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THE DETERMINATION OF MARGINAL COST PRICES UNDER A SET OF AXIOMS

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This paper presents a set of axioms which characterize a family of price mechanisms for consumption goods, including marginal cost prices and Aumann–Shapley prices. By strengthening one of the axioms, marginal cost prices are characterized and by requiring that cost is shared the Aumann–Shapley prices are characterized. A discussion of the economic interpretation of the axioms is also provided.

1. INTRODUCTION

THE MAIN PURPOSE of this paper is to provide an axiomatic approach to marginal cost (MC) pricing and to point out its similarity with Aumann–Shapley (A–S) pricing. The latter is a cost-sharing price mechanism discussed in [**3** and **6**] that is derived from a set of five natural axioms.

In this paper we consider models in which there is one producer with a given technology who faces fixed input prices and produces a finite number of consumption goods. Thus, we can uniquely derive the cost function that describes the minimal cost of producing a given vector of consumption goods.

By a price mechanism $P(\cdot,\cdot)$ we mean a rule or a function that associates with each cost function F and vector α of quantities, a vector of prices:

$$P(F,\alpha) = (P_1(F,\alpha), P_2(F,\alpha), \dots, P_m(F,\alpha)),$$

where m is the dimension of α and $P_i(F,\alpha)$ is the price of a unit of the ith commodity.

We shall consider price mechanisms which obey the following four axioms. First we require that prices should be independent of the units of measurement (Axiom 1). This is a fundamental requirement of any pricing system. We also require that the price of a commodity for which the cost function is nondecreasing be nonnegative (Axiom 4). Axiom 2 requires that two commodities having the same effect on the cost have the same price. This emphasizes the fact that the price of a commodity measures its "real value" in production.

Finally Axiom 3 enables us to calculate the prices via its factors of production: if the cost is broken into two (additive) factors, e.g., the cost of labor and the cost of raw materials, then the prices can be obtained by adding the prices attributable to the two factors separately. (In Section 4 we will show that Axiom 3 can be replaced by two other natural axioms.)

In this paper we prove that by strengthening the positivity axiom slightly (Axiom 4*), the four axioms (1, 2, 3, and 4*) uniquely characterize MC prices

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(Theorem B). On the way to proving this result we state a theorem (Theorem A) that is interesting in its own right. It characterizes the set of all price mechanisms satisfying the four "basic" axioms: 1, 2, 3, and 4. Among them are the marginal cost and the Aumann–Shapley price mechanisms. The latter as mentioned earlier, can be uniquely characterized by an additional requirement that cost equal revenue, i.e., cost is shared by the prices. This price mechanism was first proposed by Billera, Heath, and Raanan [**2**] to set telephone billing rates which would allocate the cost arising in serving the consumers; and it has been adopted for internal telephone billing at Cornell University. Later it was characterized axiomatically (independently) by Billera–Heath [**3**] and Mirman–Tauman [**6**]. Using Theorem A one can easily prove (Theorem C) that the A–S price mechanism is the unique cost-sharing mechanism which obeys Axioms 1–4. This provides an alternative proof for the main results in [**3** and **6**].

To sum up, Axioms 1–4 are the key axioms in our study. Both MC prices and A–S prices obey them; moreover, strengthening Axiom 4 yields MC prices, while adding the cost-sharing requirement yields A–S prices.

Finally, we should mention that our work stems from ideas developed in game theory. In [**2**] it is shown that for a given cost function F and vector α of quantities, one can associate a nonatomic game $v(F,\alpha)$ so that its Aumann-Shapley value measures the effect of each unit of each commodity on the cost. If this magnitude is chosen to be the price of the commodity we get exactly the Aumann–Shapley price mechanism. However, from the same game $v(F,\alpha)$ one can derive a price mechanism using, instead of the Aumann–Shapley value, a wider solution concept called the semi-value. Using the characterization of Dubey, Neyman, and Weber [**5**] for all semi-values of a large space of nonatomic games, it turns out that the corresponding set of price mechanisms derived from the set of all semi-values is exactly the set of all price mechanisms obeying Axiom 1–Axiom 4. Thus our Theorem A should be considered as the parallel result of Dubey, Neyman, and Weber formulated in purely economic terms.

2. THE AXIOMATIC APPROACH

We define the notion of a price mechanism and present four axioms by which we describe desirable mechanisms; then we characterize the set of all price mechanisms that satisfy these axioms. A price mechanism can lead to a profit as well as to a loss for the producer. However, how the profit is shared or how the loss is covered will not be discussed.

