as \quad n \to \infty.$$

PROOF. Observe first that each polynomial p in PO^M is a linear combination of functions of the form

$$g(x_1,\dots,x_M) = \left(\sum_{m=1}^{M} n_m x_m \right)^l$$

where the n_m's are nonnegative integers and l is a positive integer (e.g., see [1, p. 41]). Obviously it is sufficient to prove the lemma for polynomials g of the above form. Clearly there is a sequence of one variable piecewise linear functions $(G_n)_{n=1}^{\infty}$ s.t.

$$\|G_n - H\|_d \to 0, \qquad \text{as} \quad n \to \infty$$

where d is an arbitrary positive number, $C^{G_n} = [0,d]$ and $H(z) = z^l$ for each $z \in E_+^1$.

Thus if d is sufficiently large

$$\left\| G_n\left(\sum_{m=1}^{M} n_m x_m \right) - g(x_1, \ldots, x_M) \right\|_d \to 0, \qquad \text{as} \quad n \to \infty.$$

Finally notice that G_n is a TP cost function and thus also $G_n(\sum_{m=1}^{M} x_m)$ (which is defined on $\{x \in E_+^M \mid \sum_{m=1}^{M} x_m \leqslant d\}$). Hence the function $G_n(\sum_{m=1}^{M} n_m x_m)$ is a WDP cost function.

Let $Q(\cdot,\cdot)$ be a price mechanism on $\mathscr{F}_i$, for some i ($i = 1,2,3$), obeying Axioms 1–5.

LEMMA 8. *Let $(F_n)_{n=1}^{\infty}$ be a sequence of functions in $\mathscr{F}_i^M$ ($i = 1,2,3$) and let α be in $\bigcap_{n=1}^{\infty} C^{F_n}$. If $\|F_n\|_\alpha \to 0$ as $n \to \infty$ then $Q(F_n, \alpha) \to 0$ as $n \to \infty$.*

The proof of this lemma is analogous to the proof of Proposition 3 in [13].

COROLLARY 9. *Let $\alpha \in E_{++}^M$ and let $(F_n)_{n=1}^{\infty}$ be a sequence of functions in $\mathscr{F}_i^M$ s.t. $\alpha \in \bigcap_{n=1}^{\infty} C^{F_n}$. Then $\|F_n - F_k\|_\alpha \to 0$ as $n, k \to \infty$ implies that $Q(F_n, \alpha)$ approaches a limit as $n \to \infty$.*

This corollary is a direct consequence of Lemma 8 together with the additivity of $Q(\cdot, \alpha)$ on $\mathscr{F}_i^M$.

LEMMA 10. *The price mechanism $Q(\cdot,\cdot)$ on $\mathscr{F}_i$ can be extended to a price mechanism on $\mathscr{S}_i$ which obeys the five axioms.*

PROOF. Define a mechanism $\bar{Q}(\cdot,\cdot)$ on $\mathscr{S}_i$ by

$$\bar{Q}(F, \alpha) = \lim_{n \to \infty} Q(F_n, \alpha)$$

where $(F_n)_{n=1}^{\infty}$ is a sequence of functions in $\mathscr{F}_i$ such that $\alpha \in \bigcap_{n=1}^{\infty} C^{F_n}$ and $\|F_n - F\|_\alpha \to 0$ as $n \to \infty$. The existence of such a sequence $(F_n)_{n=1}^{\infty}$ is guaranteed by Lemma 7. Furthermore by Corollary 9 the limit $\lim_{n \to \infty} Q(F_n, \alpha)$ is independent of the choice of this sequence.

