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An ordinal solution to bargaining problems with many players ☆

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Abstract

Shapley proved the existence of an ordinal, symmetric and efficient solution for three-player bargaining problems. Ordinality refers to the covariance of the solution with respect to order-preserving transformations of utilities. The construction of this solution is based on a special feature of the three-player utility space: given a Pareto surface in this space, each utility vector is the ideal point of a unique utility vector, which we call a ground point for the ideal point. Here, we extend Shapley's solution to more than three players by proving first that for each utility vector there exists a ground point. Uniqueness, however, is not guaranteed for more than three players. We overcome this difficulty by the construction of a single point from the set of ground points, using minima and maxima of coordinates.

JEL classification: C70; C71; C78

Keywords: Bargaining problems; Ordinal utility; Bargaining solutions

1. Introduction

1.1. Covariance of solutions to bargaining problems

In his seminal paper, Nash (1950) suggested analyzing a bargaining situation by considering the corresponding bargaining problem. The latter consists of two elements: the set of utility vectors that describe the bargainers' utility from possible agreements, and the disagreement point which is the utility vector that corresponds to the outcome in

☆ A PowerPoint presentation of the article is available at http://www.tau.ac.il/~samet/safra-samet-1.pps.

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case no agreement is reached. Nash also suggested the notion of a solution to bargaining problems which is a function that assigns to each bargaining problem a utility vector. In particular, he characterized axiomatically a specific solution, which bears his name. Many other solutions to bargaining problems were subsequently proposed in the literature (see Thomson (1994) for a survey of this literature).

The Nash solution presumes that preferences are represented by Von Neumann–Morgenstern utility functions. Two such utility functions represent the same preferences if one is derived from the other by an affine positive transformation. The presumption of the Nash solution on the utility functions is demonstrated by the following property, which is one of the axioms that characterize this solution. If one bargaining problem is transformed to another by applying affine positive transformations to the players' utility functions, then the Nash solution varies correspondingly. That is, the utility transformations map the solution of the first problem to that of the second. We say that the Nash solution is *covariant* with positive affine transformation of utility.

The egalitarian solution, studied by Kalai (1977), presumes a different type of utility presentation of preferences. This solution is covariant with any order-preserving transformation of utility, the same transformation being applied to each player's utility function. The covariance of the egalitarian solution with the transformations in this group reflects the interpersonal comparison of utilities which underlies it.

The larger the group of transformations with respect to which a solution is covariant, the less the assumptions made on the nature of the presentation of preferences by utility functions. It is natural then to look for a solution which is covariant with respect to the largest possible group of transformations: the group of order-preserving transformations, where different transformations are applied to different players. Such a solution is said to be *ordinal*. For further discussion of the covariance of solutions with utility transformation see Shubik (1982).

1.2. Ordinal solutions

Obviously, there are ordinal solutions. The simplest is the one that assigns to each problem its disagreement point. The shortcoming of this solution is that it fails the efficiency test.

Consider next an efficient ordinal solution. All players except player 1 are bound to their disagreement payoff, while player 1 receives her Pareto payoff—the payoff that makes the new payoff vector lie on the Pareto surface. When order-preserving transformations are applied to the utility functions of the players, this point is transformed to a point of the same nature, that is, to the solution of the transformed problem. But this discriminatory solution is not appealing.

Another ordinal solution is one in which *each* player receives her Pareto payoff. This payoff vector is called by Kalai and Smorodinsky (1975) the ideal-point of the problem. This solution treats all player on an equal footing, but it is infeasible.

Shapley (1969) has shown that there is no ordinal solution to two-player bargaining problems which is also Pareto efficient and non-discriminating. Indeed his proof shows that the only ordinal solutions are the four solutions mentioned above: the two discriminatory ones, the infeasible one, and the inefficient one. Figure 1 sketches the proof.

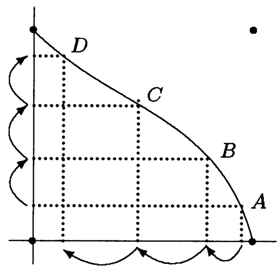

Fig. 1. The four ordinal solutions to two-player problems. The disagreement point is the origin. The curved arrows along the axes depict two order-preserving transformations of the utilities. On each axis the transformation maps the coordinates of the disagreement point and the discriminatory points to themselves. The point A on the Pareto surface moves to B, B moves to C and C to D. The bargaining problem remains the same under the transformations. Therefore the solution for this problem should be a point that is mapped to itself. The only points that are mapped to themselves are depicted by large dots.

For three players there are eight simple, but not attractive, ordinal solutions, similar to the four solutions to the two-player case. But in this case Shapley showed that it is possible to construct an ordinal solution which is also efficient and symmetric.[1]

1.3. Shapley's solution in terms of ideal points

Shapley's construction for three players lends itself to several possible extensions to more players. But so far none has been shown to lead to an ordinal solution. In a recent survey of bargaining theory, Thomson (1994) still reports on an ordinal solution to only three-player problems. Our extension here is based on the description of the three-player construction in terms of ideal points. In Safra and Samet (2001) we show how a different formulation of Shapley's construction leads to another extension for more players. This extension makes use of a solution to gradual bargaining problems introduced by O'Neill et al. (2001). Both extensions are efficient and symmetric.