We denote by E^m the m-dimensional Euclidean space, by E^m_+ the nonnegative orthant of E^m, and by E^m_{++} the positive orthant of E^m.

Let $\mathfrak{F}^m$ be the set of all real-valued functions F which are defined on E^m_+, which satisfy $F(0) = 0$, and which are continuously differentiable on E^m_+, where m is the number of commodities. A producer is characterized by a cost function[2]

[2] It is worth mentioning that for the results obtained in this section it is enough to consider only nondecreasing cost functions.

$F \in \mathfrak{F}^m$ defined on E_+^m. $F(\alpha)$ is the cost borne by him of producing the output vector α.

DEFINITION 1: A price mechanism is a function P which associates with each m, each $F \in \mathfrak{F}^m$ and each α in E_+^m a vector of prices $P(F,\alpha)$ in E^m,

$$P(F,\alpha) = (P_1(F,\alpha), \ldots, P_m(F,\alpha)).$$

We will characterize those price mechanisms which satisfy the following four axioms. The first axiom requires that the prices should be independent of the units of measurement. To illustrate it, suppose that F is a cost function of a producer who produces one commodity only. $F(x)$ is the cost of producing x units of this commodity. Assume that x is measured in kilograms. Let $G(y)$ be the cost function of the same producer where y is measured now in tons. Clearly

$$G(y) = F(1000y).$$

According to our notations, if α tons are produced the price *per one ton* is $P(G,\alpha)$. Since α tons are 1000α kg the price *per kg* is $P(F, 1000\alpha)$. Therefore, a price mechanism $P(\cdot,\cdot)$ which obeys the rescaling axiom should have the property that

$$P(G,\alpha) = 1000 \cdot P(F, 1000\alpha);$$

and in general the following should hold:

AXIOM 1 (Rescaling): *Let F be in $\mathfrak{F}^m$. Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be m positive real numbers. Let G be a function in $\mathfrak{F}^m$ defined by*

$$G(x_1, x_2, \ldots, x_m) = F(\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_m x_m).$$

Then, for each $\alpha \in E_+^m$ and each i, $1 \leqq i \leqq m$,

$$P_i(G,\alpha) = \lambda_i P_i(F, (\lambda_1 \alpha_1, \ldots, \lambda_m \alpha_m)).$$

The next axiom reflects the requirement that two commodities that are the "same" should have the same price. Since by definition a price mechanism yields prices that depend on the cost function and not on demand functions it is clear that being the "same commodity" means playing the same role in the cost function. As an illustration, consider someone who produces red and blue cars. He can represent his cost function as a two-variable function $F(x_1, x_2)$ where x_1 and x_2 are the quantities of red and blue cars respectively. But in fact, the cost of producing a red car is the same as the cost of producing a blue car. This can be formulated in the following way. There is a one-variable function G for which $G(x)$ is the cost of producing a total of x cars (red ones, blue ones, or both) and

$$F(x_1, x_2) = G(x_1 + x_2).$$

In this case the axiom asserts that the price of a blue car is the same as the price of a red car, which is the price of a car, i.e.,

$$P_1(F,(\alpha_1,\alpha_2)) = P_2(F,(\alpha_1,\alpha_2)) = P(G,\alpha_1+\alpha_2).$$

In general the following should hold:

AXIOM 2 (Consistency): *Let F be in $\mathcal{F}^m$ and let G be in $\mathcal{F}^1$. If for every $x \in E_+^m$,*

$$F(x_1,x_2,\ldots,x_m) = G\left(\sum_{i=1}^m x_i\right),$$

then, for each i, $1 \leqq i \leqq m$, and for each $\alpha \in E_+^m$,

$$P_i(F,\alpha) = P\left(G,\sum_{i=1}^m \alpha_i\right).$$

Suppose now that a given cost function F can be broken into two components, say G—the cost of raw-materials and H—the cost of labor. In that case it is reasonable to require that the prices arising from the cost F will be the sum of the prices arising from G and H. (In Section 4 we show that this axiom can be replaced by two other natural axioms.)