It is easy to verify that $\bar{Q}(\cdot,\cdot)$ obeys the cost-sharing, additivity, consistency and rescaling axioms. Let us verify the positivity property of $\bar{Q}(\cdot,\cdot)$ on $\mathscr{S}_i$. Let F be a function in $\mathscr{S}_i$ which is nondecreasing on C_α for some $\alpha \in C^F$. By Lemma 7 there exists a sequence $(F_n)_{n=1}^{\infty}$ of functions in $\mathscr{F}_i$ s.t. $\alpha \in \bigcap_{n=1}^{\infty} C^{F_n}$ and $\|F_n - F\|_\alpha \to 0$ as $n \to \infty$. Since F is nondecreasing, for each $\epsilon > 0$ and for each sufficiently large n, $F_n + \epsilon \sum_{m=1}^{M} x_m$ is nondecreasing on C_α. Hence by the positivity and the additivity of $Q(\cdot,\cdot)$ on $\mathscr{F}_i$ and by the cost sharing axiom

$$0 \leqslant Q_m\left(F_n + \epsilon \sum_{j=1}^{M} x_j, \alpha \right) = Q_m(F_n, \alpha) + \epsilon, \qquad m = 1, \ldots, M.$$

By Corollary 9, $Q(F_n, \alpha)$ approaches a limit and thus

$$\bar{Q}(F, \alpha) = \lim_{n \to \infty} Q(F_n, \alpha) \geqslant 0,$$

and the proof is completed.

Finally, to establish the proof of Theorem 6 we need the following lemma.

LEMMA 11. *For each $F \in \mathscr{F}_i$ ($i = 1,2,3$) and $\alpha \in C^F$, $\alpha \gg 0$, there exist two* *sequences* $(f^n)_{n=1}^{\infty}$ *and* $(g^n)_{n=1}^{\infty}$ *of polynomials in PO with the following properties*:

$$f^n - F \text{ is nondecreasing on } C_\alpha, \tag{4}$$

$$F - g^n \text{ is nondecreasing on } C_\alpha, \tag{5}$$

$$f^n(0) = g^n(0) = 0, \tag{6}$$

$$\int_0^1 \left| \frac{\partial f^n}{\partial x_m}(t\alpha) - \frac{\partial g^n}{\partial x_m}(t\alpha) \right| dt \to 0, \quad \text{as} \quad n \to \infty, \quad \text{for} \quad m = 1, \ldots, M. \tag{7}$$

PROOF. Since $\mathscr{F}_1 \supseteq \mathscr{F}_2 \supseteq \mathscr{F}_3$ it is sufficient to prove the lemma for F in $\mathscr{F}_1$. Let $F \in \mathscr{F}_1$ and let $\alpha \gg 0$ be in C^F. Let $H_1, \ldots, H_l$ and $F^1, \ldots, F^r$ be the hyperplanes and the functions respectively which are involved in the definition of F. Assume that $C^F \subseteq E_+^M$ and that

$$H_j = \{x \mid \lambda^j x = a^j\},$$

where $\lambda_m^j \geqslant 0$ and $\lambda^j \neq 0$.
Denote

$$L = \min\{\lambda_m^j \mid \lambda_m^j > 0, m = 1, \ldots, M, j = 1, \ldots, l\}.$$

Let F_m^k be the mth partial derivative of F^k, $k = 1, \ldots, r$. The function F_m is similarly defined. Let

$$T = \max_{1 \leqslant k \leqslant r} \max_{1 \leqslant m \leqslant M} \max_{x \in C_\alpha} |F_m^k(x)|. \tag{8}$$

It is easy to verify that for each x and y in C_α,

$$|F(x) - F(y)| \leqslant T\|x - y\|. \tag{9}$$

For any n choose $\epsilon_n > 0$, s.t. $\epsilon_n \to 0$ as $n \to \infty$. Fix n and choose $\delta > 0$ s.t. $\delta < \epsilon_n L/8T$. Let

$$\hat{H}_j^\delta = \{x \mid a^j - \delta \leqslant \lambda^j x \leqslant a^j + \delta\}, \qquad j = 1, \ldots, l,$$

$$\hat{H}^\delta = \bigcup_{j=1}^{l} \hat{H}_j^\delta, \quad \text{and}$$

$$H = \bigcup_{j=1}^{l} H_j.$$

For each $a \in E^1$, let ψ_a be a c.d. function on E^1 which obeys

$$\begin{cases} 0 \leqslant \psi_a'(t) \leqslant \dfrac{2T}{L}, & \text{for each } t \in E^1, \\ \psi_a'(t) = \dfrac{2T}{L}, & \text{for } a - \delta \leqslant t \leqslant a + \delta, \\ \psi_a(t) = 0, & \text{for } t \leqslant a - 2\delta, \\ \psi_a(t) = \epsilon_n, & \text{for } t \geqslant a + 2\delta. \end{cases} \tag{10}$$