Given a bargaining problem and a vector of utilities x we denote by $\pi(x)$ the ideal point of x, namely, the utility vector in which each player i gets her Pareto payoff given x_{-i}.[2] The point x is called the *ground* for $\pi(x)$. The ordinal solution suggested here is based on the following simple observation.

> The relation between a ground point and its ideal point is covariant with respect to order-preserving transformations.

That is, order-preserving transformations map a ground point and its ideal point to a pair of points that have the same relationship. This principle guarantees, in particular, that the solution that assigns to each bargaining problem the ideal point of the disagreement

[1] The solution was first documented in Shubik (1982). See also Thomson (1994).

[2] In Kalai and Smorodinsky (1975) the ideal point is an infeasible point defined for a feasible and inefficient disagreement point. In the sequel we make the straightforward extension of this notion for infeasible "disagreement points." See also footnote 3.

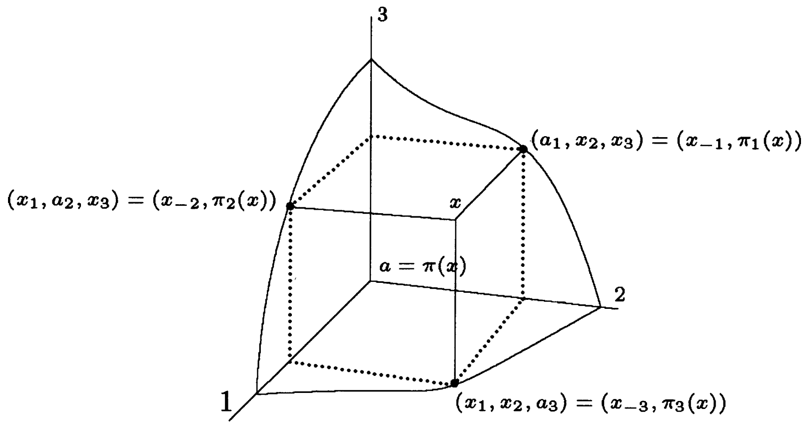

Fig. 2. The first step in Shapley's construction.

point is an ordinal solution. However, this solution does not lead to the construction of an efficient and feasible ordinal solution.

Shapley's construction uses the same principle in the reverse direction. It starts with assigning to a bargaining problem the ground point to the disagreement point of the problem. What makes the construction possible for three-player problems is the following fact.

For any three-player bargaining problem there exists a unique ground point for every utility vector.

In other words, for every y there exists a unique x such that $y = \pi(x)$. In particular, consider a bargaining problem with disagreement point (a_1, a_2, a_3) and a Pareto surface S. Then there exists a unique ground point x for the disagreement point a. Thus, for each i, a_i is the Pareto payoff of i when the other two players are bound to their payoffs at x. By the definition of Pareto payoff this means that the three points, (a_1, x_2, x_3), (x_1, a_2, x_3), and (x_1, x_2, a_3), are in S. The point x is depicted in Fig. 2.

The solution that assigns to each bargaining problem the ground point of the disagreement point is infeasible, but it is non-discriminating, and, most importantly, it is ordinal.

To construct a solution which is also on the Pareto surface of the bargaining problem we note that x is closer in each coordinate to the Pareto surface than the disagreement point a. We now solve the bargaining problem starting with x as a disagreement point.[3] Continuing

[3] Starting with a disagreement point below the Pareto surface results in a point x which is above the Pareto surface. Such a point cannot be interpreted as a disagreement point. It is possible, however, to give some other interpretation to an infeasible initial point of a bargaining problem, or simply treat it as a technical step in constructing the ordinal solution.

this way, we generate a sequence of points that converge to a point on the Pareto surface.[4] This point is the desired solution.

1.4. Extending Shapley's solution

An essential part of this construction is the *existence and uniqueness* of a ground point for any given payoff vector. We prove that existence holds also for more than three players.[5] For this we use the Alexandroff–Pasynkoff Lemma, which is an intersection theorem derived from the Sperner Lemma.

Uniqueness, however, does not necessarily hold for more than three players, as has been demonstrated by example in Sprumont (2000). A first attempt to overcome the multiplicity of ground points might be to choose one ground point for each problem, in a way that is symmetric in the players and covariant with order-preserving transformations. We doubt that there is a way to choose such a point. We solve the multiplicity problem by generating from the set of ground points of a given utility vector y a single point x, which is not necessarily a ground point, as follows. If y is feasible then x_i is the minimum payoff to player i in the set of ground points for y. When y is infeasible, then x_i is the maximum of these payoffs. Since the maximum and minimum functions are order preserving the construction of x is covariant with order preserving transformations of utility. When y has a single ground point, which is always the case for three-player problems, then the point x thus defined is the ground point for x as in Shapley's construction.

The construction follows now the same iterative process as in the case of three-player problem. The choice of the maximum function below the Pareto surface and the minimum above it guarantees that the generated sequence of points converges to the Pareto surface of the bargaining problem.