AXIOM 3 (Additivity): *Let F, G, and H be in $\mathcal{F}^m$. If for each $x \in E_+^m$,*

$$F(x_1,\ldots,x_m) = G(x_1,\ldots,x_m) + H(x_1,\ldots,x_m),$$

then for each $\alpha \in E_+^m$,

$$P(F,\alpha) = P(G,\alpha) + P(H,\alpha).$$

The last axiom asserts that if increasing production of every good results in higher costs, then the price of each good should be nonnegative. We may state this formally as follows:

AXIOM 4 (Positivity): *Let $F \in \mathcal{F}^m$ and let $\alpha \in E_{++}^m$. If F is nondecreasing at each $x \leqq \alpha$, then*[3]

$$P(F,\alpha) \geqq 0,$$

i.e., for each i, $1 \leqq i \leqq m$, $P_i(F,\alpha) \geqq 0$.

Price mechanisms which satisfy these four axioms are of special form as stated in the following theorem.

[3] By $x \leqq \alpha$ we mean $x_i \leqq \alpha_i$ for each i, $1 \leqq i \leqq m$.

THEOREM A: *$P(\cdot,\cdot)$ is a price mechanism which obeys Axioms* 1–4 *if and only if there is a nonnegative measure μ on $([0,1],\mathfrak{B})$ ($\mathfrak{B}$ is the set of all Borel subsets of $[0,1]$) such that for each m, for each $F \in \mathfrak{F}^m$ and for each $\alpha \in E_+^m$, $\alpha \neq 0$,*

$$(*) \qquad P_i(F,\alpha) = \int_0^1 \frac{\partial F}{\partial x_i}(t\alpha)\, d\mu(t) \qquad (i = 1, \ldots, m).$$

Moreover, for a given price mechanism $P(\cdot,\cdot)$ there is a unique measure μ which satisfies $(*)$. In other words, $(*)$ defines a one-to-one mapping from the set of all nonnegative measures on $([0,1],\mathfrak{B})$ onto the set of all price mechanisms obeying Axioms 1–4.

For an intuitive interpretation of the formula $(*)$, assume that the vector α is produced in an homogenous way, starting from 0 and ending at α. Suppose also that along this production process each time a "small" proportion (an infinitesimal one) of α is produced, the mth commodity is then charged its current marginal production cost. The price of the mth commodity, once α has been produced, will be the average of these marginal costs weighted by the measure μ which corresponds to the given price mechanism. If this measure happens to be the atomic probability measure whose whole mass is concentrated at the point, $t = 1$, i.e., if $\mu(\{1\}) = 1$, the associated price mechanism $P(\cdot,\cdot)$ is the well-known marginal cost price mechanism. For any m, for any $F \in \mathfrak{F}^m$ and for any $\alpha \in E_+^m$,

$$P_i(F,\alpha) = \frac{\partial F}{\partial x_i}(\alpha) \qquad (i = 1, \ldots, m).$$

If μ is chosen to be the Lebesgue measure on $[0,1]$ the associated price mechanism $P(\cdot,\cdot)$ is the Aumann–Shapley price mechanism (see [**3** and **6**]):

$$P_i(F,\alpha) = \int_0^1 \frac{\partial F}{\partial x_i}(t\alpha)\, dt \qquad (i = 1, \ldots, m).$$

These prices are the uniform average of the marginal cost along the diagonal $[0,\alpha]$.

We shall prove Theorem A through Proposition 1 to Proposition 4, below. Let $P(\cdot,\cdot)$ be a price mechanism obeying Axioms 1–4. The following is a simple consequence of the additivity and the positivity axioms.

PROPOSITION 1: *Let m be a positive integer. Let F and G be in $\mathfrak{F}^m$ and let $\alpha \in E_{++}^m$. If $F(x) = G(x)$ for each $x \leqq \alpha$, then*

$$P(F,\alpha) = P(G,\alpha).$$

Let $\alpha \in E_{++}^m$, and let C_α be the box $\{x \in E_+^m \mid x \leqq \alpha\}$. Let $\mathfrak{F}^m(C_\alpha)$ be the set of all continuously differentiable functions on C_α with $F(0) = 0$. Each

$F \in \mathfrak{F}^m(C_\alpha)$ can be extended to a function on E_+^m which is continuously differentiable (for a proof see, for example, Whitney [**8**]). If $\hat{F}$ and $\bar{F}$ are two such extensions of F we have by Proposition 1,

$$P(\hat{F}, \alpha) = P(\bar{F}, \alpha).$$

Therefore the function $P(\cdot, \alpha)$ on $\mathfrak{F}^m$ can be considered also as a function on $\mathfrak{F}^m(C_\alpha)$ which is positive and additive, i.e.,

$$P(F + G, \alpha) = P(F, \alpha) + P(G, \alpha),$$

for each F and G in $\mathfrak{F}^m(C_\alpha)$, and

$$P(F, \alpha) \geqq 0,$$

for each F which is nondecreasing on C_α.

