Games and Economic Behavior 135 (2022) 1–15

Contents lists available at ScienceDirect

Games and Economic Behavior

www.elsevier.com/locate/geb

Non-Bayesian correlated equilibrium as an expression of non-Bayesian rationality

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ARTICLE INFO

Article history:
Received 15 April 2021
Available online 18 May 2022

Keywords:
Dominance rationality
Strict dominance
Weak dominance
Common knowledge of rationality
Correlated equilibrium

ABSTRACT

We study new non-Bayesian solutions of games in strategic form, based on four notions of dominance: weak or strict domination by either a pure or a mixed strategy. For each of these types of dominance, d, we define a family of sets of strategy profiles, called *d-correlated equilibria*. We study the structure and properties of these families. A player is *d-dominance rational* when she does not play a strategy that is d-dominated relative to what she knows about the play of the other players. A set of profiles is a d-correlated equilibrium if and only if it is the set of profiles played in a model where d-dominance rationality is commonly known. When d denotes strict domination by a mixed strategy, a set of profiles is a d-correlated equilibrium if and only if it is the set of profiles played in a model where Bayesian rationality is commonly known.

1. Introduction

1.1. Non-Bayesian models

Various notions like rationality, solutions, and models are said to be Bayesian when they involve probabilistic beliefs of agents about uncertain events. In particular, in game theory such beliefs apply to the strategies that other players employ. These notions are said to be non-Bayesian when probabilistic beliefs are not assumed and used. A Bayesian agent can rank her strategies according to the expected utility they yield with respect to her probabilistic beliefs about the strategies of the other players. The agent is Bayesian rational if she plays a strategy which is maximal according to this ranking. Similarly, a non-Bayesian agent can partially rank her strategies by dominance, where a dominated strategy ranks lower than the strategy that dominates it. The agent is non-Bayesian rational when she plays a strategy that is maximal in this partial ranking, that is, when it is undominated.

We propose new non-Bayesian solutions for games in strategic form and characterize them by common knowledge of non-Bayesian rationality. We consider four types of dominance and label them as in Hillas and Samet (2020): strict dominance by a mixed strategy (sm) strict dominance by a pure strategy (sp), weak dominance by a mixed strategy (wm), and weak dominance by a pure strategy (wp). We refer to these types of dominance as d-dominance, for d in $\{sm, sp, wm, wp\}$. We define four solutions of *non-Bayesian correlated equilibrium*, one for each type of dominance. These solutions can be viewed as the non-Bayesian analogue of the Bayesian notion of correlated equilibrium in Aumann (1987).

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1 Samet acknowledges financial support of the Israel Science Foundation, grant 722/18.

https://doi.org/10.1016/j.geb.2022.05.005
0899-8256/

We assume that the payoffs in the games we study are given by a von Neumann and Morgenstern (1947) utility function. The von Neumann-Morgenstern utility of a lottery is the expected utility with respect to the lottery. This allows us to consider mixed strategies which are lotteries over the pure strategies of the player. Note, however, that these lotteries, as well as the lotteries on prizes which are required to derive the von Neumann-Morgenstern utility function, are objective physical probabilities, like those that are realized in a roulette. They are to be distinguished from subjective probability functions. The derivation of the latter requires a heavy machinery like the one used by Savage (1954). That objective probabilities are simpler than subjective probability is demonstrated in Anscombe and Aumann (1963). They use lotteries over prizes as a primitive which they use to derive subjective probability. Thus the non-Bayesian approach, can keep objective probability without the use of subjective probability. Even when we assume von Neumann-Morgenstern utilities we can justify domination by pure strategies alone if we assume the player does not have access to a randomizing device that will enable her to play mixed strategies. But for dominance by pure strategies we could assume instead that the utility function is ordinal.

1.2. *d*-Correlated equilibrium

For a given game, for each notion of d-dominance, a *d-correlated equilibrium* for this game is a subset C of strategy profiles that we describe analogously to a (Bayesian) correlated equilibrium. Consider a mediator who chooses a strategy profile s from the set C. For each i, the mediator recommends s_i to Player i. We denote by $T_{-i}(s_i)$ the strategy profiles in C of the players other than i that can be recommended to them when s_i is recommended to Player i. For C to be a d-correlated equilibrium we require that for each i and each strategy s_i recommended to Player i, there is no strategy of Player i that d-dominates s_i, given that the other players follow the recommendation, and so play a strategy profile in $T_{-i}(s_i)$. The set is 'correlated' in the sense that it may not be a product of sets of individual strategies. It is an 'equilibrium' because a player who was recommended to play s_i has no reason, in terms of d-domination, to deviate from the recommended strategy when the other players follow the recommendations they received.

1.3. Characterization of *d*-correlated equilibrium

We characterize the four solutions of d-correlated equilibrium by common knowledge of d-dominance rationality. A player is *d-dominance rational* if she does not play a strategy that is d-dominated given what she knows about the strategies played by the other players. To formalize knowledge and common knowledge we use a standard *knowledge structure*, defined by Aumann (1976). It consists of a state space and a partition for each player that defines the events that the player knows in each state. A *model of the game* is a knowledge structure with an association of a strategy profile to each state in the state space, the strategy profile that is played in that state. Thus, in a model of a game we can describe the knowledge players have about the profiles played. This in turn, makes it possible to identify the event that a player is d-dominance rational and finally, the event that d-dominance rationality is commonly known.

Our main result is that a set of profiles C is a d-correlated equilibrium if and only if there exists a model of the game in which d-dominance rationality of the players is commonly known, and C is the set of the strategy profiles played in the states of the model.

1.4. Local and global characterizations

A comparison is called for to Hillas and Samet (2020), which also studies the implications of common knowledge of d-dominance rationality. There we described four processes of elimination of strategy profiles, one for each d, extending a procedure suggested by Stalnaker (1994).[2] We showed that a strategy profile survives the elimination process for d-dominance if and only if it is played in some state in a model of the game at which the players commonly know that they are d-dominance rational. Since this characterization involves what happens at a particular state we can label it as *local*. We also proved a *global* result, one that involves the whole model. As is shown in the proof of Theorem 1 in Hillas and Samet (2020), there exists a model of the game in which there is common knowledge of d-dominance rationality and such that the set of strategy profiles played in the model is the set of all strategy profiles that survive the process of elimination for d. The local result is behavioral. It states what players *play* in a state. The global result has epistemic content: If C is the set of all the strategy profiles that are played in the states of the model, then the players not only *play* a strategy profile in C, but they *commonly know* that the strategy profile they play is in C.

Here we study the global question in full and characterize the sets of strategy profiles C that are played in a model of the game in which the players commonly know that they are d-dominance rational. In other words, we ask what are the sets C that players can commonly know that the profile strategy they play is in C, when d-dominance rationality is commonly known.

[2] A comparison of Hillas and Samet (2020) to Stalnaker (1994) is found in the introduction of Hillas and Samet (2020).

1.5. Properties of *d*-correlated equilibria

The family of d-correlated equilibria is shown to be closed under unions and therefore has a maximal member. The maximal d-correlated equilibrium provides the link between the local result and the global one: it consists of all the profiles that can be played when d-dominance rationality is commonly known, namely the strategy profiles that survive the elimination process corresponding to d, described in Hillas and Samet (2020).

All versions of d-correlated equilibrium are weaker notions than correlated equilibrium. That is, the support of a correlated equilibrium is a d-correlated equilibrium for each type of domination d. We show by example that the inclusion may be strict, that is, there are games in which there are d-correlated equilibria that are not the support of any correlated equilibrium.

Finally, we address the global question that has not been dealt with in the literature. Namely, what are the sets of strategy profiles that are played in a model where players are Bayesian rational in each state. Although we deal with non-Bayesian rationality the question belongs here because the answer to the question can be given in terms of dominance rationality; these sets of profiles are the sm-correlated equilibria.

2. Literature review

The first study of the implications of common knowledge of rationality is Aumann (1987), Aumann defined a model of a game that we utilize and describe in Section 8. He assumed that there exists a common prior from which the beliefs of each player are derived by conditioning on the player's knowledge. Aumann showed that when Bayesian rationality is commonly known in all the states, then the distribution of the strategy profiles induced by the common prior is a correlated equilibrium. This result is global in the sense that it deals with the whole model and not with a specific state. Using the same model as Aumann (1987), Aumann and Dreze (2008) dealt with a local question of the possible beliefs that players can have in a particular state.