In Section 2 we present the algebraic properties of Pareto surfaces. We define the two sides of such surfaces in terms of ideal points and prove (rather than assume) the continuity of these surfaces. We formulate the ordinality of ideal points and introduce the properties of the constructed solution: ordinality, efficiency, and symmetry. In Section 3 we describe Shapley's solution and its extension and prove the existence of ground points. The more technical proofs are in Appendix A.

2. Preliminaries

2.1. Pareto surfaces

Consider a finite set N of n players, with $n \geqslant 2$. A point in $\mathbb{R}^N$ describes the utility levels of the players. For $x = (x_i)_{i \in N}$ and $y = (y_i)_{i \in N}$ in $\mathbb{R}^N$ we write $x \geqq y$ when $x_i \geqslant y_i$ for each $i \in N$, $x \geqslant y$ when $x \geqq y$ and $x \neq y$, and $x > y$ if $x_i > y_i$ for each $i \in N$. The inequalities $\leqq$, $\leqslant$, and $<$ are similarly defined. For each proper subset M of N, we denote

[4] Taking the ideal point of the ideal point of ... the disagreement point, rather then the ground points, would not lead to a solution because this sequence is getting *further away* from the surface.

[5] Sprumont (2000) claims without proving that there are such points for four-player problems.

by x_{-M} a point in $\mathbb{R}^{N\setminus M}$. For $x = (x_i)_{i\in N}$ in $\mathbb{R}^N$, the vector x_{-M} is the projection of x on $\mathbb{R}^{N\setminus M}$, i.e., the vector $(x_i)_{i\in N\setminus M}$. When M is a singleton we omit the curly brackets and write x_{-i} and $N \setminus i$.

Definition 1. A subset $S \subset \mathbb{R}^N$ is a *Pareto surface* (a *surface* for short) if the following two conditions hold:

(1) if $x, y \in S$ and $x \geqq y$ then $x = y$;
(2) for each i, the projection of S on $\mathbb{R}^{N\setminus i}$ is $\mathbb{R}^{N\setminus i}$.[6]

Observation 1. *Let S be a Pareto surface. Then for each i and $x \in \mathbb{R}^N$ there is a unique number denoted by $\pi_i^S(x)$ such that $(x_{-i}, \pi_i^S(x)) \in S$. This defines a function $\pi_i^S : \mathbb{R}^N \to \mathbb{R}^i$, which satisfies the following*:

- *π_i^S is strictly decreasing in x_j for $j \neq i$,*
- *π_i^S does not change with x_i.*

We call $\pi_i^S(x)$, i's *Pareto payoff*, at x. Following Kalai and Smorodinsky (1975), the point $\pi^S(x)$ is called the *ideal point for x*. The point x is called the *ground point for $\pi^S(x)$*. In Kalai and Smorodinsky (1975) the ideal point for x was defined for x below S (a relation we define below). Here we use it also for x above S.

We omit the superscript S from π_i^S, when the surface S is clear from the context.

There are only three possibilities for the relation between the points x and $\pi(x) = (\pi_i(x))_{i\in N}$.

Observation 2. *Let S be a Pareto surface. Then for each $x \in \mathbb{R}^N$, the vector $\pi(x)$ satisfies either $x < \pi(x)$, or $x = \pi(x)$, or $x > \pi(x)$.*

Using this proposition, we define the two sides of a surface S.

Definition 2. For a Pareto surface S and $x \in \mathbb{R}^N$,

- if $x < \pi(x)$ we say that x is *below* S and denote it by $x \prec S$ or $S \succ x$,
- if $x > \pi(x)$ we say that x is *above* S and denote it by $x \succ S$ or $S \prec x$.

Obviously, $x = \pi(x)$ iff $x \in S$ in which case we say that x is on S. We write $a \preceq S$ when either $a \prec S$ or $a \in S$. The notations $a \succeq S$, $S \preceq a$, $S \succeq a$ are similarly defined.

The following proposition provides an equivalent alternative criterion for being above or below a surface S, which is sometimes easier to use.

[6] The second condition in Definition 1 is essential in this work: It is necessary for the existence of the Pareto-payoff functions which are defined next. In particular it implies that surfaces are unbounded. This property, however, is not essential. The construction of the solution in this paper can be also carried out for bounded Pareto surfaces. We preferred unbounded surfaces for tractability and simplicity of notation.

Observation 3. *For $x \notin S$, x is above (below) S iff there is $y \in S$ such that $x \geqslant y$ ($y \geqslant x$).*

Our construction hinges on continuity properties that are guaranteed by the definition of Pareto surfaces.

Proposition 1.

- *A Pareto surface is closed.*
- *The functions π_i are continuous.*

2.2. Order-preserving transformations

A strictly increasing continuous function from $\mathbb{R}$ onto $\mathbb{R}$ is called a *scalar order-preserving transformation*. An *order-preserving transformation* is a vector of scalar order-preserving transformations. The order-preserving transformation $\mu = (\mu_i)_{i\in N}$ defines a map from $\mathbb{R}^N$ onto $\mathbb{R}^N$ by $\mu(x) = (\mu_i(x_i))_{i\in N}$. It is easy to see that μ maps Pareto surfaces to Pareto surfaces. That is, for any Pareto surface S, the set $\mu(S) = \{\mu(x) \mid x \in S\}$ is also a Pareto surface. The following key observation states the covariance of π^S with order-preserving transformations.