Henceforth, we will refer to $P(\cdot, \alpha)$ as a function on $\mathfrak{F}^m$ as well as a function on $\mathfrak{F}^m(C_\alpha)$.

PROPOSITION 2: *There exists a nonnegative measure μ on* $([0,1], \mathfrak{B})$ *such that*

$$P(F, \alpha) = \int_0^1 \frac{\partial F}{\partial x}(t\alpha)\, d\mu(t),$$

for each $F \in \mathfrak{F}^1$ and $\alpha > 0$. Moreover, the measure μ is uniquely determined by the above equation.

PROOF: We will first prove the proposition in case $\alpha = 1$. By the last remark we will consider here $P(\cdot, 1)$ as a function on $\mathfrak{F}^1([0,1])$ (the set of all functions in $\mathfrak{F}^1$ restricted to $[0,1]$) and we will prove this proposition for functions F in $\mathfrak{F}^1([0,1])$.

There is 1-1 linear mapping τ from $C[0,1]$ (the class of continuous real functions on $[0,1]$) onto $\mathfrak{F}^1([0,1])$, defined by

$$\tau f(x) = \int_0^x f(t)\, dt, \qquad f \in C[0,1].$$

$P(\cdot, 1)$ defines a functional ψ on $C[0,1]$ by

$$\text{(1)} \qquad \psi f = P(\tau f, 1), \qquad f \in C[0,1].$$

By the additivity and the positivity of $P(\cdot, 1)$ we get the additivity and the positivity of ψ (positivity means here that $\psi f \geqq 0$ whenever $f \geqq 0$). By the additivity of ψ, $\psi(rf) = r\psi(f)$ for any rational number r. Using the positivity axiom it is easy to verify that the last equation holds for any real number. Thus ψ is a linear and positive functional on $C[0,1]$. Applying the Riesz Representation Theorem for ψ (see, for example, [7, p. 40]) yields the existence of a unique nonnegative measure μ on $([0,1], \mathfrak{B})$ such that

$$\psi(f) = \int_0^1 f(t)\, d\mu(t).$$

This, together with (1), implies that

$$P(F,1) = \int_0^1 \frac{\partial F}{\partial x}(t)\, d\mu(t). \tag{2}$$

Now let $F \in \mathscr{F}^1$ and let $\alpha > 0$. Define a function G in $\mathscr{F}^1$ by $G(x) = F(\alpha x)$; then by (2),

$$P(G,1) = \int_0^1 \frac{\partial G}{\partial x}(t)\, d\mu(t)$$

and by the rescaling axiom

$$P(G,1) = \alpha \cdot P(F,\alpha).$$

Therefore

$$P(F,\alpha) = \frac{1}{\alpha}\int_0^1 \frac{\partial G}{\partial x}(t)\, d\mu(t) = \int_0^1 \frac{\partial F}{\partial x}(t\alpha)\, d\mu(t).$$

DEFINITION 2: Let $C = C_\beta$ for $\beta \in E^m_{++}$. The norm $\|\ \|_1$ on $\mathscr{F}^m(C)$ (the set of all continuously differentiable functions F on C with $F(0) = 0$) is defined by

$$\|F\|_1 = \sum_{i=1}^{m} \sup\left|\frac{\partial F}{\partial x_i}\right|$$

where the sup is taken over C.

It is easy to check that $\|\ \|_1$ indeed defines a norm on $\mathscr{F}^m(C)$. The property that $\|F\|_1 = 0$ implies $F = 0$, follows from $F(0) = 0$.

PROPOSITION 3: *Let $C = C_\beta$ for $\beta \in E^m_{++}$. For each $\alpha \in C$ the function $P(\cdot,\alpha)$ is continuous in the norm $\|\ \|_1$ on $\mathscr{F}^m(C)$.*