Define l functions $G^1, \ldots, G^l$ by

$$G^j(x) = \psi_{a^j}(\lambda^j x), \qquad j = 1, \ldots, l. \tag{11}$$

The function G^j satisfies,

$$\text{(i)} \qquad \text{For each} \quad x \in C_\alpha, \qquad 0 \leqslant G^j_m(x) \leqslant \frac{\lambda^j_m \cdot 2T}{L},$$

$$\text{(ii)} \qquad \text{If} \quad x \in \hat{H}^\delta_j, \quad \text{then} \quad G^j_m(x) = \frac{\lambda^j_m \cdot 2T}{L}, \tag{12}$$

$$\text{(iii)} \qquad \text{If} \quad x \notin \hat{H}^{2\delta}_j, \quad \text{then} \quad G^j_m(x) = 0.$$

Let us choose $\eta > 0$ s.t. $\eta < \min(\delta/\|\alpha\|, \epsilon_n/T, 1)$ and

$$\|x - y\| < \eta\|\alpha\| \Rightarrow |F^k_m(x) - F^k_m(y)| < \frac{\epsilon_n}{r}, \qquad k = 1, \ldots, r, \quad m = 1, \ldots, M. \tag{13}$$

Now, for each continuous function defined on a box C, let g^* be the function on C defined by

$$g^*(x) = \frac{1}{\lambda(C)} \int_{z \in C} g((1-\eta)x + \eta z)\, dz,$$

where λ is the Lebesgue measure on C. For each $x \in C$, $g^*(x)$ is the average values of g over the box $(1-\eta)x + \eta C$. If $C = \times^M_{m=1}[c_m, d_m]$, then

$$g^*(x) = \frac{1}{\eta^M \lambda(C)} \int \cdots \int_{(1-\eta)x_m + \eta c_m}^{(1-\eta)x_m + \eta d_m} \cdots \int g(z)\, dz_1 \ldots dz_m.$$

Thus it is easy to verify that g is a c.d. function on C. Moreover if the derivative g_m exists on $(1-\eta)x + \eta C$, then

$$(g^*)_m(x) = (1-\eta)(g_m)^*(x). \tag{14}$$

The diameter of ηC_α is $\eta\|\alpha\| < \delta$ and the diameter of $(1-\eta)x + \eta C_\alpha$ is the same. Hence, for each $x \in C_\alpha \backslash \hat{H}^\delta$ there is a neighborhood of $(1-\eta)x + \eta C_\alpha$ which is contained in $C_\alpha \backslash H$. On this neighborhood F coincides with F^k for some k, $1 \leqslant k \leqslant r$. Thus replacing g by F and C by C_α we have, by (14), that for each m, $1 \leqslant m \leqslant M$,

$$(F^*)_m(x) = (1-\eta)(F_m)^*(x), \qquad x \in C_\alpha \backslash \hat{H}^\delta. \tag{15}$$

Moreover, in this neighborhood, F_m is continuous and thus averaging F_m on the box $(1-\eta)x + \eta C_\alpha$ yields a value of F_m at some point of this box. Therefore, from the choice of η we have, by (13), that for $x \in C_\alpha \backslash \hat{H}^\delta$,

$$|(F_m)^*(x) - F_m(x)| < \epsilon_n/r \leqslant \epsilon_n. \tag{16}$$

Define the function $\hat{f}^n$ on C_α by,

$$\hat{f}^n = F^* + \sum_{j=1}^{l} G^j + 2\epsilon_n u,$$

where $u(x) = \sum^M_{m=1} x_m$. Let f_n be the function on C_α defined by $f_n(x) = \hat{f}_n(x) - \hat{f}_n(0)$. Clearly $f_n(0) = 0$. To prove that $f^n - F$ is nondecreasing on C_α, let x be in $C_\alpha \backslash H$ and let m, $1 \leqslant m \leqslant M$, be fixed. We will show that $(f^n - F)_m(x) \geqslant 0$. Consider two possibilities:

I. If $x \in C_\alpha \backslash \hat{H}^\delta$ we have by (8), (15) and (16)

$$\begin{aligned} |(F^* - F)_m(x)| &\leqslant |(F^*)_m(x) - (F_m)^*(x)| + |(F_m)^*(x) - F_m(x)| \\ &\leqslant \eta|(F_m)^*(x)| + \epsilon_n \leqslant \eta T + \epsilon_n < 2\epsilon_n. \end{aligned}$$

Together with (12) we have

$$f_m^n(x) - F_m(x) = (F^*)_m(x) - F_m(x) + \sum_{j=1}^{l} G_m^j(x) + 2\epsilon_n$$

$$\geqslant \sum_{j=1}^{l} G_m^j(x) \geqslant 0.$$

II. If $x \in \hat{H}^\delta \backslash H$ then there are two possible cases to check:

(a) There is a j_0, $1 \leqslant j_0 \leqslant l$, s.t. $x \in \hat{H}_{j_0}^\delta \backslash H_{j_0}$ and $\lambda_m^{j_0} > 0$. Since by (9) F satisfies Lipshitz condition with constant T, so does F^*, and therefore $|(F^*)_m| \leqslant T$. Thus by (12) we have

$$f_m^n(x) - F_m(x) = (F^*)_m(x) - F_m(x) + \sum_{j=1}^{l} G_m^j(x) + 2\epsilon_n$$

$$\geqslant -2T + G_m^{j_0}(x) + 2\epsilon_n \geqslant -2T + 2T + 2\epsilon_n > 0.$$

(b) For each j with $x \in \hat{H}^\delta \backslash H_j$, $\lambda_m^j = 0$. In this case the box $C = (1-\eta)x + \eta C_\alpha$ does not intersect any H_j with $\lambda_m^j > 0$. Thus if H_j is a hyperplane which intersects C and if F^{k_1} and F^{k_2} are two functions defined on the two sides of $H_j \cap C$ then $F_m^{k_1}$ and $F_m^{k_2}$ coincide on H_j. Hence, F_m is well defined on C and is continuous there. Therefore

$$(F^*)_m(x) = (1-\eta)(F_m)^*(x). \tag{17}$$

Since the diameter of C is $\eta\|\alpha\|$ we have by (13) that for any two points x and y in C,

$$|F_m(x) - F_m(y)| < r \cdot \frac{\epsilon_n}{r} = \epsilon_n .$$

This together with (17) imply, as in case I,

$$|(F_m)^*(x) - F_m(x)| < 2\epsilon_n$$

and the proof of this part can now be completed as in case I.

Thus, we have proved that for any $x \in C_\alpha \backslash H$, $(f_m^n - F_m)(x) \geqslant 0$. Together with the continuity of $f^n - F$ it follows that $f^n - F$ is nondecreasing on C_α.

In the same manner one can define the sequences $(\hat{g}^n)_{n=1}^\infty$ and $(g^n)_{n=1}^\infty$ on C_α by

$$\hat{g}^n = F^* - \sum_{j=1}^{l} G^j - 2\epsilon_n u,$$

$$g^n(x) = \hat{g}(x) - \hat{g}^n(0),$$

to obtain that $g^n(0) = 0$ and that $F - g^n$ is a nondecreasing function on C_α.

It is well known that any function which is c.d. on some box in E^M can be extended to a c.d. function on E^M. Thus, we may assume that the functions f^n and g^n are defined on C^F and belong to $\mathscr{F}_0$. It remains to prove that

$$\int_0^1 |g_m^n(t\alpha) - f_m^n(t\alpha)|\, dt \to 0, \qquad \text{as} \quad n \to \infty.$$