The novelty in Aumann (1987), as he states it, is the doing away with the dichotomy between the Bayesian approach and the game theoretic approach. According to the first approach, every event, even one that describes the strategies played by the players, should be assigned a probability. According to the second approach, games should be analyzed using some theory of equilibrium that reflects the players' rationality, prior to any assignment of probabilities. Aumann (1987) showed that the game theoretic notion of correlated equilibrium emerges from the Bayesian description of the players, who act rationally given their information and beliefs. In this respect, our paper follows in the steps of Aumann (1987) by doing away with the dichotomy between the epistemic approach that specifies the *knowledge* of the players, even with respect to what the players are actually playing, and the game theoretic approach that studies notions of equilibrium that precede any specification of the knowledge of the players. The d-correlated equilibrium concepts are defined in terms of the game itself, without reference to the players' knowledge. We show, however, that they emerge from a description of the knowledge of players who act rationally with respect to what they know. The analogy between this paper and Aumann's is the reason we adopted the non-Bayesian analogy of Aumann's title.

When players do not have a common prior then we cannot ask what is *the* distribution of strategy profiles played in a model. We can ask though what are the possible sets of strategy profiles that can be played in a model when there is a common knowledge of Bayesian rationality. Obviously, we want a characterization of these sets in terms of the game itself, as is done in Aumann (1987) and as we do here, and not in terms of the model of the game. This is done in Section 8.

The local question concerning the characterization of the strategy profiles that can be played in a state when dominance rationality is commonly known, has been studied in a number of works. Aumann (1995) and Aumann (1998) used non-Bayesian models to study games in extensive form with complete information. Chen et al. (2007) showed that for sp-dominance rationality these profiles are those that survive iterative elimination of strategies that are strictly dominated by pure strategies. The case of wp-dominance rationality was studied by Bonanno (2008) and Bonanno and Tsakas (2018), for common belief.[3] Trost (2013) studied sp-dominance rationality for common knowledge. In Hillas and Samet (2020) we offered a unified approach, by characterizing, uniformly for all d, the set of profiles that can be played when d-dominance rationality is common knowledge. That characterization is made in terms of iterative elimination of *d-flaws*, a notion that extends the notion of inferior profiles defined by Stalnaker (1994).

The local question for Bayesian rationality was studied by Tan and Werlang (1988), and Brandenburger and Dekel (1987). The belief model in Tan and Werlang is the universal type space of Mertens and Zamir (1985). Brandenbuger and Dekel considered a posteriori equilibrium, following Aumann (1974), which is equivalent to the standard Bayesian model of the game used here except for differences in interpretation and terminology. Both papers showed that the strategy profiles that can be played when Bayesian rationality is commonly known, are the correlatedly rationalizable profiles, which are also the profiles that survive iterative elimination of strategies that are strictly dominated by mixed strategies. Correlatedly rationalizable profiles are those that survive iterative elimination of strategies that are not best response to some belief of the players. They are a generalization of the rationalizable profiles of Bernheim (1984) and Pearce (1984), in which each

[3] Bonanno and Tsakas (2018) showed that common belief characterizes the elimination process corresponding to wp-dominance when belief satisfies the axiom of truth, that is, when belief is indeed knowledge. Moreover, they showed that this does not hold for common belief proper.

player is allowed to have correlated beliefs about the strategy profiles of the other players. Lemma 3 of Pearce (1984) provides the essential tool linking Bayesian rationality to dominance rationality.

3. Basic definitions and notation

3.1. The game

Let G be a game with a finite set of players, I, and, for each i, a finite set, S_i, the strategies of Player i. The set of strategy profiles is $S = \times_i S_i$, and the set of the profiles of Player i's opponents is $S_{-i} = \times_{j \neq i} S_j$. The payoff function for Player i is $h_i \colon S \to \mathbb{R}$. We let Σ_i be the set of Player i's mixed strategies, and $\Sigma = \times_{i \in I} \Sigma_i$. Depending on context, s_i denotes either an element of S_i or the projection on S_i of a profile s in S. Similarly, again depending on context, s_{-i} denotes either an element of S_{-i} or the projection on S_{-i} of a profile s in S. As usual, for each i, we extend h_i from S to Σ by taking expectations.

3.2. Knowledge structures

To express the knowledge and common knowledge of the players, we use a *knowledge structure* which is a pair (Ω, Π), where Ω is a finite set called the *state space*, and $\Pi = (\Pi_i)_{i \in N}$ is a profile of partitions of Ω. For $\omega \in \Omega$, we denote by $\Pi_i(\omega)$, the element of Player i's partition that contains ω. Subsets of Ω are called *events*. We say that Player i *knows* an event E at a state ω if $\Pi_i(\omega) \subseteq E$. Thus, the event that i knows E is $\{\omega \mid \Pi_i(\omega) \subseteq E\}$ which is the union of all elements of Π_i contained in E.

The *meet* of the partitions Π_i is the partition Π^m which is the finest among all partitions that are coarser than each Π_i. Like the partitions of the players, the partition Π^m defines knowledge which we call *common knowledge*. Thus, we say that the event E is *commonly known* at ω if $\Pi^m(\omega) \subseteq E$. Hence, the event that E is commonly known is the union of all the elements of the meet, Π^m, that are contained in E. We refer to the meet also as the *common knowledge partition*.

3.3. Models of the game

In order to describe rationality by an event in a knowledge structure, we need the behavior of the players to be described by events. This is done in a *model of the game* G which is a triplet $(\Omega, \Pi, \mathbf{s})$, where (Ω, Π) is a knowledge structure, and $\mathbf{s} \colon \Omega \to S$ specifies the strategy profile that is played in each of the states.

The strategy played by Player i in each state is given by the function $\mathbf{s}_i \colon \Omega \to S_i$, defined by $\mathbf{s}_i(\omega) = (\mathbf{s}(\omega))_i$. The event that Player i plays the strategy $\mathbf{s}_i(\omega)$, is $\{\omega' \mid \mathbf{s}_i(\omega') = \mathbf{s}_i(\omega)\}$. We assume that each player knows at each state which strategy she plays. That is, for each ω, $\Pi_i(\omega) \subseteq \{\omega' \mid \mathbf{s}_i(\omega') = \mathbf{s}_i(\omega)\}$. This means that for each state ω, Player i plays the same strategy in all the states in $\Pi_i(\omega)$. For any event E we write $\mathbf{s}(E)$ for $\{\mathbf{s}(\omega) \mid \omega \in E\}$ and $\mathbf{s}_{-i}(E)$ for $\{\mathbf{s}_{-i}(\omega) \mid \omega \in E\}$.

Let $T_{-i} = \mathbf{s}_{-i}(\Pi_i(\omega))$ be the set of strategy profiles played in $\Pi_i(\omega)$ by players other than Player i. Thus, $\Pi_i(\omega) \subseteq \{\omega' \mid \mathbf{s}_{-i}(\omega') \in T_{-i}\}$ and therefore, Player i knows at ω that the other players play a strategy profile in T_{-i}. Moreover, for each $t_{-i} \in T_{-i}$ there exists $\omega' \in \Pi_i(\omega)$ such that $\mathbf{s}_{-i}(\omega') = t_{-i}$, and thus, at ω, for each element of T_{-i}, Player i cannot exclude the possibility that that element is played.

4. Dominance and *d*-correlated equilibrium

We define four types of domination: strict domination by a pure strategy, strict domination by a mixed strategy, weak domination by a pure strategy, and weak domination by a mixed strategy. We denote these four types of domination by the abbreviations *sp*, *sm*, *wp*, and *wm* and use d to refer to any of these abbreviations.

Definition 1 *(relative domination).* Let T_{-i} be a nonempty subset of S_{-i}. (We do not assume that T_{-i} is a product set $\times_{j \neq i} T_j$.)

Strict domination. A mixed strategy σ_i in Σ_i *strictly dominates* s_i *in* S_i *relative to* T_{-i} if $h_i(\sigma_i, t_{-i}) > h_i(s_i, t_{-i})$ for all t_{-i} in T_{-i}, in which case we also say that s_i is *sm-dominated by* σ_i *relative to* T_{-i}. If σ_i is a pure strategy, we also say that s_i is *sp-dominated by* σ_i *relative to* T_{-i}.

Weak domination. A mixed strategy σ_i in Σ_i *weakly dominates* s_i *in* S_i *relative to* T_{-i} if $h_i(\sigma_i, t_{-i}) \geq h_i(s_i, t_{-i}))$ for all t_{-i} in T_{-i}, and the inequality is strict for at least one t_{-i} in T_{-i}, in which case we also say that s_i is *wm-dominated by* σ_i *relative to* T_{-i}. If σ_i is a pure strategy, we also say that s_i is *wp-dominated by* σ_i *relative to* T_{-i}.

For each type of domination d we define certain subsets of strategy profiles as d-correlated equilibria.

Definition 2 *(d-correlated equilibrium).* A non-empty set of strategy profiles $C \subseteq S$ is a *d-correlated equilibrium* if, for each s in C and each Player i, s_i is not d-dominated relative to $\{t_{-i} \mid (s_i, t_{-i}) \in C\}$.

We illustrate d-correlated equilibria in a simple example.

		Player 2	
		L	R
Player 1	T	$1,1$	$1,1$
	B	$1,1$	$0,0$

Fig. 1. $s*$-correlated equilibrium may differ from $w*$-correlated equilibrium.