Observation 4. *Let S be a surface and μ an order-preserving transformation on $\mathbb{R}^N$. Then for each x,*

$$\pi^{\mu(S)}\big(\mu(x)\big) = \mu\big(\pi^S(x)\big). \tag{1}$$

To see this we need to show that for each i and x, $\pi_i^{\mu(S)}(\mu(x)) = \mu_i(\pi_i^S(x))$. Indeed, as $(x_{-i}, \pi_i^S(x)) \in S$, it follows that

$$\big(\mu_{-i}(x_{-i}), \mu_i\big(\pi_i^S(x)\big)\big) \in \mu(S).$$

By definition $(\mu_{-i}(x_{-i}), \pi_i^{\mu(S)}(\mu(x)) \in \mu(S))$, which establishes the required equality by the uniqueness of $\pi_i^{\mu(S)}(\mu(S))$.

2.3. Bargaining problems and solutions

Definition 3. A *bargaining problem* (a *problem* for short) is a pair (a, S), where S is a Pareto surface and $a \in \mathbb{R}^N$.[7] The set of all problems is denoted by $\mathcal{B}$. A *solution* is a function $\Psi : \mathcal{B} \to \mathbb{R}^N$.

Consider the following properties of a solution Ψ to bargaining problems.

[7] The usual definition of a bargaining problems requires that a is a feasible point, i.e., $a \preceq S$. Here we allow a, for convenience, to be infeasible. See footnote 3.

Ordinality: For each problem (a, S) and order-preserving transformation μ,

$$\Psi\big(\mu(a), \mu(S)\big) = \mu\big(\Psi(a, S)\big).$$

Efficiency: For each problem (a, S), $\Psi(a, S) \in S$.

Let τ be a permutation on N. For $x \in \mathbb{R}^N$, τx is the vector defined by $(\tau x)_i = x_{\tau(i)}$. For a surface S, $\tau S = \{\tau x \mid x \in S\}$.

Symmetry: For each problem (a, S) and permutation τ of N,

$$\Psi(\tau a, \tau S) = \tau \Psi(a, S).$$

There are simple ordinal solutions that have only two of these properties. Thus, the solution $(a, S) \to \pi(a)$ is ordinal and symmetric but not efficient. The solution $(a, S) \to (a_{-i}, \pi_i(a_{-i}))$ is ordinal and efficient but not symmetric. The solution $(a, S) \to a$ is both symmetric and ordinal but not efficient. In the next section we construct a solution to bargaining problems with at least three players that is ordinal, efficient, and symmetric.

3. An ordinal solution

3.1. *Shapley's solution for $n = 3$*

Shapley's ordinal solution for three players is based on the following proposition, which we prove in the next subsection.

Proposition 2. *For each Pareto surface S in $\mathbb{R}^3$ and $a \in \mathbb{R}^3$ there exists a unique ground point for a. That is, there exists a unique point x such that $\pi^S(x) = a$.*[8]

The equality in Proposition 2 is equivalent to the list of equalities $\pi_i(x) = a_i$ for $i = 1, 2, 3$. By the definition of π, this means that for each i, $(x_{-i}, a_i) \in S$. In Fig. 2 the unique point x is depicted, for which a is an ideal point.

For a given problem (a, S), define a sequence of points $(a^k)_{k \geqslant 0}$ in $\mathbb{R}^N$, such that $a^0 = a$, and for each $k \geqslant 0$, a^{k+1} is the unique point that satisfies $\pi(a^{k+1}) = a^k$. This sequence converges to a point $\Psi(a, S)$ on S. The solution Ψ thus defined is ordinal, efficient, and symmetric.

3.2. *The existence of ground points*

Sprumont (2000) has shown, by a simple example of four players, that the uniqueness in Proposition 2 is special for the case $n = 3$. We show here that existence holds for any $n \geqslant 3$.

[8] In particular, π^S is a homeomorphism of $\mathbb{R}^3$.

Proposition 3. *For each Pareto surface S in $\mathbb{R}^N$ and $a \in \mathbb{R}^N$, the set of ground points for a, $\{x \mid \pi^S(x) = a\}$, is nonempty and closed.*

To prove it we use a closed-covering theorem, known as the AP Lemma, by Alexandroff and Pasynkoff (1957). See Ichiishi and Idzik (1990) for a discussion of closed-covering theorems and their applications to cooperative game theory.

The AP Lemma. *Let $(A_i)_{i \in N}$ be a family of closed sets that cover the unit simplex in $\mathbb{R}^N$ such that A_i contains the face of the simplex in which $x_i = 0$. Then $\bigcap_{i \in N} A_i \neq \emptyset$.*

We identify a subset of a surface which is homeomorphic to the simplex. For a surface S in $\mathbb{R}^N$ and $a \prec S$ let $S_a = \{x \in S \mid x \geqslant a\}$. For $a \succ S$ let $S_a = \{x \in S \mid x \leqslant a\}$.

