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NOTE

Common Priors and Separation of Convex Sets

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We observe that the set of all priors of an agent is the convex hull of his types. A prior common to all agents exists if the sets of the agents' priors have a point in common. We give a necessary and sufficient condition for the nonemptiness of the intersection of several closed convex subsets of the simplex, which is an extension of the separation theorem. A necessary and sufficient condition for the existence of common prior is a special case of this. *Journal of Economic Literature* Classification Numbers: C70, D82.

A *type space* for a set of agents $I = \{1, \dots, n\}$ is a tuple $\langle \Omega, (\Pi_i, t_i)_{i \in I} \rangle$, where Ω is a finite set of states, Π_i is a partition of Ω, and t_i is a function $t_i \colon \Omega \to \Delta^\Omega$, which associates with each state ω the *type* of i at ω, i.e., a point in Δ^Ω, the simplex in R^Ω, which we consider as the set of probability distributions over Ω. The type function t_i satisfies the following two conditions: for each $\omega \in \Omega$, $t_i(\omega)(\Pi_i(\omega)) = 1$, where $\Pi_i(\omega)$ is the element of the partition Π_i which contains ω; t_i is constant over each element of Π_i.

A *prior* for $i \in I$ is a probability distribution $p \in \Delta^\Omega$, such that for each $Q \in \Pi_i$, if $p(Q) > 0$, then $t_i(\omega)(\cdot) = p(\cdot \mid Q)$. Clearly, each type of i, $t_i(\omega)$, is a prior for i, and thus, the following observation is straightforward.

OBSERVATION. The set of all priors of i, denoted by P_i, is the convex hull of all of i's types.

A probability distribution $p \in \Delta^\Omega$ is a *common prior* on the type space if $p \in \bigcap_{i=1}^n P_i$.

Consider first the case of two agents. A condition for the existence of a common prior in this case is a straightforward result of the separation theorem. For $f \in R^\Omega$, denote by $E_i f$ the element of R^Ω defined by $(E_i f)(\omega) = t_i(\omega) f$.

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CLAIM. *When there are two agents* $(n = 2)$, *then there exists a common prior, iff there is no* f *in* R^{Ω}, *such that* $E_1 f > 0 > E_2 f$.

Proof. There is no common prior iff P_1 and P_2 can be strongly separated, that is, iff there are $g \in R^{\Omega}$ and $c \in R$, such that $x_1 g > c > x_2 g$, for each $x_1 \in P_1$, and $x_2 \in P_2$. Subtracting c from all the coordinates of g yields a vector f, such that $x_1 f > 0 > x_2 f$ for each $x_1 \in P_1$, and $x_2 \in P_2$. However, these inequalities hold iff they hold for the extreme points of P_1 and P_2, i.e., iff $E_1 f > 0 > E_2 f$. ■

We now show that a generalization of this claim to any number of agents is the immediate result of a generalization of the separation theorem that gives a necessary and sufficient condition for the intersection of several convex, closed subsets of the simplex to have a nonempty intersection.

Two closed, convex subsets, K_1 and K_2, of the simplex Δ^m in R^m are disjoint iff for some $f \in R^m$, $x_1 f > 0 > x_2 f$ for each $x_1 \in K_1$ and $x_2 \in K_2$. Denoting $f_1 = f$ and $f_2 = -f$ we can rewrite the separating condition, symmetrically, as $K_1 \cap K_2 = \emptyset$ iff there are f_1, f_2 in R^m, such that $f_1 + f_2 = 0$, and $x_i f_i > 0$, for each $x_i \in K_i$, for $i = 1, 2$.

The following proposition generalizes this condition for n sets.

PROPOSITION. *Let* $K_1, \dots, K_n$ *be convex, closed, subsets of* Δ^m. *Then,* $\cap_{i=1}^{n} K_i = \emptyset$ *iff there are* $f_1, \dots, f_n$ *in* R^m, *such that* $\sum_{i=1}^{n} f_i = 0$, *and* $x_i f_i > 0$ *for each* $x_i \in K_i$, *for* $i = 1, \dots, n$.