By the definition of f^n and g^n,

$$|g_m^n(t\alpha) - f_m^n(t\alpha)| = 2 \sum_{j=1}^{l} G_m^j(t\alpha) + 4\epsilon_n , \tag{18}$$

and (11) implies that

$$G_m^j(t\alpha) = \lambda_m^j \psi'_{a^j}(t\lambda^j\alpha).$$

The length of the interval of t's for which $a^j - 2\delta \leqslant t\lambda^j\alpha \leqslant a^j + 2\delta$ is $4\delta/\lambda^j\alpha$, and by (11), $0 \leqslant \psi'_{a^j} \leqslant 2T/L$ in this interval. Outside this interval $\psi'_{a^j} = 0$. Thus, by the definition of δ we get

$$\sum_{j=1}^{l} \int_0^1 G_m^j(t\alpha)\,dt \leqslant \sum_{j=1}^{l} \lambda_m^j \cdot \frac{4\delta}{\lambda^j\alpha} \cdot \frac{2T}{L} \leqslant \epsilon_n \sum_{j=1}^{l} \frac{\lambda_m^j}{\lambda^j\alpha}.$$

The last summation is independent of n. Therefore by (18) we conclude

$$\int_0^1 |g_m^n(t\alpha) - f_m^n(t\alpha)|\,dt \to 0, \qquad \text{as} \quad n \to \infty. \tag{19}$$

Hence the two sequences $(f^n)_{n=1}^\infty$ and $(g^n)_{n=1}^\infty$ obey the four requirements (4), (5), (6) and (7). Thus to complete the proof of the lemma it is left to prove that these two sequences can be replaced by polynomials which still obey the above four requirements. But this follows from the fact that the polynomials in M variables are dense in the set of all continuously differentiable functions on C^F with the norm $\|\cdot\|_\alpha$ for each $\alpha \in C^F$ (see [13]).

We are now ready to complete the proof of Theorem 6.

Proof of Theorem 6. As was already mentioned it is easy to verify that the A-S formula indeed defines a price mechanism which obeys the five axioms on each of the sets $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}_3$. For the uniqueness part let $Q(\cdot,\cdot)$ be a price mechanism which obeys the five axioms on $\mathcal{F}_i$ for some i ($i = 1,2,3$). By Lemma 11 $Q(\cdot,\cdot)$ can be extended to a price mechanism on $\mathcal{S}_i$ which obeys the five axioms. Now since $\mathcal{S}_i$ contains the polynomials and since on PO the A-S price mechanism is the only price mechanism that obeys the five axioms (see [8] and [13])

$$Q_m(p,\alpha) = \int_0^1 \frac{\partial p}{\partial x_m}(t\alpha)\,dt, \qquad m = 1,\dots,M, \tag{20}$$

for each $p \in \mathrm{PO}^M$ and $\alpha \in E_+^M$.

Now let $F \in \mathcal{F}_i$ and let $\alpha \in C^F$, $\alpha \gg 0$. By Lemma 11 there exist two sequences of polynomials $(f^n)_{n=1}^\infty$ and $(g^n)_{n=1}^\infty$ which obey (4), (5), (6) and (7). By the positivity of $Q(\cdot,\cdot)$ on $\mathcal{S}_i$

$$Q(f^n - F,\alpha) \geqslant 0 \quad \text{and}$$
$$Q(F - g^n,\alpha) \geqslant 0.$$

Hence by the additivity of $Q(\cdot,\cdot)$

$$Q(g^n,\alpha) \leqslant Q(F,\alpha) \leqslant Q(f^n,\alpha), \qquad n = 1,2,\dots.$$

By (20)

$$\int_0^1 \frac{\partial g^n}{\partial x_m}(t\alpha)\,dt \leqslant Q_m(F,\alpha) \leqslant \int_0^1 \frac{\partial f^n}{\partial x_m}(t\alpha)\,dt, \qquad m = 1,\dots,M.$$

This, together with (7), implies

$$Q_m(F,\alpha) = \lim_{n\to\infty} \int_0^1 \frac{\partial f^n}{\partial x_m}(t\alpha)\,dt, \qquad m = 1,\dots,M. \tag{21}$$

Since the sequence $(f^n)_{n=1}^\infty$ depends on F and α and not on the price mechanism $Q(\cdot,\cdot)$, the proof of the theorem follows now by (21).