Lemma 1. *Let S be a Pareto surface in $\mathbb{R}^N$, and $a \notin S$. Then the map $h(x) = (x - a)/\sum_j (x_j - a_j)$ is a homeomorphism of S_a onto Δ, the unit simplex in $\mathbb{R}^N$. Moreover, for each $j \in N$ the set $\{x \in S_a \mid x_j = a_j\}$ is mapped by h homeomorphically onto the face of Δ where $x_j = 0$.*

Proof of Propositions 2 and 3. For $a \in S$ the two propositions trivially hold, as $\{x \mid \pi(x) = a\} = \{a\}$. Assume, then, that $a \prec S$. The proof for $a \succ S$ is similar.

The condition $\pi(x) = a$ is equivalent to requiring that $(x_{-j}, a_j) \in S$ for each j. We fix a player i, and rewrite this requirement as conditions (2) and (3) below.

$$(x_{-i}, a_i) \in S, \tag{2}$$

$$(x_{-\{i,j\}}, a_j, x_i) \in S, \quad \text{for each } j \neq i. \tag{3}$$

We further rewrite (3) as

$$\pi_i(x_{-\{i,j\}}, a_j, a_i) = x_i, \quad \text{for each } j \neq i. \tag{4}$$

Consider the Pareto surface $T = \{x_{-i} \in \mathbb{R}^{n \setminus i} \mid (x_{-i}, a_i) \in S\}$ in $\mathbb{R}^{N \setminus i}$. By Lemma 1 the set $T_{a_{-i}} = \{x_{-i} \in T \mid x_{-i} \geqslant a_{-i}\}$ is mapped homeomorphically to the unit simplex in $\mathbb{R}^{N \setminus i}$, with the subset $\{x_{-i} \in T_{a_{-i}} \mid x_j = a_j\}$ being mapped homeomorphically to the face of this simplex, where $x_j = 0$.

Define for each $j \neq i$, $\zeta_j : T_{a_{-i}} \to \mathbb{R}^i$ by $\zeta_j(x_{-i}) = \pi_i(x_{-\{i,j\}}, a_j, a_i)$. Then, by (2) and (4) x is a ground point for a iff $x_{-i} \in T$ and all the functions ζ_j for $j \neq i$ coincide at x_{-i} and their common value is x_i. Figure 3 describes, for the case $n = 3$, the set $T_{a_{-i}}$ as well as the value of the functions ζ_j at some point in this set.

Let $A_j = \{x_{-i} \in T_{a_{-i}} \mid \zeta_j(x_{-i}) = \min_{k \neq i} \zeta_k(x_{-i})\}$. Then the sets A_j are closed and $\bigcup_{j \neq i} A_j = T_{a_{-i}}$. Moreover, $\{x_{-i} \in T_{a_{-i}} \mid x_j = a_j\} \subseteq A_j$. Indeed, if $x_j = a_j$, then $\zeta_j(x_{-i}) = \pi_i(x_{-i}, a_i)$. Since π_i is decreasing in x_k, it follows that for any $k \notin \{i, j\}$, $\pi_i(x_{-i}, a_i) \leqslant \pi_i(x_{-\{i,k\}}, a_k, a_i)$. But the latter is just $\zeta_k(x_{-i})$, which shows that $x_{-i} \in A_j$. By Lemma 1 and the AP Lemma there exists a point x_{-i} in $\bigcap_{j \neq i} A_j$. By the definition of the sets A_j the functions ζ_j coincide at x_{-i}. The point (x_{-i}, x_i), where x_i is the common value of the function ζ_j is a ground point for a.

The set $\{x \mid \pi(x) = a\}$ is closed by the continuity of π.

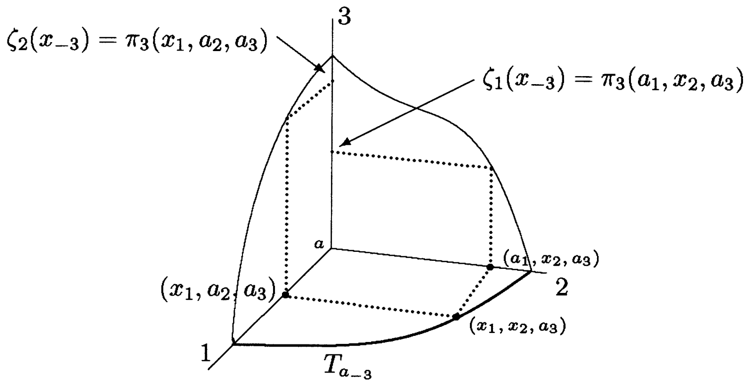

Fig. 3. The existence and uniqueness of a ground point for $n = 3$. In this picture we fix player 3. The set $T_{a_{-3}}$, which is one-dimensional, is depicted by a thick line. The arrows indicate the values of $\zeta_1(x_{-3})$ and $\zeta_2(x_{-3})$ on the axis of player 3. When the point (x_1, x_2, a_3) varies along $T_{a_{-3}}$ from right to left, ζ_1 strictly increases, and ζ_2 strictly decreases. Therefore, there exists a unique point where they coincide.