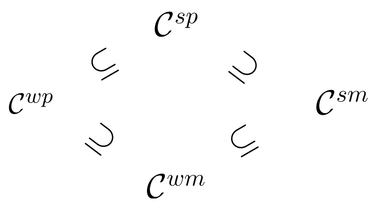

Fig. 2. Inclusion relations between families of d-correlated equilibria.

Example 1. Consider the game given in Fig. 1. Since each player has only two strategies, domination by pure strategies and domination by mixed strategies are the same. So we refer to $w*$-correlated equilibria and $s*$-correlated equilibria where $*$ can be either p or m.

The set of profiles $C = \{(T, L), (T, R), (B, L)\}$ is a d-correlated equilibrium for all d. If Player 1 is recommended to play B then the set of strategies that might have been recommended to Player 2 is $\{L\}$ and, relative to this set, strategy B is not d-dominated by T. If Player 1 is recommended to play T, then the set of strategies that might have been recommended to Player 2 is $\{L, R\}$ and, relative to this set, T is not d-dominated by B. The same considerations apply symmetrically to Player 2 and so C is a d-correlated equilibrium for all d.

The set $C = \{(T, R), (B, L), (B, R)\}$ is an $s*$-correlated equilibrium but *not* a $w*$-correlated equilibrium. When Player 1 is recommended to play B then the set of strategies that might have been recommended to Player 2 is $\{L, R\}$ and, relative to this set, the recommended strategy B is $w*$-dominated by T. Therefore C is not a $w*$-correlated equilibrium. However, B is not $s*$-dominated by T. When Player 1 is recommended to play T then the set of strategies that might have been recommended to Player 2 is $\{R\}$ and, relative to this set T is not $s*$-dominated. Again, the same considerations apply symmetrically to Player 2. Thus C is an $s*$-correlated equilibrium but not a $w*$-correlated equilibrium.

Similarly the set C of all four strategy profiles is an $s*$-correlated equilibrium but not a $w*$-correlated equilibrium.

It is easy to see that the three Nash equilibria, $\{(T, L)\}$, $\{(T, R)\}$, and $\{(B, L)\}$ are the only singletons that are d-correlated equilibria for any d, and these are d-correlated equilibria for all d. Any subset of the set containing these three singletons is a d-correlated equilibrium for all d.

We denote by $\mathcal{C}^d$ the family of all d-correlated equilibria. If a strategy is strictly dominated relative to some set it is also weakly dominated relative to the same set and if a strategy is dominated by a pure strategy relative to some set it is also dominated by a mixed strategy, namely the one that puts probability one on that pure strategy. Thus both $\mathcal{C}^{wp}$ and $\mathcal{C}^{sm}$ are nested by set inclusion between $\mathcal{C}^{wm}$ and $\mathcal{C}^{sp}$ (see Fig. 2).

Example 1 shows that two of the inclusions shown in Fig. 2 may be strict, namely that $\mathcal{C}^{wp} \subsetneq \mathcal{C}^{sp}$ and $\mathcal{C}^{wm} \subsetneq \mathcal{C}^{sm}$. Indeed, the four strategy profile d-correlated equilibrium is in $\mathcal{C}^{sp} = \mathcal{C}^{sm}$ but not in $\mathcal{C}^{wp} = \mathcal{C}^{wm}$. In Example 3 in Section 9, we show that the other two inclusions may also be strict, namely, $\mathcal{C}^{sm} \subsetneq \mathcal{C}^{sp}$ and $\mathcal{C}^{wm} \subsetneq \mathcal{C}^{sm}$. It is possible to use those examples to construct a single game in which all the inclusions are simultaneously strict.

5. Dominance rationality and *d*-correlated equilibrium

For each type of domination d we also define a notion of dominance rationality in models of the game.

Definition 3 *(d-dominance rationality).* Player i is *d-dominance rational* at state ω if the strategy she plays at ω is not d-dominated relative to the set of profiles of the other players' strategies that she considers possible at ω. That is, there is no strategy of hers that d-dominates $\mathbf{s}_i(\omega)$ relative to the set $\mathbf{s}_{-i}(\Pi_i(\omega))$.

We denote by R_i^d the event that Player i is d-dominance rational, and by $R^d = \cap_i R_i^d$ the event that all players are d-dominance rational.

We want to characterize the set C of strategy profiles that are played in all the states of the event Ω' that the players commonly know R^d. Obviously, in this event the players commonly know that the played profile is in C. By the definition of common knowledge, Ω' is a union of elements of the common knowledge partition, the meet. Thus, $(\Omega', \Pi', \mathbf{s}')$, where Π_i' consists of the elements of Π_i contained in Ω', and $\mathbf{s}'$ is the restriction of $\mathbf{s}$ to Ω', is a model of the game G. Moreover, in this model all the players are rational in each state. Thus, the question of which profiles are played in *events* that the players commonly know R^d is equivalent to the question of which profiles are played in a *model* of the game in which players are rational in *each* state. The answer to this question is given in the next theorem.

Theorem 1. *For all d, a non-empty set of strategy profiles C is a d-correlated equilibrium if and only if* $C = \mathbf{s}(\Omega)$ *in some model of G where* $\Omega = R^d$.

Proof. We shall prove the result for d in $\{wp, wm\}$. The proof for the other two cases of d is similar.

Suppose that there is some model of G such that $C = \mathbf{s}(\Omega)$ with $\Omega = R^d$. We need to show that C is a d-correlated equilibrium. We show this by contradiction. Suppose that C is not a d-correlated equilibrium. Then there is some s in C, i, and σ_i such that s_i is weakly dominated by σ_i relative to $\{t_{-i} \mid (s_i, t_{-i}) \in C\}$.

Thus for all ω such that $\mathbf{s}_i(\omega) = s_i$,

$$h_i(\sigma_i, \mathbf{s}_{-i}(\omega)) \geq h_i(\mathbf{s}(\omega)) \tag{1}$$

with the inequality strict for at least one such $\bar{\omega}$. Since $\mathbf{s}_i(\bar{\omega}) = s_i$, it follows that $\mathbf{s}_i(\omega') = s_i$ for all ω' in $\Pi_i(\bar{\omega})$. Hence, equation (1) holds for all states in $\Pi_i(\bar{\omega})$ with strict inequality at $\bar{\omega}$. Thus Player i is not d-dominance rational at $\bar{\omega}$, contrary to the supposition that $\Omega = R^d$.

Conversely, suppose that C is a d-correlated equilibrium. We construct a knowledge space for which $\mathbf{s}(\Omega) = C$. The space Ω consists of states ω_s, one for each s in C. The partitions Π_i are defined by, $\Pi_i(\omega_s) = \{\omega_{\bar{s}} \mid \bar{s} \in C \text{ and } \bar{s}_i = s_i\}$. Finally, $\mathbf{s}(\omega_s) = s$. By the definition of C, for each i, Player i is d-dominance rational in each state, and therefore, $\Omega = R^d$ and, by construction, $\mathbf{s}(\Omega) = C$. □

If it is common knowledge in a model of G that the strategy profile played is in a set D, then $\mathbf{s}(\omega)$ is in D for all ω in Ω, that is $\mathbf{s}(\Omega) \subseteq D$. Thus Theorem 1 has the following Corollary.

Corollary 1. *For all d, there exists a model of the game in which d-dominance rationality is commonly known and the players commonly know that the played profile is in D if and only if D contains a d-correlated equilibrium.*

6. The structure of the set of *d*-correlated equilibria

6.1. Maximal d-correlated equilibrium

The set of correlated equilibria is convex. That is, given any two correlated equilibria, any mixture of the two is also a correlated equilibrium. For d-correlated equilibria the analogous result is that the union of two d-correlated equilibria is a d-correlated equilibrium. This property holds for each d, as we now show.

Proposition 1. *For all d,* $\mathcal{C}^d$ *is closed under unions.*

Proof. Let d be in $\{wp, wm\}$. We need to show that if C_1 and C_2 are d-correlated equilibria then $C_1 \cup C_2$ is too. We prove the contrapositive.

Suppose that $C_1 \cup C_2$ is not a d-correlated equilibrium. Thus there is some s in $C_1 \cup C_2$, some i, and some strategy, σ_i, of Player i, such that s_i is d-dominated by σ_i relative to $\{t_{-i} \mid (s_i, t_{-i}) \in C_1 \cup C_2\}$. Thus there is $\bar{t}_i$ in $\{t_{-i} \mid (s_i, t_{-i}) \in C_1 \cup C_2\}$ such that $h_i(\sigma_i, \bar{t}_i) > h_i(s_i, \bar{t}_i)$. Let k in $\{1, 2\}$ be such that $(s_i, \bar{t}_i)$ is in C_k. Then s_i is weakly dominated by σ_i relative to $\{t_{-i} \mid (s_i, t_{-i}) \in C_k\}$ and so C_k is not a d-correlated equilibrium. Thus it is not the case that both C_1 and C_2 are d-correlated equilibria, as required.