4. The economic model. A transportation model consists of N origins $B_1,\dots,B_N$, M destinations $A_1,\dots,A_M$, L consumers, and $K+1$ commodities denoted by

$0, 1, \ldots, K$. Commodities $1, \ldots, K$ are transported from origins to destinations, and the other good called the input is used to measure the transportation cost (for example it can be money). For any k, $1 \leqslant k \leqslant K$, $C^k = (c^k_{nm})$ is a matrix of order $N \times M$ where c^k_{nm} is the cost, in terms of the input, of transporting a unit of the kth commodity from B_n to A_m. This cost can include the price of the commodity as well as the profit of the transportating agency.

Each consumer l, $1 \leqslant l \leqslant L$, can consume, in a given consumption set D^l in E_+^{1+KM}, a vector $x(l) = (x^0(l), x_1(l), \ldots, x_m(l), \ldots, x_M(l))$. Here, for each m, $1 \leqslant m \leqslant M$, $x_m(l) = (x^1_m(l)), \ldots, x^K_m(l))$ is a bundle in A_m consisting of the K commodities and $x^0(l) \in E^1_+$ is the amount of input he consumes, i.e., each consumer consumes the input and the K commodities at each destination. The lth consumer has a preference ordering on D^l expressed by a utility function U^l, and is endowed with some amount w^l of the input. The origins can be thought of as public monopolies, the consumers can be central distribution agencies, chains or individuals located at the destinations. Each consumer has a utility function which depends only on the coordinates corresponding to his location.

For each n, $1 \leqslant n \leqslant N$, and each k, $1 \leqslant k \leqslant K$, origin B_n has a limited capacity of b^k_n units of the kth commodity. Let Σ be the subset of E_+^{KM} defined by,

$$\Sigma = \left\{ x = (x_1, \ldots, x_M) \,\middle|\, \sum_{m=1}^{M} x^k_m \leqslant \sum_{n=1}^{N} b^k_n \text{ for every } k,\ 1 \leqslant k \leqslant K \right\}.$$

Define the cost function $F : \Sigma \to E^1$ by

$$F(x) = \sum_{k=1}^{K} F_k(x^k),$$

where for $k = 1, \ldots, K$, $x^k = (x^k_1, \ldots, x^k_M)$, F_k is defined on

$$C^{F_k} = \left\{ x \in E_+^M \,\middle|\, \sum_{m=1}^{M} x_m \leqslant \sum_{n=1}^{N} b^k_n \right\},$$

by

$$F_k(x^k) = \min \sum_{n,m} c^k_{nm} y^k_{nm},$$

subject to

$$\sum_{n=1}^{N} y^k_{nm} = x^k_m, \qquad m = 1, \ldots, M,$$

$$\sum_{m=1}^{M} y^k_{nm} \leqslant b^k_n, \qquad n = 1, \ldots, N,$$

$$y^k_{nm} \geqslant 0, \qquad n = 1, \ldots, N, \quad m = 1, \ldots, M,$$

i.e., F_k is the TP cost function for the kth commodity.

Each vector x in D^l will be represented by a pair $(x^0, \bar{x})$ where x^0 is the input's quantity and $\bar{x} \in E_+^{KM}$ is the bundle of the K commodities at each destination. For each l, $1 \leqslant l \leqslant L$, and each price vector p in E^{KM} (price per commodity in each destination) define the budget set $B_l(p)$ of l by

$$B_l(p) = \{(x^0, \bar{x}) \in D^l \mid x^0 + \bar{x}p \leqslant w^l\}.$$

A vector $x \in B_l(p)$ is *maximal* in $B_l(p)$ if $U^l(x) \geqslant U^l(y)$, for each $y \in B_l(p)$. Finally, a vector $(x^0, \bar{x})$ is feasible if $F(\bar{x}) \leqslant x^0$.

DEFINITION OF EQUILIBRIUM. A point of the form $(x(1), \ldots, x(L), p_1, \ldots, p_M)$, where $x(l)$ is a vector of bundles in E_+^{1+KM} and $p_m \in E^K$ is a price vector for each of the K commodities at the mth destination A_m, is an equilibrium if

I. For each l, $1 \leqslant l \leqslant L$, $x(l)$ is maximal in $B_l(p)$ where $p = (p_1, \ldots, p_M)$.

II. $(w - x^0, \bar{x})$ is feasible, where $w = \sum_{l=1}^{L} w^l$, $x^0 = \sum_{l=1}^{L} x^0(l)$ and $\bar{x} = \sum_{l=1}^{L} \bar{x}(l)$.