Consider now the case $N = \{1, 2, 3\}$. Fix player 3. As was shown, x is ground point for (a_1, a_2, a_3) iff $(x_1, x_2) \in T_{a_{-3}}$ and $\zeta_1(x_{-3}) = \zeta_2(x_{-3}) = x_3$. The functions ζ_1 and ζ_2 are strictly decreasing with x_2 and x_1 correspondingly. But along the one-dimensional Pareto surface $T_{a_{-3}}$, x_1 is a strictly decreasing function of x_2. Thus, ζ_2 is strictly increasing with x_1. But then, ζ_1 and ζ_2 can coincide in $T_{a_{-3}}$ in only one point. □

3.3. Constructing the ordinal solution

The uniqueness of x that satisfies $\pi(x) = a$ when $n = 3$ is essential to the construction of the solution in that case.

We overcome the lack of uniqueness for $n > 3$ as follows. For each problem (a, S) we define a solution Φ such that for each player i

$$\Phi_i(a, S) = \begin{cases} \min\{x_i \mid \pi^S(x) = a\} & \text{if } a \prec S, \\ a_i & \text{if } a \in S, \\ \max\{x_i \mid \pi^S(x) = a\} & \text{if } a \succ S. \end{cases}$$

Proposition 4. *The solution Φ is ordinal and symmetric.*

Proof. Obviously, the solution Φ is symmetric. To see that it is ordinal, let μ be an order-preserving map, and suppose that $a \prec S$. Then

$$\begin{aligned} \Phi_i\big(\mu(a), \mu(S)\big) &= \min\{y_i \mid \pi^{\mu(S)}(y) = \mu(a)\} \\ &= \min\{\mu_i(x_i) \mid \pi^{\mu(S)}\big(\mu(x)\big) = \mu(a)\} \\ &= \min\{\mu_i(x_i) \mid \mu\big(\pi^S(x)\big) = \mu(a)\} = \min\{\mu_i(x_i) \mid \pi^S(x) = a\} \\ &= \mu_i\big(\min\{x_i \mid \pi^S(x) = a\}\big) = \mu_i\big(\Phi_i(a, S)\big). \end{aligned}$$

The first and last equalities follow from the definition of Φ. The second holds since μ is on $\mathbb{R}$. The third uses the ordinality of π in Observation 4. The forth equality holds since μ

is one to one. And finally, the fifth holds since the minimum is covariant with scalar order preserving transformations. The case of $a \succ S$ is shown similarly, and the case $a \in S$ is trivial. $\square$

The solution Φ is defined in terms of Φ like Shapley's solution for three players. Define a sequence $(a^k)_{k \geqslant 0}$ by $a^0 = a$ and for each $k \geqslant 0$, $a^{k+1} = \Phi(a^k, S)$.

Proposition 5. *For each problem (a, S) in $\mathbb{R}^N$, with $n \geqslant 3$, the sequence $(a^k)_{k \geqslant 0}$ converges to a point in S.*

Two remarks concerning this convergence are in order. Note, first that the existence of ground points holds also for $n = 2$. It is the convergence in Proposition 5 that holds only for $n \geqslant 3$, which explains why there is no ordinal, efficient symmetric solution for $n = 2$. More specifically, it is the sharp inequalities in Lemma 3 in Appendix A which hold only for $n \geqslant 3$.

Second, we note that the convergence in the case $n = 3$ has a special feature. It is simple to show that for any point $x \notin S$, $\pi^S(x)$ and x are on different sides of S. Therefore, for three-player problems, starting with $a_0 \prec S$, all the even elements of the sequence (a^k) are below S, and all the odd ones are above S. This must not be the case for $n > 3$, since x and $\Phi(x, S)$ may be on the same side of S.

Theorem 1. *The solution $\Psi(a, S) = \lim a^k$ is efficient, ordinal, and symmetric.*

Proof. Efficiency is guaranteed by Proposition 5. Symmetry is obvious from the symmetry of Φ. To prove ordinality observe that $\mu(a^{k+1}) = \mu(\Phi(a^k, S)) = \Phi(\mu(a^k), \mu(S))$. The first inequality follows from the definition of the sequence a^k. The second from the ordinality of Φ. But then, by the definition of Ψ, $\lim \mu(a^k) = \Psi(\mu(a), \mu(S))$. By the continuity of μ, $\lim \mu(a^k) = \mu(\lim a^k) = \mu(\Psi(a, S))$. $\square$

Appendix A

Proof of Observation 1. The existence of the number $\pi_i^S(x)$ is stated in part (2) of Definition 1. The uniqueness follows from part (1) of this definition.