The proof for d in $\{sp, sm\}$ is similar. □

This proof uses directly the definition of d-correlated equilibrium. Alternatively, we could use Theorem 1 by choosing, for $k = 1, 2$, a model of G in which C_k is the set strategies played in the event that there is common knowledge of d-dominance rationality, and then construct a model of G in which the two knowledge structures are sub-structures.

The following is a straightforward corollary of Proposition 1.

Corollary 2. *For all d, the union of all d-correlated equilibria, denoted* $C^d_{\max}$*, is a d-correlated equilibrium.*

We now characterize the maximal d-correlated equilibrium, $C^d_{\max}$. In Hillas and Samet (2020) we define, in Definition 6 of that paper, a strategy profile s to be *compatible with common knowledge of d-dominance rationality* if there is some model of the game and some state at which the profile s is played with d-dominance rationality being commonly known at that state. Equivalently, s is compatible with common knowledge of d-dominance rationality if it is played at some state in a model of the game where d-dominance rationality is commonly known in each state. By Theorem 1, s is compatible with common knowledge of d-dominance rationality if and only if s belongs to some d-correlated equilibrium. This with Corollary 2 implies the following characterization of $C^d_{\max}$.

Corollary 3. *For all d, the maximal d-correlated equilibrium, $C^d_{\max}$, consists of all strategy profiles that are compatible with common knowledge of d-dominance rationality.*

Hillas and Samet (2020) provide us also with an algorithm to compute $C^d_{\max}$ for each d. For a given set C of strategy profiles we define certain elements of C as *d-flaws* (Definition 3). We then define the process of the *iterative elimination of d-flaws* (Definition 4), extending a process proposed by Stalnaker (1994) for $d = sm$. By Theorem 1 of that paper, the set of strategy profiles that are compatible with common knowledge of d-dominance rationality is the terminal set of the processes of elimination of d-flaws. Thus this process gives an algorithm to compute $C^d_{\max}$ since, by Corollary 3 of this paper, the set of strategy profiles that are compatible with common knowledge of d-dominance rationality is $C^d_{\max}$. The proof delivers more than is stated in the theorem; it shows that $C^d_{\max} = \mathbf{s}(\Omega)$ in some model of G where $\Omega = R^d$. This is the claim of Theorem 1 for the case $C = C^d_{\max}$.

By Claim 3 in Hillas and Samet (2020), for d in $\{sp, sm\}$, the terminal set of the processes of elimination of d-flaws is the terminal set of the processes of elimination of d-dominated strategies. In the latter process, the set of remaining strategy profiles, after each step of elimination, is a product for individual strategy sets. Thus, in particular the terminal set $C^d_{\max}$ is such a product. Not all d-correlated equilibria for such d are product sets but, the following proposition describes a property of all d-correlated equilibria that extends the claim that $C^d_{\max}$ is a product set. For any d-correlated equilibrium, C, the smallest product set containing C is a d-correlated equilibrium. In fact, a stronger result holds: any set between, in terms of set inclusion, C and the smallest product set containing C is a d-correlated equilibrium.

For a set of profiles C we denote by C_i the projection of C on S_i.

Proposition 2. *For d in $\{sp, sm\}$, if C is a d-correlated equilibrium then any set C' such that $C \subseteq C' \subseteq \times_i C_i$ is a d-correlated equilibrium.*

Proof. If C' is not a d-correlated equilibrium then for some s in C' and i, s_i is d-dominated relative to $\{t_{-i} \mid (s_i, t_{-i}) \in C'\}$. Since $C'_i \subseteq C_i$, there exists t_{-i} such that (s_i, t_{-i}) is in C. Thus, $\emptyset \neq \{t_{-i} \mid (s_i, t_{-i}) \in C\} \subseteq \{t_{-i} \mid (s_i, t_{-i}) \in C'\}$. Since domination is strict, s_i is also d-dominated relative to $\{t_{-i} \mid (s_i, t_{-i}) \in C\}$ which is impossible for the d-correlated equilibrium C. □

Proposition 2 extends the claim that $C^d_{\max}$ is a product set because it implies that $C^d_{\max}$ is a subset of $\times_i C^d_{\max,i}$, and that the latter is a d-correlated equilibrium. By the maximality of $C^d_{\max}$, $\times_i C^d_{\max,i}$ is a subset of $C^d_{\max}$, and thus $C^d_{\max} = \times_i C^d_{\max,i}$.

For d in $\{wp, wm\}$, $C^d_{\max}$ may not be a product set, and, in particular, Proposition 2 fails to hold. We illustrate this, and a number of other results shown in this section, in the context of Example 1 in Subsection 6.3.

6.2. Minimal and extreme d-correlated equilibrium

The family $\mathcal{C}^d$ of d-correlated equilibria is partially ordered by inclusion. The d-correlated equilibrium $C^d_{\max}$ is the maximal element in this partial order and it contains all the d-correlated equilibria in $\mathcal{C}^d$. This partial order suggests the definition of minimal d-correlated equilibria.

Definition 4. A d-correlated equilibrium C is *minimal* if there is no d-correlated equilibrium that is a proper subset of C.

Since $\mathcal{C}^d$ is a finite family of sets, there exists a minimal d-correlated equilibrium. Since, by Proposition 1, $\mathcal{C}^d$ is close under unions, all the unions of the minimal d-correlated equilibria are in $\mathcal{C}^d$. However, as we shall see later, in subsection 6.3, there are d-correlated equilibria which are not such unions.

To find a family of d-correlated equilibria that are small in some sense, and generate $\mathcal{C}^d$ we use the analogy between d-correlated equilibria and correlated equilibria. The set of correlated equilibria is convex, that is, it is closed under convex combinations, and it is generated by taking convex combinations of the extreme points of this set. We define extreme d-correlated equilibria, analogously to the extreme points in a convex set, where unions are used instead of convex combinations.

Definition 5. A d-correlated equilibrium C is *extreme* if it is not a union of two d-correlated equilibria both different from C.

Again, by the finiteness of $\mathcal{C}^d$ there exists an extreme d-correlated equilibrium. This finiteness easily implies also that the extreme d-correlated equilibria play an analogous role to that of the extreme points of the set of correlated equilibria.

Proposition 3. *For all d, each d-correlated equilibrium is a union of extreme d-correlated equilibria.*

The following proposition follows immediately by the closedness of $\mathcal{C}^d$ with respect to unions, and it suggests an equivalent definition of extreme d-correlated equilibria.

Proposition 4. *For all d, a d-correlated equilibrium C is extreme if and only if it is not the union of any number of d-correlated equilibria all of which are different from C.*

We now characterize the minimal and the extreme d-correlated equilibria by models of the game, in terms of the minimal commonly known events, namely the elements of the meet.

Proposition 5. *Consider a d-correlated equilibrium, C.*

(a) *C is* extreme *if and only if for each model of the game, $(\Omega, \Pi, \mathbf{s})$, such that $C = \mathbf{s}(\Omega)$ and $\Omega = R^d$, for* some *element M of the common knowledge partition (the meet) $C = \mathbf{s}(M)$.*
(b) *C is* minimal *if and only if for each model of the game, $(\Omega, \Pi, \mathbf{s})$, such that $C = \mathbf{s}(\Omega)$ and $\Omega = R^d$, for* each *element M of the common knowledge partition (the meet) $C = \mathbf{s}(M)$.*

Proof. To prove (a), suppose that C is an extreme d-correlated equilibrium and consider a model of the game in which $C = \mathbf{s}(\Omega)$ and $\Omega = R^d$. We need to show that for some element M of the meet $C = \mathbf{s}(M)$. Suppose to the contrary that $C \neq \mathbf{s}(M)$ for all members of the meet. Since $C = \mathbf{s}(\Omega)$, Ω is not an element of the meet and thus the meet is not a singleton. We let the elements of the meet be $M_1, M_2, \ldots, M_m$ with $m \geq 2$ and $C \neq \mathbf{s}(M_k)$ for each k. We can consider each M_k as a model of the game in which there is common knowledge of d-dominance rationality in all states, and thus, by Theorem 1, $\mathbf{s}(M_k)$ is a d-correlated equilibrium for each k. Thus C is the union of d-correlated equilibria all unequal to C, contradicting, by Proposition 4, the supposition that C is extreme. Thus there is an element M of the meet such that $C = \mathbf{s}(M)$, as required.

For the converse, suppose that C is a d-correlated equilibrium that is not extreme. Then $C = C^1 \cup C^2$ with both C^1 and C^2 d-correlated equilibria unequal to C. We construct a model of the game in which $C = \mathbf{s}(\Omega)$ and $\Omega = R^d$ but for which no element M of the meet has $\mathbf{s}(M) = C$.