III. $p = P(F, \bar{x})$, i.e., the price vector p is the vector of A-S prices associated with the cost function F and total vector transported $\bar{x}$.

ASSUMPTIONS. (i) *For every l, $1 \leqslant l \leqslant L$, there is a vector $d^l = (d_0^l, d_1^l, \ldots, d_M^l)$ in E_+^{1+KM}, where $d_0^l \in E_+^1$ and $d_m^l = (d_{m1}^l, \ldots, d_{mK}^l) \in E_+^K$, $1 \leqslant m \leqslant M$, such that the consumption set D^l is the box*

$$D^l = \left\{ x \in E_+^{1+KM} \mid x \leqslant d^l \right\};$$

i.e., l can consume at most d_0^l units of input and at most d_{mk}^l units of the kth commodity at the mth destination.

(ii) *For every l, $1 \leqslant l \leqslant L$, $0 \leqslant w^l \leqslant d_0^l$.*

(iii) *For every l, $1 \leqslant l \leqslant L$, U^l is a continuous and quasiconcave function on D^l.*

(iv) *For every k, $1 \leqslant k \leqslant K$, $\sum_{m=1}^{M} \sum_{l=1}^{L} d_{mk}^l \leqslant \sum_{n=1}^{N} b_n$, i.e., the total capacity of each commodity available for all consumers at all destinations together, does not exceed the total capacity of the commodity available in all origins together.*

THEOREM 12. *The existence of an equilibrium is guaranteed under assumptions* (i)–(iv).

PROOF. The theorem can be proved along the lines of Theorem 2.1 of [8]. For each k, $1 \leqslant k \leqslant K$, F_k is a continuous piecewise linear function with $F_k(0) = 0$. By Corollary 4 above, F_k is c.d. along the line segment $[0, \alpha^{(k)}]$, except perhaps for finitely many points. Since $F = \sum_{k=1}^{K} F_k$, the above properties remain true if F_k is replaced by F and $\alpha^{(k)}$ by α. In particular $F(0) = 0$ (which is Assumption (5) of [8]). Since the kinks of F lie on a finite number of hyperplanes the partial derivations of F are all bounded (on C^F). Thus for each $\alpha \in C^F$, the function g_α: $[0, 1] \to E^1$ defined by $g_\alpha(t) = \partial F(t\alpha)/\partial x_m$ is bounded and continuous for each t, $0 \leqslant t \leqslant 1$, except perhaps at a finite number of points. By the Lebesgue bounded convergence theorem, if $\beta \to \alpha$ then $\int_0^1 g_\beta(t)\,dt \to \int_0^1 g_\alpha(t)\,dt$. This means that $P_m(F, \alpha)$ is continuous at α (for each $\alpha \in C^F$). Assumptions (6) and (7) of [8] are used only to obtain the continuity of A-S prices [9]. Thus it remains to check that assumptions (1)–(4) of [8] are fulfilled in our model. Indeed these four assumptions are assumptions (i), (ii), (iii), and (iv) above. Hence, the existence of an equilibrium in our model is guaranteed by Theorem 2.1 of [8].

REMARKS. (1) Note that if for each l, $1 \leqslant l \leqslant L$, U^l is increasing then it is easy to verify that the equilibrium $(x(1), \ldots, x(L), p_1, \ldots, p_M)$ will be locally efficient in the sense that $w - x^0 = F(\bar{x})$. This means that the amount spent by all the consumers equals the total transportation cost. Consequently there is no waste.

(2) Consider a consumer l who is an individual at the mth destination and has an increasing utility function, dependent only on bundles in the mth destination. Under the assumption that transportation costs are always positive, this consumer gets, in equilibrium, a consumption vector of the form $x(l) = (x^0, 0, \ldots, 0, x_m, 0, \ldots, 0)$, i.e., he receives commodities only at his destination.