By the definition of π_i it does not change with x_i. Suppose that for $j \neq i$, $x'_j > x_j$. Since both $(x_{-\{i,j\}}, x'_j, \pi_i(x'_j, x_{-j}))$ and $(x_{-\{i,j\}}, x_j, \pi_i(x_j, x_{-j}))$ are in S, it follows by property 1 that $\pi_i(x'_j, x_{-j}) < \pi_i(x_j, x_{-j})$. $\square$

Proof of Observation 2. Obviously, if $x \in S$, then $\pi(x) = x$. Suppose that for some i, $\pi_i(x) > x_i$. By definition $y = (x_{-i}, \pi_i(x)) \in S$. Similarly, for $j \neq i$, $z = (x_{-j}, \pi_j(x)) \in S$. Now, $z_i < y_i$, and for each $k \notin \{i, j\}$, $z_k = y_k$. Therefore, by property 1 of Pareto surfaces, $z_j > y_j$, which means that $\pi_j(x) > x_j$. The proof for the case $\pi_i(x) < x$ is similar. $\square$

Proof of Observation 3. If $x \succ S$, then by Observation 2, $\pi(x) < x$. The point $y = (x_{-i}, \pi(x))$ is in S and $x \geqslant y$. Conversely, suppose $x \geqslant y$ for y in S. Then $x \geqslant y = \pi(y) \geqslant \pi(x)$ which shows that $x \succ S$. The proof of the other half of the proposition is similar. $\square$

Proof of Proposition 1. We observe first that for each i, and number a_i, the set $S_{-i} = \{x_{-i} \mid (x_{-i}, a_i) \in S\}$ is a Pareto surface in $\mathbb{R}^{N\setminus i}$.

We prove, by induction on n, that if $a \prec S$ then there exists c such that $a < c \in S$.

Let $n = 2$ and let $b = (\pi_1(a), a_2)$. Then $b_1 > a_1$. Choose c_1 such that $b_1 > c_1 > a_1$. By part (2) of Definition 1, there exists c_2 such that $c = (c_1, c_2) \in S$. Thus, by part (1) of Definition 1 $b \not\geqslant c$, and therefore $c_2 > b_2$. Hence, $c > a$.

Suppose we proved our claim for $n - 1$, and let N be of size n. Since $a \prec S$ it follows that $a_{-i} \prec S_{-i}$, and hence by the induction hypothesis, there exists $b_{-i} \in S_{-i}$, such that $b_{-i} > a_{-i}$. Let $c_{-i} = (a_{-i} + b_{-i})/2$. Then $b_{-i} > c_{-i} > a_{-i}$. There exists a number c_i such that the point $c = (c_{-i}, c_i) \in S$. Since $(b_{-i}, a_i) \in S$, it follows that $(b_{-i}, a_i) \not\geqslant c$, which implies that $c_i > a_i$. Hence, $c > a$.

We can similarly prove that if $a \succ S$, then there exists c such that $a > c \in S$.

Assume now that a is an accumulation point of S. Suppose $a \prec S$. Then, there is c such that $a < c \in S$. But then there exists a point $d \in S$ close enough to a such that $d < c$, which contradicts condition (1) in Definition 1. A similar contradiction follows if $a \succ S$. Thus $a \in S$.

Consider the function $\hat{\pi}_i : \mathbb{R}^{N\setminus i} \to \mathbb{R}^i$ defined by $\hat{\pi}_i(x_{-i}) = \pi_i(x_{-i}, x_i)$. Since π_i is independent of x_i, $\hat{\pi}_i$ is well defined and the equality holds for any x_i. The graph of $\hat{\pi}_i$ is S. As $\hat{\pi}_i$ is decreasing, it is bounded in the neighborhood of each point, and therefore the closedness of S implies the continuity of $\hat{\pi}_i$ and therefore that of π_i. □

Proof of Lemma 1. Since $a \notin S$ the denominator does not vanish on S and h is well defined and continuous. Suppose $h(x) = h(y)$ and assume without loss of generality that $\sum_j (x_j - a_j) \geqslant \sum_j (y_j - a_j)$, then for each i, $x_i - a_i \leqslant y_i - a_i$ and therefore $x = y$. Finally, to see that h is on let y be a point in Δ with $y_j > 0$. Consider the function $f(t) = \pi_i(a + ty) - (a_i + ty_i)$. Suppose that $a \prec S$. Then $f(0) > 0$. For big enough t, $a + ty \geqslant (a_{-j}, \pi_j(a)) \in S$. Therefore for such t, $a + ty \succ S$ and therefore $f(t) < 0$. Thus for some $\bar{t}$, $f(t) = 0$ and hence $a + \bar{t}y \in S$. It is easy to see that $h(a + \bar{t}y) = y$. The proof for the case $a \succ S$ is similar. It is straightforward to show the homeomorphism of the faces of the simplex to the said sets. □

The following two lemmas are used to prove the convergence in Proposition 5.

Lemma 2. *If $a \prec S$ ($a \succ S$) and $\pi(x) = a$, then $a < x < \pi(a)$ ($a > x > \pi(a)$).*

Proof. We prove the case $a \prec S$. The other case is similarly proved. If to the contrary $a \not< x$ then by Observation 2 $a \geqq x$. Hence, $\pi(a) \leqq \pi(x)$, and thus $\pi(a) \leqq a$ which by Observation 2 contradicts the assumption $a \prec S$. To show that $x_i < \pi_i(a)$ for each i, choose $j \neq i$. By what we have shown, $a \leqslant (x_{-j}, a_j)$ with strict inequalities for all $k \neq j$. Also, since $\pi(x) = a$, $(x_{-j}, a_j) \in S$. Since π_i is decreasing in all $k \neq i$ and there are players other than i and j, as $n \geqslant 3$, it follows that $\pi_i(a) > \pi_i(x_{-j}, a_j) = x_i$. □

The following proposition indicates the sense in which $\Phi(a, S)$ is closer to the surface S than a.