PROOF: Since $P(\cdot,\alpha)$ is additive it is sufficient to prove that if $(F_n)_{n=1}^{\infty}$ is a sequence of functions in $\mathscr{F}^m(C)$ satisfying $\|F_n\|_1 \to 0$, as $n \to \infty$, then,

$$P_i(F_n,\alpha) \to 0, \qquad \text{as } n \to \infty,$$

for each i, $1 \leqq i \leqq m$. From the additivity axiom one can easily verify that for each rational number λ and for each $F \in \mathscr{F}^m$,

$$P(\lambda F,\alpha) = \lambda P(F,\alpha).$$

For each integer n let us choose a positive rational number ϵ_n such that

(3) $\qquad \epsilon_n \to 0, \quad \text{as } n \to \infty \quad \text{and} \quad \|F_n\|_1 < \epsilon_n.$

Let R be the function in $\mathcal{F}^m$ defined by

$$R(x_1, x_2, \dots, x_m) = \sum_{j=1}^{m} x_j.$$

By (3), for each i, $1 \leqq i \leqq m$, and for each $x \in C$,

$$\frac{\partial(\epsilon_n R - F_n)}{\partial x_i}(x) = \epsilon_n - \frac{\partial F_n}{\partial x_i}(x) > 0$$

and

$$\frac{\partial(\epsilon_n R + F_n)}{\partial x_i}(x) = \frac{\partial F_n}{\partial x_i}(x) + \epsilon_n > 0.$$

Therefore, $\epsilon_n R - F_n$ and $\epsilon_n R + F_n$ are nondecreasing functions on C. By the positivity and the additivity axioms we have,

$$P_i(F_n, \alpha) \leqq P_i(\epsilon_n R, \alpha) = \epsilon_n P_i(R, \alpha)$$

and

$$P_i(F_n, \alpha) \geqq -P_i(\epsilon_n R, \alpha) = -\epsilon_n P_i(R, \alpha).$$

Thus

$$P_i(F_n, \alpha) \to 0 \quad \text{as } n \to \infty,$$

and the proof of Proposition 3 is completed.

PROPOSITION 4: *For any polynomial p in $\mathcal{F}^m$ and for any $\alpha \in E_+^m$, $\alpha \neq 0$,*

$$P_i(p, \alpha) = \int_0^1 \frac{\partial p}{\partial x_i}(t\alpha)\, d\mu(t), \qquad 1 \leqq i \leqq m.$$

PROOF: Any polynomial in $\mathcal{F}^m$ is a linear combination of polynomials of the form

(4) $\qquad F(x_1, \dots, x_m) = (n_1 x_1 + \dots + n_m x_m)^l$

where the n_i's are nonnegative integers and l is a positive integer (e.g. see [**1**, p. 41]). By the additivity axiom it is sufficient to prove the proposition for functions of this form. Let us assume first, that for each i, $1 \leqq i \leqq m$, $n_i > 0$.

Let L be the function in $\mathcal{F}^1$ defined by

$$L(x) = x^l.$$

Then by (4)

$$F(x_1, \ldots, x_m) = L\left(\sum_{j=1}^{m} n_j x_j\right). \tag{5}$$

By the rescaling and the consistency axioms, for each $\alpha \in E^m_+$ and each i, $1 \leqq i \leqq m$,

$$P_i(F, (\alpha_1, \ldots, \alpha_m)) = n_i P_i\left(L, \sum_{j=1}^{m} n_j \alpha_j\right). \tag{6}$$

Since $\sum_{j=1}^{m} n_j \alpha_j > 0$ we can use Proposition 2 to obtain

$$P\left(L, \sum_{j=1}^{m} n_j \alpha_j\right) = \int_0^1 \frac{dL}{dx}\left(t \cdot \sum_{j=1}^{m} n_j \alpha_j\right) d\mu(t).$$

The proof then follows by (5) and (6).

In the general case, however some of the n_i's might be zero. In that case we define for each $\epsilon > 0$ a function F_ϵ in $\mathcal{F}^m$ by

$$F_\epsilon(x) = ((n_1 + \epsilon)x_1 + \ldots + (n_m + \epsilon)x_m)^l.$$

Let $C = C_\beta$ for some $\beta \in E^m_{++}$. Clearly

$$\|F_\epsilon - F\|_1 \to 0 \quad \text{as } \epsilon \to 0. \tag{7}$$

Since the coefficients of the x_i's in F_ϵ are all positive we have by the previous case

$$P_i(F_\epsilon, \alpha) = \int_0^1 \frac{\partial F_\epsilon}{\partial x_i}(t\alpha)\, d\mu(t) \qquad (i = 1, \ldots, m). \tag{8}$$

The left hand side of (8) tends, by (7) and Proposition 3, to $P_i(F, \alpha)$ when $\epsilon \to 0$. The right hand side of (8) tends to $\int_0^1 (\partial F/\partial x_i)(t\alpha)\, d\mu(t)$, and so

$$P_i(F, \alpha) = \int_0^1 \frac{\partial F}{\partial x_i}(t\alpha)\, d\mu(t).$$

Hence the proof of Proposition 4 is complete.