Proof. Consider the bounded, closed, and convex subsets of R^{mn}, $X = \times_{i=1}^{n} K_i$, and $Y = \{(p, \dots, p) \in R^{mn} \mid p \in \Delta^m\}$. Clearly, $\bigcap_{i=1}^{n} K_i = \emptyset$ iff X and Y are disjoint, but these two sets are disjoint iff there is a constant c and $g = (g_1, \dots, g_n) \in R^{nm}$, where $g_i \in R^m$ for each i, such that for each $x = (x_1, \dots, x_n)$ in X and $y = (p, \dots, p)$ in Y, $xg > c > yg$. Moreover, we may assume that $c = 0$ (by subtracting c/n from all the components of g). Hence, $\sum_{i=1}^{n} x_i g_i > 0$ and $\sum_{i=1}^{n} p g_i < 0$. The last inequality holds for all $p \in \Delta^m$ and therefore it is equivalent to $\sum_{i=1}^{n} g_i < 0$. Moreover, whereas the coordinates of x_i are nonnegative, increasing the coordinates of the g_i does not change the first inequality, and hence the intersection of the K_is is empty iff there is g such that $\sum_{i=1}^{n} g_i = 0$, and $\sum_{i=1}^{n} x_i g_i > 0$.

Now, let $\bar{x}_i$ be the point that minimizes $x_i g_i$ over K_i. Whereas $\sum_{i=1}^{n} \bar{x}_i g_i > 0$, there are constants c_i such that $\bar{x}_i g_i + c_i > 0$ for $i = 1, \dots, n$, and $\sum_{i=1}^{n} c_i = 0$. Denote by e the vector of 1s in R^m and define $f_i = g_i + c_i e$. Then $\sum_{i=1}^{n} f_i = \sum_{i=1}^{n} g_i = 0$ and for each $x_i \in K_i$, $x_i f_i \geq \bar{x}_i f_i = \bar{x}_i g_i + c_i \bar{x}_i e > 0$, as $\bar{x}_i e = 1$. ■

The following characterization of the existence of common prior is a special case of the proposition.

COROLLARY. *There exists a common prior on the type space iff there are no* $f_1, \ldots, f_n$ *in* R^{Ω}, *such that* $\sum_{i=1}^{n} f_i = 0$, *and* $E_i f_i > 0$ *for all* $i \in I$.

The functions f_i, which sum to zero, can be interpreted as a bet among the agents. The condition $(E_i f)(\omega) > 0$ for each state ω amounts to saying that the positivity of $E_i f_i$ is always common knowledge among the agents. Thus, the last corollary is equivalent to the following: a necessary and sufficient condition for the existence of a common prior is that there is no bet for which it is always common knowledge that all agents expect a positive gain.

This result already was proved by Morris (1995) for finite type spaces and independently by Feinberg (1995, 1996) for compact type spaces. It also follows from Bonanno and Nehring (1996) for finite type spaces with two agents.

The convex analysis of priors and common priors is very simple and rather crude. It does not tell us anything about the structure of the sets of priors and common priors; for example, it does not suggest the conditions that guarantee the uniqueness of a common prior. A finer analysis, which takes into account the stochastic nature of type functions (see Samet (1998)), is more telling.

REFERENCES

Bonanno, G., and Nehring, K., (1996). ''Fundamental Agreement: A New Foundation for the Harsanyi Doctrine,'' Working Paper 96-02, University of California, Davis.

Feinberg, Y. (1995). ''A Converse to the Agreement Theorem,'' Discussion Paper 83, Center for Rationality and Interactive Decision Theory, The Hebrew University, Jerusalem.

Feinberg, Y. (1996). ''Characterizing the Existence of a Common Prior via the Notion of Disagreement,'' Ph.D. Thesis, The Hebrew University, Jerusalem.

Morris, S. (1995). ''Trade with Heterogenous Prior Beliefs and Asymmetric Information,'' *Econometrica* **62**, 1327–1347.

Samet, D. (1998). ''Iterated Expectations and Common Priors,'' *Games Econ. Behav.* **24**, 132–142.