(3) Note that one can redefine F by replacing the TP cost functions by similar WDP cost functions and still retain all the properties needed for the existence of an equilibrium.

5. An algorithm for computing A-S prices for TP cost functions. In this section we refer to an algorithm which evaluates the A-S prices for a given TP and feasible vector $\alpha = (\alpha_1, \ldots, \alpha_M)$. The algorithm can also be used to solve the associated TP problem for $x = \alpha$. Consequently, to compute the A-S prices vector there is no need to solve TP separately in order to obtain the transportation plan.

Since F is piecewise linear, by Corollary 4 we have that the A-S price vector $P(F,\alpha)$ is a finite sum of the gradients of the linear "pieces" of F along the line segment $[0,\alpha]$, where each of these is weighted by the normalized length of the sub-interval in which F has a constant gradient. Consequently it is sufficient to consider $F(t\alpha)$ for $0 \leqslant t \leqslant 1$ and find the values t^j, $j = 0, \ldots, J$, where $t^0 = 0$, $t^J = 1$, for which $F(t\alpha)$ changes gradient. The algorithm which computes the points t^j, the fixed gradient of F in each interval $(t^j\alpha, t^{j+1}\alpha)$ and solves the TP at α is described in detail in [14]. This algorithm is an application to the transportation problem of the general parametric programming approach suggested by Srinivasan and Thompson [15], [16].

To illustrate the computations of the A-S prices consider the example of §2. For this problem, shown in Figure 1,

$$C^F = \{(x_1, x_2) \mid x_1 + x_2 \leqslant 40, x_1 \geqslant 0, x_2 \geqslant 0\},$$

and the kinks are located on the hyperplanes $x^1 + x^2 = 20$ and $x^2 = 20$.

In fact

$$F(x_1, x_2) = \begin{cases} 10x_1 + 15x_2, & x_1 \geqslant 0, x_2 \geqslant 0, x_1 + x_2 \leqslant 20, \\ 1000x_1 + 1005x_2 - 19800, & 0 \leqslant x_2 \leqslant 20, 20 \leqslant x_1 + x_2 \leqslant 40, \\ 1000x_1 + 1500x_2 - 29700, & x_1 \geqslant 0, x_2 \geqslant 20, x_1 + x_2 \leqslant 40. \end{cases}$$

Consequently the algorithm will determine that

$$\nabla F(t\alpha) = (10, 15) \quad \text{for} \quad 0 \leqslant t < \tfrac{1}{2}, \quad \text{and}$$

$$\nabla F(t\alpha) = (1000, 1005) \quad \text{for} \quad \tfrac{1}{2} < t \leqslant 1,$$

thus

$$P(F,\alpha) = \tfrac{1}{2}\begin{pmatrix} 10 \\ 15 \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} 1000 \\ 1005 \end{pmatrix} = \begin{pmatrix} 505 \\ 510 \end{pmatrix}.$$

Finally let us mention that the algorithm can also, in the view of the analysis in [15], [16], handle the capacitated transportation problem. This is a TP where the values of the variables y_{nm} are constrained to be bounded on both sides. Moreover, any cost function derived from this problem belongs to $\mathscr{F}_2$ and thus, by Theorem 6 above, the A-S prices are applicable in this case as well. We also note that Balanchandran and

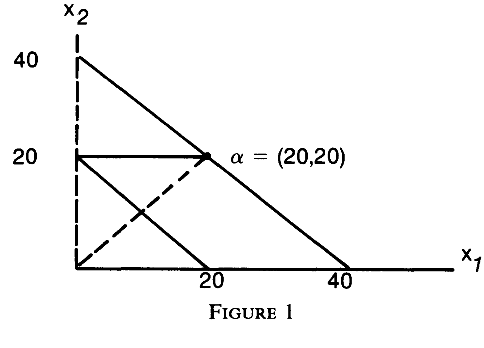

FIGURE 1

Thompson [2], [3] and [4] provide a complete parametric analysis for the generalized transportation problem. Their algorithms can be used to obtain an algorithm to compute the A-S prices for WDP cost functions.

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