We are now ready to prove Theorem A.

Proof of Theorem A: Let $P(\cdot, \cdot)$ be a price mechanism obeying the four axioms. Let $F \in \mathcal{F}^m$ and let $\alpha \neq 0$ be in E^m_+. Choose β with $\beta \gg \alpha$ and denote

$C = C_\beta$. The polynomials in m variables are dense in $\mathfrak{F}^m(C)$ with C^1 norm (for a proof see [4, p. 68]). (The C^1 norm is defined by

$$\|F\|_{C^1} = \sup|F| + \sum_{i=1}^{m} \sup\left|\frac{\partial F}{\partial x_i}\right|.\Big)$$

Therefore, there exists a sequence of polynomials $(\hat{p}_n)_{n=1}^{\infty}$ such that $\|\hat{p}_n - F\|_{C^1} \to 0$ as $n \to \infty$. Thus if

$$p_n = \hat{p}_n - \hat{p}_n(0),$$

then $p_n(0) = 0$ and $\|p_n - F\|_1 \to 0$ as $n \to \infty$. Since the polynomials in m variables are dense in $\mathfrak{F}^m(C)$ the first part of the theorem follows by Proposition 3 and 4. The second part can easily be verified.

3. THE DETERMINATION OF THE MARGINAL COST PRICES BY A SET OF AXIOMS

Let us strengthen the positivity axiom (Axiom 4) as follows.

AXIOM 4*: *Let $F \in \mathfrak{F}^m$ and let $\alpha \in E^m_{++}$. If F is nondecreasing at each $x \leqq \alpha$ in a neighborhood of α, then $P(F,\alpha) \geqq 0$.*

In other words, we require that the prices be nonnegative at α even if F is nondecreasing in a neighborhood of α only.

It is clear that Axiom 4* implies Axiom 4 and therefore by Theorem A a price mechanism $P(\cdot,\cdot)$ which satisfies Axioms 1, 2, 3, and 4* is of the form

$$P_i(F,\alpha) = \int_0^1 \frac{\partial F}{\partial x_i}(t\alpha)\, d\mu(t).$$

But in fact the available set of mechanisms is now much smaller.

THEOREM B: *A price mechanism $P(\cdot,\cdot)$ satisfies Axioms* 1, 2, 3, *and* 4* *if and only if there is a constant $c \geqq 0$ such that for each m, $F \in \mathfrak{F}^m$ and $\alpha \in E^m_+ (\alpha \neq 0)$,*

$$P_i(F,\alpha) = c \cdot \frac{\partial F}{\partial x_i}(\alpha) \qquad (i = 1, \ldots, m).$$

PROOF: It is obvious that a price mechanism $P(\cdot,\cdot)$ defined by

$$P_i(F,\alpha) = c \cdot \frac{\partial F}{\partial x_i}(\alpha), \qquad c \geqq 0,$$

obeys the four axioms.

Assume now that a price mechanism $P(\cdot,\cdot)$ satisfies the four axioms. Then, by

Theorem A there exists a nonnegative measure μ on $([0,1], \mathcal{B})$ such that

$$P_i(F,\alpha) = \int_0^1 \frac{\partial F}{\partial x_i}(t\alpha)\, d\mu(t),$$

for each m, $F \in \mathcal{F}^m$ and $\alpha \in E_+^m$ $(\alpha \neq 0)$.

Notice that if F is constant in a neighborhood of α then $P_i(F,\alpha) = 0$ (apply Axiom 4* for F and $-F$). Define, for each ϵ, $1 > \epsilon > 0$, a function f_ϵ: $E_+^1 \to E_+^1$ by

$$f_\epsilon(x) = \begin{cases} 1, & 0 \leqq x \leqq 1-\epsilon, \\ -\dfrac{2}{\epsilon}x + \dfrac{2}{\epsilon} - 1, & 1-\epsilon \leqq x \leqq 1 - \dfrac{\epsilon}{2}, \\ 0, & 1 - \dfrac{\epsilon}{2} \leqq x. \end{cases}$$