Since C^1 and C^2 are d-correlated equilibria, there exist, by Theorem 1, for $k = 1, 2$, a model of the game, $(\Omega^k, \Pi^k, \mathbf{s}^k)$ such that all the players are d-dominance rational in each state of Ω^k, and $\mathbf{s}^k(\Omega^k) = C^k$. We can assume that $\Omega^1 \cap \Omega^2 = \emptyset$. We consider a model of the game $(\Omega, \Pi, \mathbf{s})$, where $\Omega = \Omega^1 \cup \Omega^2$, Π_i consists of all the elements of Π_i^1 and Π_i^2, and $\mathbf{s}(\omega) = \mathbf{s}^k(\omega)$ when $\omega \in \Omega^k$. Obviously all the players are d-dominance rational in each state of Ω and $\mathbf{s}(\Omega) = \mathbf{s}(\Omega^1) \cup \mathbf{s}(\Omega^2) = C^1 \cup C^2 = C$. Each element M in the meet partition Π^m is a subset of either Ω^1 or Ω^2. Thus, for some k, we have $\mathbf{s}(M) \subseteq \mathbf{s}(\Omega_k) = C_k \subsetneq C$ and so $\mathbf{s}(M) \neq C$, as required.

To prove (b), suppose that C is a minimal d-correlated equilibrium and consider a model of the game in which $C = \mathbf{s}(\Omega)$ and $\Omega = R^d$. If M is an element of the meet, then $\mathbf{s}(M) \subseteq \mathbf{s}(\Omega)$, and by the minimality of C, $\mathbf{s}(M) = C$. Conversely, if C is not minimal, then there exists a d-correlated equilibrium C' which is a proper subset of C. We construct a model as in the proof of (a), where for some element of the meet, M, $\mathbf{s}(M) \subseteq C' \neq C$. □

6.3. Illustrating the structure of d-correlated equilibria

We consider again the game in Example 1. In our earlier consideration of this game we showed directly that, for each d, the sets $\{(T, L)\}$, $\{(T, R)\}$, and $\{(B, L)\}$ were the only singleton d-correlated equilibria. (These are also the only pure strategy equilibria. We shall later show, in Proposition 7, that, for all d, a singleton set is a d-correlated equilibrium if and only if it contains only a pure strategy Nash equilibrium.) Also, as stated in Proposition 1, each union of these equilibria is a d-correlated equilibrium.

These are the only $w*$-correlated equilibria. The extreme $w*$-correlated equilibria are the three pure strategy equilibria. The maximal $w*$-correlated equilibrium is the set $\{(T, L), (T, R), (B, L)\}$, which is not a product set.

As we showed in our initial consideration of this game in Section 1, the set $\{(T, R), (B, L), (B, R)\}$ and the set of all profiles are also $s*$-correlated equilibria (along with all the $w*$-correlated equilibria). Thus the extreme $s*$-correlated equilibria are the set $\{(T, R), (B, L), (B, R)\}$ and each singleton set consisting of one of the three pure strategy equilibria. The $s*$-correlated equilibrium $\{(T, R), (B, L), (B, R)\}$ is another example of an extreme equilibrium that is not minimal. (No proper subset of $\{(T, R), (B, L), (B, R)\}$ containing (B, R) is a $s*$-correlated equilibrium.)

The union of all extreme $s*$-correlated equilibria, which is the set of all profiles, is the maximal $s*$-correlated equilibrium. It is a product set, as is implied by the fact stated earlier that for d in $\{sp, sm\}$ the set $C^d_{\max}$ is necessarily a product set.

The fact that the set $\{(T, R), (B, L), (B, R)\}$ is an $s*$-correlated equilibrium, which we earlier showed directly, also follows from Proposition 2 and the fact that $\{(T, R)\}$, and $\{(B, L)\}$, and hence also $\{(T, R), (B, L)\}$, are $s*$-correlated equilibria, since

$$\{(T, R), (B, L)\} \subseteq \{(T, R), (B, L), (B, R)\} \subseteq \{T, B\} \times \{L, R\}.$$

Clearly, a minimal d-correlated equilibrium is necessarily extreme, but the converse is not true, as is evident from Proposition 5. This is demonstrated simultaneously for all d in the following example.

Example 2. Consider the game in Fig. 3. This is the normal form of a two stage centipede game. This is a special case of

		Player 2	
		L	R
Player 1	T	2, 0	2, 0
	B	1, 3	4, 2

Fig. 3. d-correlated equilibria may not be the union of minimal d-correlated equilibria.

the general T stage centipede game that we examine in Example 6 in Section 10, though we rename the strategies here.

The sets $\{(T, L)\}$ and $\{(T, L), (T, R)\}$ are d-correlated equilibria for each d. In these sets, when Player 1 is recommended to play T the set of strategies that might have been recommended to Player 2 is either $\{L\}$ or $\{L, R\}$ and, relative to either of these sets, the recommended strategy T is not d-dominated by B for any d. And, if the set of strategies that might have been recommended to Player 1 is $\{T\}$ then neither of L and R d-dominate the other for any d. However, $\{(T, R)\}$ is not a d-correlated equilibrium for any d. In this set when Player 1 is recommended to play T the set of strategies that might have been recommended to Player 2 is $\{R\}$ and, relative to this set, the recommended strategy T is d-dominated by B for all d.

Thus, for each d, $\{(T, L), (T, R)\}$ is not a minimal d-correlated equilibrium, since it properly contains the d-correlated equilibrium $\{(T, L)\}$. But $\{(T, L), (T, R)\}$ is not a union of two d-correlated equilibria which are both different from it and therefore it is an extreme d-correlated equilibrium. Also, $\{(T, L), (T, R)\}$ is not a union of minimal d-correlated equilibria, and therefore the minimal d-correlated equilibria do not generate C^d by unions.

7. Correlated equilibrium and *d*-correlated equilibrium

We recall that a correlated equilibrium of the game G is a probability distribution p on S such that for all i, and all s_i and σ_i in S_i,

$$\sum_{s_{-i}\in S_{-i}} p(s_i, s_{-i})h_i(s_i, s_{-i}) \geq \sum_{s_{-i}\in S_{-i}} p(s_i, s_{-i})h_i(\sigma_i, s_{-i}). \tag{2}$$

Observe that by the linearity of expectation, these inequalities hold also if σ_i is a mixed strategy in Σ_i.

Proposition 6. *For all d, the support of a correlated equilibrium is a d-correlated equilibrium.*

Proof. By the relations given in Fig. 2, it is enough to show it for $d = wm$. Let p be a correlated equilibrium, the support of which is C. Suppose that contrary to our claim, for some s in C, s_i is wm-dominated by σ_i in Σ_i relative to $R = \{t_{-i} \mid (s_i, t_{-i}) \in C\}$. Then, for all t_{-i} in R, $h_i(\sigma_i, t_{-i}) \geq h_i(s_i, t_{-i})$ with at least one strict inequality. By the definition of C, $p(s_i, t_{-i}) > 0$ if and only if t_{-i} is in R. Therefore,

$$\begin{aligned}\sum_{t_{-i}\in S_{-i}} p(s_i, t_{-i})h_i(s_i, t_{-i}) &= \sum_{t_{-i}\in R} p(s_i, t_{-i})h_i(s_i, t_{-i})\\ &< \sum_{t_{-i}\in R} p(s_i, t_{-i})h_i(\sigma_i, t_{-i})\\ &= \sum_{t_{-i}\in S_{-i}} p(s_i, t_{-i})h_i(\sigma_i, t_{-i}),\end{aligned}$$

contrary to our assumption that p is a correlated equilibrium. □

The converse of Proposition 6 does not hold. That is, a d-correlated equilibrium is not necessarily the support of a correlated equilibrium. The game in Example 1 demonstrates it easily for d in $\{wp, wm\}$. The set of all four strategy profiles is a d-correlated equilibrium for such d. However, (B, R) is not in the support of any correlated equilibrium. Example 4 in Section 9 demonstrates it for all d, and shows, moreover, that a d-correlated equilibrium can be disjoint from the largest support of a correlated equilibrium.

Since a correlated equilibrium of G exists (see the comment of Aumann, 1974 on page 77 following Proposition 4.3 or, for a more elementary proof that does not rely on the existence of Nash equilibrium, Hart and Schmeidler, 1989) we have the following corollary.

Corollary 4. *For all d, for any finite game there exists a d-correlated equilibrium.*

Corollary 5 follows more directly from the algorithm we gave earlier, following the proof of Proposition 5, for constructing $C^d_{\max}$, an algorithm that always terminates in a non-empty set.

The support of a mixed strategy equilibrium is the set of strategy profiles that are played with positive probability. Since the distribution on profiles of a mixed strategy equilibrium is a correlated equilibrium Proposition 6 implies the following additional corollary.