Lemma 3. *If $a \prec S$ then $a < \Phi(a,S) < \pi(a)$ and $\pi(\Phi(a,S)) \geqq a$.*
If $a \succ S$ then $a > \Phi(a,S) > \pi(a)$ and $\pi(\Phi(a,S)) \leqq a$.

Proof of Lemma 3. Suppose $a \prec S$. The proof for $a \succ S$ is similar. Since the inequalities in Lemma 2 hold for every point in $\{x \mid \pi(x) = a\}$ and this set is closed, it holds also if we replace x by $\Phi(a,S)$. If x satisfies $\pi(x) = a$, then $x \geqq \Phi(a,S)$ and hence $\pi(\Phi(a,S)) \geqq \pi(x) = a$. □

Proof of Proposition 5. If a point a^k is in S, then $a^m = a^k$ for all $m > k$ and we are done. Assume, then, that none of the points of the sequence are in S.

Claim 1. *The subsequence of $(a^k)_{k \geqslant 0}$ of all the points below S (above S) is strictly increasing (decreasing) and bounded, and therefore converges to a point b (c).*

If $a^k, a^{k+1} \prec S$, then $a^{k+1} = \Phi(a^k, S) > a^k$, by Lemma 3, and similarly for a^k, $a^{k+1} \succ S$, $a^{k+1} < a^k$. Thus, as long as the sequence a^k stays on the same side it remains monotonic: increasing while below S and decreasing while above it.

Suppose a^k, $a^{m+1} \prec S$ with $m > k$, while all the points $a^{k+1}, \ldots, a^m$ are above S. As we showed, if there is more than one of them, these points are strictly decreasing. As $a^m \succ S$, $a^{m+1} > \pi(a^m)$ by Lemma 3. As π is decreasing $\pi(a^m) \geqq \pi(a^{k+1})$ (the inequality is weak, since possibly $m = k+1$). Again by Lemma 3, $\pi(a^{k+1}) \geqq a^k$. These three inequalities amount to $a^{m+1} > a^k$. Thus, returning to the side below S after an incursion to the other side, ends at a point greater than the last one below S.

To see that this subsequence is bounded, let $a^k \prec S$, then $a^k < \pi(a^k) < \pi(a^{k_1})$ where a^{k_1} is the first point below S. The proof for the points above S is similar.

Claim 2. *$b = c$ and this point is in S.*

Obviously $b \preceq S$, since for any i, $\pi_i(b) - b_i = \lim_l(\pi_i(a^{k_l}) - a_i^{k_l}) \geqslant 0$, where (a^{k_l}) is the subsequence of points below S. Similarly, $c \succeq S$.

Case 1. There are only finitely many points of (a^k) above S.

In this case, $a^k \to b$. Fix player i. Since $\Phi_i(a^k, S) = a_i^{k+1}$, there exists x^k such that $\pi(x^k) = a^k$ and $x_i^k = a_i^{k+1}$. By Lemma 2, $a^k < x^k < \pi(a^k)$. By the monotonicity of a^k, $a^{k_0} \leqslant a^k$ and $\pi(a^k) \leqslant \pi(a^{k_0})$, for some k_0 and all $k > k_0$. Thus, (x^k) is uniformly bounded from above and below. Assume, without loss of generality, that $x^k \to x$. Thus, $a^k = \pi(x^k) \to \pi(x)$ and hence $\pi(x) = b$. Moreover, $a_i^{k+1} = x_i^k \to x_i$ and therefore $x_i = b_i$. Hence, $\pi_i(x) = b_i = x_i$ which implies that $x \in S$. Thus, $\pi(x) = x$, and therefore, $b = x \in S$. The proof is similar for the case when there are only finitely many a^k below S.

Case 2. There are infinitely many points of (a^k) on both sides of S.

We note that by Lemma 3, if $a^k \prec S$, then $a^{k+1} = \Phi(a^k, S) < \pi(a^k)$, and $\pi(a^{k+1}) = \pi(\Phi(a^k, S)) \geqq a^k$, and if $a^k \succ S$, then $a^{k+1} = \Phi(a^k, S) > \pi(a^k)$, and $\pi(a^{k+1}) = \pi(\Phi(a^k, S)) \leqq a^k$.

For infinitely many k, $a^k \prec S \prec a^{k+1}$. Thus by the above inequalities, $c \leqq \pi(b)$, and $\pi(c) \geqq b$. Also, for infinitely many k, $a^{k+1} \prec S \prec a^k$, and therefore, $b \geqq \pi(c)$, and $\pi(b) \leqq c$. Thus, $\pi(b) = c$ and $\pi(c) = b$. But then it is impossible that $b \prec S$, because in this case $c < \pi(b)$ by Lemma 2. Hence, $b \in S$ and $c = \pi(b) = b$. □

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