Since f_ϵ is continuous the function F_ϵ defined by

$$F_\epsilon(x) = \int_0^x f_\epsilon(t)\, dt$$

is in $\mathcal{F}^1$. F_ϵ is constant in a neighborhood of $\alpha = 1$; therefore

$$P(F_\epsilon, 1) = 0.$$

Hence

$$\int_0^1 f_\epsilon(t)\, d\mu(t) = 0.$$

On the other hand

$$\int_0^1 f_\epsilon(t)\, d\mu(t) \geqq \mu([0, 1-\epsilon]) \geqq 0.$$

Therefore, for each $0 < \epsilon < 1$,

$$\mu([0, 1-\epsilon]) = 0.$$

Thus, $\mu([0,1)) = 0$ which implies that $\mu([0,1]) = \mu(\{1\})$, and the proof is completed.

COROLLARY: *If in addition to Axioms* 1, 2, 3, *and* 4* *we require for the identity* (*one variable*) *function* $H(x) = x$ *that* $P(H,1) = 1$ *then the MC pricing is the only price mechanism which obeys these requirements* (*i.e. in this case the constant* c *of the Theorem B must equal* 1).

PROOF: According to Theorem B, $P(H,1) = c \cdot (dH/dx)(1) = 1$. Thus $c = 1$.

Finally, let us return to the four original Axioms 1–4 and add the axiom which requires cost sharing (total cost equal total revenue).

AXIOM 5 (Cost Sharing): *For each m, each $F \in \mathcal{F}^m$, and each $\alpha \in E^m_+$,*

$$\alpha \cdot P(F,\alpha) = F(\alpha).$$

THEOREM C: *There is a unique price mechanism $P(\cdot,\cdot)$ which satisfies Axioms 1–5. $P(\cdot,\cdot)$ is given by*

$$P(F,\alpha) = \int_0^1 \frac{\partial F}{\partial x_i}(t\alpha)\,dt.$$

$P(\cdot,\cdot)$ is the Aumann–Shapley price mechanism.

This result was previously stated (independently) by Billera–Heath [3] and by Mirman–Tauman [6]. However it is also an immediate corollary of Theorem A above. Indeed, assume that $P(\cdot,\cdot)$ is a price mechanism obeying Axioms 1–5. By Theorem A there is a nonnegative measure μ on $([0,1]), \mathcal{B})$ such that

$$P_i(F,\alpha) = \int_0^1 \frac{\partial F}{\partial x_i}(t\alpha)\,d\mu(t), \tag{9}$$

for each m, $1 \leqq i \leqq m$, $F \in \mathcal{F}^m$ and $\alpha \in E^m_+$, $(\alpha \neq 0)$. Since $P(\cdot,\cdot)$ satisfies Axiom 5,

$$P(F,1) = \frac{F(1)}{1} = \int_0^1 \frac{dF}{dx}(t)\,dt,$$

for each $F \in \mathcal{F}^1$. Therefore by (9) we get

$$\int_0^1 \frac{dF}{dx}(t)\,dt = \int_0^1 \frac{dF}{dx}(t)\,d\mu(t).$$

It then follows that the measure μ and the Lebesgue measure coincide on $C[0,1]$ as linear functionals on $C[0,1]$. Therefore by the Riesz Representation Theorem these two measures are the same.

4. SEPARABILITY REPLACING ADDITIVITY

In this section we show that the additivity axiom (Axiom 3) can be replaced by two natural axioms. The first is very similar in spirit to the consistency axiom (Axiom 2), and the second deals with a set of commodities that can be separated into two subsets, which can be produced independently.

For the first axiom consider the production of two commodities with the cost function $F(x, y)$. The producer can decide to generate a new commodity consisting of the other two such that each unit of the new commodity consists of a unit of the first commodity and a unit of the second one. The cost function G for the new commodity satisfies

$$G(x) = F(x,x).$$

It is only natural to ask that the price per unit of the new commodity will be the sum of the two prices of the original commodities. In general:

AXIOM 1 (Aggregation): *Let* $F(x_{11}, \ldots, x_{1n_1}, x_{21}, \ldots, x_{2n_2}, \ldots, x_{m1}, \ldots, x_{mn_m})$ *be in* $\mathcal{F}^l$ *where* $l = \sum_{i=1}^m n_i$. *Let* G *be the function in* $\mathcal{F}^m$ *defined by*

$$G(x_1, \ldots, x_m) = F\Big(\underbrace{x_1, \ldots, x_1}_{n_1}, \underbrace{x_2, \ldots, x_2}_{n_2}, \ldots, \underbrace{x_m, \ldots, x_m}_{n_m}\Big).$$