Corollary 5. *For all d, the support of a mixed strategy equilibrium is a d-correlated equilibrium.*

It is trivial to check directly that, for each d, every pure strategy equilibrium is a d-correlated equilibrium. Of course, Corollary 5 also implies this. But in this case it is easy to see that we have the following stronger result.

Proposition 7. *For all d, a singleton $\{s\}$ is a d-correlated equilibrium if and only if s is a pure strategy equilibrium.*

Thus, for singleton sets, all versions of d-correlated equilibrium coincide.

8. Common knowledge of Bayesian rationality

Analogously to d-correlated equilibrium we now define a new Bayesian solution that we call *b-correlated equilibrium*. We first define the notion of relative best response.

Definition 6. A strategy s_i of Player i is *best response relative to a set of strategy profiles $T_{-i} \subseteq S_{-i}$*, if there exists a probability p in $\Delta(T_{-i})$ such that $\mathrm{E}_p h_i(s_i, t_{-i}) \geq \mathrm{E}_p h_i(\sigma_i, t_{-i})$ for each σ_i in S_i.

Definition 7. A set of strategy profiles C is a b-correlated equilibrium if for each i and each s in C, s_i is a best response relative to the set $\{t_{-i} \mid (s_i, t_{-i}) \in C\}$.

We next characterize b-correlated equilibria in terms of common knowledge of Bayesian rationality. For this we describe a Bayesian belief model of the game. We start with a model of the game and turn it into a *belief model* by defining the probabilistic beliefs of each player in each state as follows. For each i and each π in Π_i we let p^i_π be a probability distribution over π. That is, $p^i_\pi(\pi) = 1$. We then associate with each state ω a probability distribution of i, $p^i_\omega = p^i_{\Pi_i(\omega)}$ that we call *i's type at ω*.[4]

Bayesian rationality in a belief model of the game is standardly defined.

Definition 8. Player i is *Bayesian rational* at state ω if the strategy $\mathbf{s}_i(\omega)$ maximizes i's expected payoff. That is, for each σ_i in S_i,

$$\mathrm{E}_{p^i_\omega} h_i(\mathbf{s}_i(\omega), \mathbf{s}_{-i}) \geq \mathrm{E}_{p^i_\omega} h_i(\sigma_i, \mathbf{s}_{-i}).$$

We denote by R^b the event that all players are Bayesian rational.

Proposition 8. *A non-empty set of strategy profiles C is a b-correlated equilibrium if and only if $C = \mathbf{s}(\Omega)$ in some belief model of the game with $\Omega = R^b$.*

Before we prove this proposition we observe that an equivalent concept to that of b-correlated equilibrium has already previously introduced in this paper. The following lemma is a simple extension of Lemma 3 of Pearce (1984).

Lemma 1. *A strategy s_i is a best response relative to a set T_{-i} if and only if s_i is not sm-dominated relative to T_{-i}.*

An immediate corollary is that b-correlated equilibrium is not actually a new solution concept.

Corollary 6. *A set of strategy profiles C is a b-correlated equilibrium if and only if it is an sm-correlated equilibrium.*[5]

[4] With this definition, $\Pi_i(\omega)$ is the event that the type of i is p^i_ω, and thus Player i knows her type at ω.

[5] In light of the equivalence between *sm*-correlated equilibrium and best response correlated equilibrium, stated in Corollary 6, we might have suspected that the converse holds, in some sense, for *sm*-correlated equilibrium and *subjective* correlated equilibrium. The latter can be defined, as in Aumann (1987), by the same inequalities as correlated equilibrium but with the inequalities of each Player i written with a probability p^i rather than the common prior p. However, Example 1 shows that this is not the case, as for all subjective correlated equilibria of this game the probability of (B, R) is 0 for each of the players.

		Player 2	
		H	T
Player 1	H	$4, -4$	$-4, 4$
	T	$-4, 4$	$4, -4$
	B	$-1, 1$	$-1, 1$

Fig. 4. Dominance by pure strategies and dominance by mixed strategies.

Proof of Proposition 8. Suppose that $C = \mathbf{s}(\Omega)$ in some belief model of the game where $\Omega = R^b$. Fix s in C. Then, $s = \mathbf{s}(\omega)$ for some state ω. As ω is in R_i^B, and as the support of p_ω^i is $\Pi_i(\omega)$, it follows that for each i, s_i is a best response relative to $\mathbf{s}_{-i}(\Pi_i(\omega))$. In particular it is also a best response relative to the larger set $\{t_{-i} \mid (s_i, t_{-i}) \in C\}$. This proves that C is a v-correlated equilibrium.

Conversely, suppose that C is a b-correlated equilibrium. Then, by Theorem 1 and Corollary 6, there exists a model of the game for which $C = \mathbf{s}(\Omega)$, and $\Omega = R^{sm}$. Fix ω in Ω and i. As ω is in R_i^{sm}, $\mathbf{s}_i(\omega)$ is not sm-dominated relative to $\Pi_i(\omega)$. By Lemma 1, $\mathbf{s}_i(\omega)$ is a best response relative to $\Pi_i(\omega)$ by a probability p in $\Delta(\Pi_i(\omega))$. Let $p^i_{\Pi_i(\omega)} = p$. Thus, we have constructed a belief model of the game in which all players are Bayesian rational at each state and $C = \mathbf{s}(\Omega)$. $\square$

The terminal set of the iterative elimination of non-best response strategies is the set of strategy profiles called *correlatedly rationalizable strategies* (see, Brandenburger and Dekel, 1987). By Lemma 1, this process is the same as the iterative elimination of strictly dominated strategies by mixed strategies. The latter terminates in the maximal sm-correlated equilibrium, and thus by Corollary 6, the set of correlatedly rationalizable strategies is the maximal b-correlated equilibrium. Tan and Werlang (1988) proved that the set of correlatedly rationalizable strategies is the set of strategy profiles that can be played at state in a belief model of the game where common knowledge of Bayesian rationality holds in all states.

9. Examples

We have used Examples 1 and 2 that we examined earlier in the paper to show a number of facts about d-correlated equilibria. Example 1 showed that there are games in which the set of d-correlated equilibria defined with weak dominance is strictly smaller than those defined with strict dominance. It also showed that for d in $\{sp, sp\}$ there are extreme d-correlated equilibria that are not maximal and that for d in $\{wp, wp\}$ the maximal d-correlated equilibrium may not be a product set. Example 2 showed that that for *all* d there are extreme d-correlated equilibria that are not maximal.

In this section we consider a number of other simple examples that illustrate other properties of d-correlated equilibria. The game of Example 3 shows that there are games in which the set of d-correlated equilibria defined with dominance by mixed strategies is strictly smaller than those defined with dominance by pure strategies.

In Proposition 6 we showed that the support of a correlated equilibrium is a d-correlated equilibrium. We commented that the converse was not true. In the game of Example 4 there is a set that is a d-correlated equilibrium for each d that is disjoint from the largest support of any correlated equilibrium.

In the game of Example 3 the set of d-correlated equilibria defined with dominance by mixed strategies is strictly smaller than that defined with dominance by pure strategies. One might think that such an example could not be found in the strategic form of extensive form games with perfect information, since mixed strategies usually play little role in the analysis of such games. Somewhat surprisingly, the game of Example 5 shows that, even for these games, such an example can be found.

9.1. The difference between dominance by pure and mixed strategies

Example 3 shows that the inclusions of $\mathcal{C}^{sm}$ in $\mathcal{C}^{sp}$ and of $\mathcal{C}^{wm}$ in $\mathcal{C}^{wp}$ may both be strict.

Example 3. The game in Fig. 4 is a two person zero sum game. It may be thought of as the strategic form of an extensive form game in which Player 1 has the option of playing B, paying 1 to Player 2, and ending the game or continuing and playing a matching pennies game with stakes of 4. The set $\{(B,H),(B,T)\}$ is a d-correlated equilibrium for d in $\{sp, wp\}$ since B is not weakly (or strictly) dominated by either H or T relative to $\{H,T\}$. However B is strictly (and weakly) dominated by $(\frac{1}{2}, \frac{1}{2}, 0)$ relative to $\{H,T\}$. In this game $\mathcal{C}^{sm} = \mathcal{C}^{wm}$ and both consist of the single set of profiles, $\{(H,H),(H,T),(T,H),(T,T)\}$, which is also, of course, the unique extreme d-correlated equilibrium for these d. That set is also an extreme d-correlated equilibrium for d in $\{sp, wp\}$ and $\{(B,H),(B,T)\}$ is the only other extreme d-correlated equilibrium for these d.

Thus, we have shown the other two of the possibilities that we claimed, namely that $\mathcal{C}^{sm} \subsetneq \mathcal{C}^{sp}$ and $\mathcal{C}^{wm} \subsetneq \mathcal{C}^{wp}$.