Then for each i, $1 \leqq i \leqq m$,

$$P_i(G,(\alpha_1, \ldots, \alpha_m)) = \sum_{j=1}^{n_i} P_{ij}(F,(\alpha_1, \ldots, \alpha_1, \alpha_2, \ldots, \alpha_2, \ldots, \alpha_m, \ldots, \alpha_m)).$$

For the second axiom assume that $n_1 + n_2 = m$ commodities can be produced with cost function F. The first n_1 commodities are n_1 types of cars and the remaining are n_2 types of shoes, which are independently produced, i.e., there are two cost functions $G(x_1, \ldots, x_{n_1})$ and $H(y_1, \ldots, y_{n_2})$ such that

$$F(x_1, \ldots, x_{n_1}, x_{n_1+1}, \ldots, x_m) = G(x_1, \ldots, x_{n_1}) + H(x_{n_1+1}, \ldots, x_m).$$

The axiom we state requires that the price of each commodity should depend only on that part of the cost function that it affects. In order to formulate this axiom we use the following notation. Let $N = \{i_1, i_2, \ldots, i_n\}$, where $i_1 < i_2 < \ldots < i_n$, be a subset of $\{1, \ldots, m\}$, and let $x \in E^m$. Denote by x_N the vector in E^n defined by:

$$X_N = (x_{i_1}, \ldots, x_{i_n}).$$

AXIOM 2 (Separability): *Let* N_1 *and* N_2 *be disjoint sets with* n_1 *and* n_2 *elements respectively such that* $N_1 \cup N_2 = \{1, \ldots, m\}$. *Let* F, G, *and* H *be functions defined on* E^m_+, $E^{n_1}_+$, *and* $E^{n_2}_+$ *respectively. If for each* $x \in E^m_+$,

$$F(x) = G(x_{N_1}) + H(x_{N_2});$$

then for each $\alpha \in E^m_+$,

$$P_{N_1}(F,\alpha) = P(G,\alpha_{N_1})$$

and

$$P_{N_2}(F,\alpha) = P(H,\alpha_{N_2}).$$

PROPOSITION 5: *Axioms* 1 *and* 2 *imply the additivity axiom* 3.

PROOF: Let F, G, and H be functions defined on E_+^m such that

$$F(x) = G(x) + H(x)$$

for each $x \in E_+^m$. Define a function L on E_+^{2m} by

$$L(x_1, x_2, \ldots, x_{2m}) = G(x_1, x_3, \ldots, x_{2m-1}) + H(x_2, x_4, \ldots, x_{2m}).$$

Denote by N_1 and N_2 the sets of odd and even numbers respectively, in the set $\{1, \ldots, 2m\}$. By the separability axiom for each $\alpha \in E_+^m$,

$$(10) \qquad \begin{cases} P_{N_1}(L,\alpha) = P(G,\alpha_{N_1}), \\ P_{N_2}(L,\alpha) = P(H,\alpha_{N_2}). \end{cases}$$

For $x = (x_1, \ldots, x_m)$, let us denote $\hat{x} = (x_1, x_1, \ldots, x_m, x_m)$. By (10), for each $\alpha \in E_+^m$,

$$(11) \qquad \begin{cases} P_{N_1}(L,\hat{\alpha}) = P(G,\alpha), \\ P_{N_2}(L,\hat{\alpha}) = P(H,\alpha). \end{cases}$$

By the definition of L it follows that for each $x \in E_+^m$,

$$L(\hat{x}) = G(x) + H(x) = F(x).$$

From the aggregation axiom we deduce that for each i, $1 \leqq i \leqq m$, and for each $\alpha \in E_+^m$,

$$(12) \qquad P_i(F,\alpha) = P_{2i-1}(L,\hat{\alpha}) + P_{2i}(L,\hat{\alpha}).$$

However, by (11),

$$(13) \qquad \begin{cases} P_{2i-1}(L,\hat{\alpha}) = P_i(G,\alpha), \\ P_{2i}(L,\hat{\alpha}) = P_i(H,\alpha). \end{cases}$$

Therefore from (12) and (13),

$$P_i(F,\alpha) = P_i(G,\alpha) + P_i(H,\alpha),$$

for each $\alpha \in E_+^m$ and $1 \leqq i \leqq m$. Thus the proof of Proposition 5 is completed.

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