9.2. The difference between a d-correlated equilibrium and the support of a correlated equilibrium

In Example 1, a set is the support of a correlated equilibrium if and only if it is a $w*$-correlated equilibrium. Example 4 exhibits a set that is a d-correlated equilibrium for each d but is not the support of a correlated equilibrium, and is, in fact, disjoint from the support of any correlated equilibrium.

		Player 2: H	Player 2: T
Player 1	H	$4, -4$	$-4, 4$
	T	$-4, 4$	$4, -4$
	B	$1, -1$	$1, -1$

Fig. 5. d-correlated equilibria and correlated equilibria.

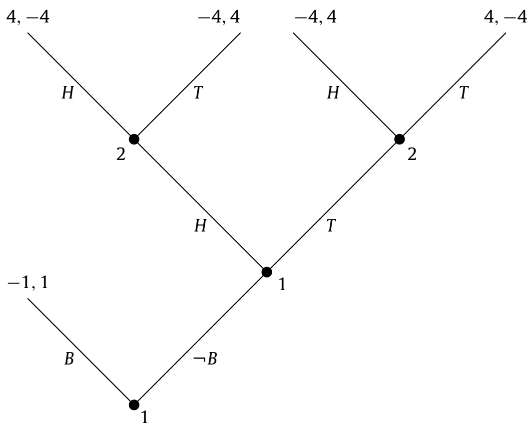

Fig. 6. Domination by pure or mixed strategies in a game with perfect information. Extensive Form.

Example 4. The game in Fig. 5 is similar to the previous example but, when playing B and ending the game, Player 1 receives 1 from Player 2, rather than paying 1. Since the game is a two person zero sum game, in a correlated equilibrium the conditional probability in each column, when the column has a positive probability, is an optimal strategy of Player 1 (see Forges, 1990). The value of the game is 1, and the only optimal strategy of Player 1 is B. Thus, the support of any correlated equilibrium is in $\{(B, H), (B, T)\}$. (The set of correlated equilibria are all distributions over these two profiles such that each has probability at least $3/8$.) However, it is easy to see that the set of the four top profiles is a d-correlated equilibrium for each d.

9.3. Mixed and pure domination in perfect information games

In the next example the game is the strategic form of an extensive form game with perfect information. Aumann (1995) and Aumann (1998) studied rationality and common knowledge of rationality in such games, defining rationality using pure strategy domination. Though it is not the way he describes it, one may think of Aumann (1995) as defining rationality as sp-dominance rationality but requiring it not just for the game but also for all subgames of the game. Since Aumann also assumes that the game is in general position this means that for each choice sp-dominance rationality implies a unique chosen action and so the same choice will result from assuming any other d-dominance rationality. Aumann (1998) studies only the centipede game, which is, in the relevant aspects, in general position and, again, may be thought of as requiring sp-dominance rationality but only for some subgames. And, again, no other d-dominance rationality will give stronger results.

Also, for a Bayesian rational player, if some mixed strategy is better than a given strategy then at least one of the pure strategies in the support of the mixed strategy is also better. But for dominance notions this is not true. A strategy can be dominated by a mixed strategy but not by any of the pure strategies in the support of that mixed strategy. Our next example shows that, even in the strategic form of an extensive form game with perfect information the $*p$-correlated equilibria and $*m$-correlated equilibria may be different.

Example 5. Consider the extensive form game with perfect information given in Fig. 6 with the associated strategic form given in Fig. 7. This game is similar to the game in Example 3, but, following the rejection of the outside option by Player 1, rather than playing a simultaneous move matching pennies game the players move sequentially, with Player 1 moving first.

Consider the set $C = \{(B, HH), (B, TT)\}$. Relative to $\{B\}$ all of Player 2's strategies are equivalent so none are dominated relative to $\{B\}$. And for Player 1, B is not weakly dominated by either H or T relative to $\{HH, TT\}$; so, for d in $\{sp, wp\}$, C is a d-correlated equilibrium. However, B is (even strongly) dominated relative to $\{HH, TT\}$ by the mixed strategy that puts weight a half on each of H and T; so, for d in $\{sm, wm\}$, C is not a d-correlated equilibrium. It is worth pointing out that, even for d in $\{sm, wm\}$, there is a d-correlated equilibrium in which Player 1 plays B, namely $\{(B, TH)\}$. It is straightforward

		Player 2			
		HH	HT	TH	TT
	H	$4,-4$	$4,-4$	$-4,\ 4$	$-4,\ 4$
Player 1	T	$-4,\ 4$	$4,-4$	$-4,\ 4$	$4,-4$
	B	$-1,\ 1$	$-1,\ 1$	$-1,\ 1$	$-1,\ 1$

Fig. 7. Domination by pure or mixed strategies in a game with perfect information. Strategic Form.

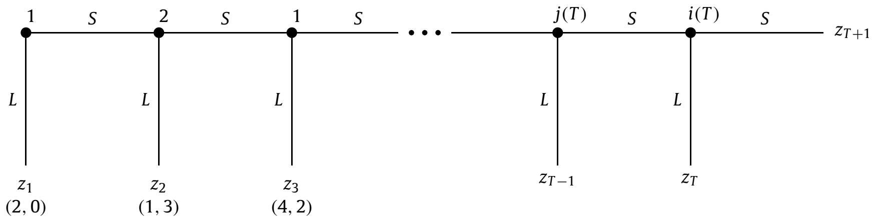

Fig. 8. The T Stage Centipede Game in extensive form.

to see this is a d-correlated equilibrium. In fact, (B, TH) is a pure strategy Nash equilibrium, and so, by Proposition 7, is a d-correlated equilibrium for all d.

The game given is not in general position. But if we perturb all the payoffs by any small amount the set C remains a $*p$-correlated equilibrium but not a $*m$-correlated equilibrium

10. The centipede game

In the previous section we considered a number of simple examples that were explicitly designed to show, in the simplest possible way, certain properties of the sets we had defined. Here we consider a larger game, which has been much discussed in the literature, in order to demonstrate the usefulness of our definitions in analyzing games of substantial and independent interest.

Example 6. We consider a centipede game, introduced by Rosenthal (1981), with T stages. This is a two player game in which Player 1 chooses in the odd stages and Player 2 in the even stages. We can think of the payoffs as accumulating. At each stage there are two possible increments to the players' utilities, a larger increment and a smaller one. The player who moves at that stage chooses between action L, adding the larger increment to her utility, adding the smaller increment to the other player's utility, and ending the game or action S, adding the smaller increment to her utility, adding the larger increment to the other player's utility, and, except at stage T, advancing the game to the next stage. If the player chooses L at stage t the game ends with outcome z_t. If no player ever chooses L then the outcome is z_{T+1}. At stage T the game ends whatever the player who chooses at stage T chooses. If she chooses L then the large increment is added to her payoff and the small increment to the other player's payoff; if she chooses S then twice the small increment is added to her payoff and twice the large increment to the other player's payoff. The game is a standard centipede game if the increments are non-negative and the large increment is greater than twice the small increment. We denote the player who chooses at stage t as $i(t)$ and the other player as $j(t)$. In a standard centipede game $i(t)$ prefers z_{t+2}, and all later outcomes, to z_t and prefers z_t to z_{t+1}.

Our analysis does not depend on the values of the increments, beyond assuming that the game is a standard centipede game, but we shall, for concreteness, assume that the small increment is $1/3$ of a unit of utility and the large increment is $5/3$ units of utility. We shall also assume that Player 1 starts the game with $1/3$ of a unit of utility and Player 2 with $-1/3$ of a unit of utility. These apparently untidy assumptions will result in very tidy payoffs to the outcomes z_t. In fact, with these increments, the total accumulated payoff at z_t is $(t+1, t-1)$ if t is odd and $(t-1, t+1)$ if t is even. The game tree is given in Fig. 8.

If T is even then each player has $(T/2)+1$ classes of Kuhn-equivalent strategies that we label $1, 3, 5, \ldots, T+1$ for Player 1 and $2, 4, 6, \ldots, T+2$ for Player 2. If T is odd then Player 1 has $((T+1)/2)+1$ classes of Kuhn-equivalent strategies and Player 2 has $(T+1)/2$ classes of Kuhn-equivalent strategies that we label $1, 3, 5, \ldots, T+2$ for Player 1 and $2, 4, 6, \ldots, T+1$ for Player 2. The equivalence class t, for $t = 1, \cdots, T$, contains all strategies in which a player $i(t)$ chooses L for the first time at stage t. The classes $T+1$ and $T+2$ consist of the single strategy of always choosing S. We shall now refer to these equivalence classes as strategies. The strategy pair (s_1, s_2) induces the outcome z_t where $t = \min\{s_1, s_2\}$.

The strategic form of the game for the case that T is even is given in Fig. 9.

Claim 1. *In the centipede game, a set C of strategy pairs is a d-correlated equilibrium for d in $\{wp, wm\}$ if and only if C contains only pairs of the form $(1, t)$, that is, Player 1 terminates the game immediately, and the pair $(1, 2)$ is in C, that is, it is possible that Player 2 terminates the game at his first opportunity.*

Player 1 \ Player 2	2	4	6	$\cdots$	T	$T+2$
1	z_1 (2, 0)	z_1 (2, 0)	z_1 (2, 0)		z_1 (2, 0)	z_1 (2, 0)
3	z_2 (1, 3)	z_3 (4, 2)	z_3 (4, 2)		z_3 (4, 2)	z_3 (4, 2)
5	z_2 (1, 3)	z_4 (3, 5)	z_5 (6, 4)		z_5 (6, 4)	z_5 (6, 4)
$\vdots$						
$T-1$	z_2 (1, 3)	z_4 (3, 5)	z_6 (5, 7)		z_{T-1}	z_{T-1}
$T+1$	z_2 (1, 3)	z_4 (3, 5)	z_6 (5, 7)		z_T	z_{T+1}

Fig. 9. The T Stage Centipede Game in strategic form for even T.

Proof. Let d in $\{wp, wm\}$ and C be a d-correlated equilibrium. There is a $\bar{t}$ such that all profiles in C induce an outcome z_t with $t \le \bar{t}$, namely $\bar{t} = T + 1$. Let $\underline{t}$ be the smallest such $\bar{t}$. We claim that $\underline{t} = 1$. Suppose that $\underline{t} > 1$. There is some profile (s_1, s_2) in C that induces the outcome $z_{\underline{t}}$. This profile must have $s_{i(\underline{t})} = \underline{t}$ and $s_{j(\underline{t})} > \underline{t}$. (Otherwise the outcome would not be $z_{\underline{t}}$.) Also there are no profiles (s'_1, s'_2) in C with $s'_{j(\underline{t})} = s_{j(\underline{t})}$ and $s'_{i(\underline{t})} > \underline{t}$, since then there would be a profile inducing an outcome z_t with $t > \underline{t}$. But now strategy $s_{j(\underline{t})}$ is weakly dominated by $\underline{t} - 1$ relative to $\{s'_{i(\underline{t})} \mid (s'_1, s'_2) \in C, s'_{j(\underline{t})} = s_{j(\underline{t})}\}$ contradicting the assumption that C is a d-correlated equilibrium.

Thus all members of C must induce the outcome z_1 and so Player 1's strategy must be 1 in all profiles in C. When Player 1's strategy is 1, Player 2's payoff is fixed, and thus no strategy of his is dominated relative to strategy 1. If $(1, 2)$ is in C then, since in this profile Player 1 strictly prefers 1 to any other strategy her strategy is not dominated relative to C. However if $(1, 2)$ is not in C then 1 is strictly dominated relative to C by 3 for Player 1 and so C is not a d-correlated equilibrium. □

The proof just given is similar, in essence, to the proof in Aumann (1998). As Aumann describes it,

> Rather than reasoning from what happens after a given vertex, it reasons from what happens *before*: If some vertex is the last that can possibly be reached, then already the one before it should have been the last.

Claim 2. *In the centipede game, a set C of profiles is an sm-correlated equilibrium if and only if for each Player i and each strategy t of i that is played in some profile in C, the set of strategies t' played by her opponent against t in C is not a subset of $\{2, \dots, t-2, t-1\} \cup \{t+3, t+4, \dots\}$, that is, the set of strategies played by $j(t)$ against t must include either strategy 1 or strategy $t + 1$.*

Proof. Let $d = sm$. We start with some observations. Consider strategy t of Player $i(t)$. Against strategies t' of Player $j(t)$ with $t' < t$, Player $i(t)$ never loses by choosing a strategy greater than t. Against $t - 1$, Player $i(t)$ gains 1 by playing $t - 2$, loses 1 by playing $t - 4$, loses 3 by playing $t - 6$, and so on. Against $t - 3$, Player $i(t)$ does equally well by playing $t - 2$, gains 1 by playing $t - 4$, loses 1 by playing $t - 6$, and so on. Against $t - 2k - 1$, Player $i(t)$ neither gains nor loses by playing $t - 2k'$ for $k' \le k$, gains by playing $t - 2(k + 1)$, and loses by playing strategies less than this. Against strategies t', of Player $j(t)$ with $t' \ge t + 3$, Player $i(t)$ gains 2 by playing strategy $t + 2$ and loses by playing strategies $t - 2$ or smaller.

So consider the mixed strategy that plays strategy $t + 2$ with probability K, strategy $t - 2$ with probability εK, strategy $t - 4$ with probability $\varepsilon^2 K$, and so on, where K is the value that will make the probabilities add to one, that is, $K = 1/(1 + \varepsilon + \varepsilon^2 + \dots)$. Relative to $\{2, \dots, t-2, t-1\} \cup \{t+3, t+4, \dots\}$, for small enough ε, this mixed strategy strictly dominates t.

On the other hand against strategy 1 of Player 1, there is no mixed strategy that strictly dominates any other strategy for Player 2 and against strategy $t + 1$ for Player $j(t)$ strategy t is the unique best response for Player $i(t)$, so there is no mixed strategy that dominates it.

Thus the claimed result is shown. □

Claim 3. *In the centipede game, a set C of strategy profiles is an sp-correlated equilibrium if and only if for each i and each strategy t of Player i that is played in some profile in C, the set of strategies t' played by her opponent against t in C is not a subset of $\{t+3, t+5, \dots\}$ or a singleton t' with $1 < t' < t$.*

Proof. Let $d = sp$. The sets of $j(t)$'s strategies relative to which t is strictly dominated are: (a) subsets of $\{t + 3, t + 5, \dots\}$, where the outcomes are strictly preferred by Player $i(t)$ to z_t, and therefore Player $i(t)$ can do strictly better against such strategies of Player $j(t)$ by playing $t + 2$; (b) singletons t' with $1 < t' < t$ which enable Player $i(t)$ to improve upon t by playing $t' - 1$. Thus, the condition of the claim exactly describes the fact that C is a sp-correlated equilibrium. □

The above three claims provide a characterization of the d-correlated equilibria for each d. However it is not immediately obvious what sets of strategy pairs constitute d-correlated equilibria in each case. The case that d in $\{wp, wm\}$ is clear cut. In these cases, any d-correlated equilibrium includes only pairs of the form $(1, t)$ and always includes $(1, 2)$ and the extreme

d-correlated equilibria are the sets containing either one or two profiles. Iterated elimination of weakly dominated strategies is even more restrictive; it results in a single profile $(1, 2)$ which, unlike d-correlated equilibrium, determines not only Player 1's strategy, but also Player 2's strategy.

The cases for d in $\{sp, sm\}$ are less clear. If C is a sm-correlated equilibrium we see that the requirement if Player 1 plays t in some element of C then either 1 or $t+1$ is in the strategies that Player 2 plays against t means that $t+1$ is, since strategy 1 is not one of Player 2's strategies. Thus we have that if t is played by Player 1 then $(t, t+1)$ is in C. If T is odd then the highest strategy of Player 1 is $T+2$ and so we can say that Player 1 does not play $T+1$ in any pair in C since $T+3$ is not one of Player 2's strategies. Let $\bar{t}$ be the highest numbered strategy of Player 1 in C. Claim 2 implies that $(\bar{t}, \bar{t}+1)$ is in C. But, since $\bar{t}$ is the highest numbered strategy of Player 1 played in C, $(\bar{t}+2, \bar{t}+1)$ is not in C, and so $(1, \bar{t}+1)$ is in C. And thus strategy 1 is played by Player 1 in C and thus $(1, 2)$ is in C. Every sm-correlated equilibrium contains $(1, 2)$. Also, for every strategy t of Player 1, if t is played in the sm-correlated equilibrium C then $(t, t+1)$ is in C. And for any strategy t' played by Player 2 in C either $(1, t')$ or $(t'+1, t')$ is in C, and if $t' > \bar{t}$, with $\bar{t}$ the highest numbered strategy of Player 1 in C, then $(1, t')$ is in C. When T is even the set of all strategy pairs is an sm-correlated equilibrium and when T is odd the set of all strategy pairs where Player 1 does not play $T+2$ is an sm-correlated equilibrium. The collection of all sm-correlated equilibria, of course, contains all the d-correlated equilibria for d in $\{wp, wm\}$.

Even more sets of strategy pairs are sp-correlated equilibria. All sets of the form $\{(s, t) \mid s \le s_0 \text{ and } t \le t_0\}$ with $t_0 > 2$ are sp-correlated equilibria. And so are sets of the form $\{(s, t) \mid s \ge s_0 \text{ and } t \ge t_0\}$ with $s_0, t_0 \le T$ with $s_0 - 1 \le t_0 \le s_0 + 1$. These latter gives us our first examples of d-correlated equilibria that do not contain $(1, 2)$ nor indeed any pair of strategies in which the game ends immediately.

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