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S5 knowledge without partitions

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Abstract We study set algebras with an operator (SAO) that satisfy the axioms of S5 knowledge. A necessary and sufficient condition is given for such SAOs that the knowledge operator is defined by a partition of the state space. SAOs are constructed for which the condition fails to hold. We conclude that no logic singles out the partitional SAOs among all SAOs.

Keywords Epistemic logic · Modal logic · S5 · Partitions · Boolean algebras with operators

1 Introduction

The standard structure used in economic theory, game theory, and decision theory to describe the knowledge of an agent is a set of *states* Ω endowed with a partition Π.[1] An informal justification of this modeling of knowledge uses the notion of a *signal*. The agent is said to observe a signal that may depend on the state. The partition of Ω into sets of states with the same signal results in Π. Thus, in each state ω the agent cannot tell which of the states obtains in $\Pi(\omega)$—the element of the partition that contains ω—because she observes the same signal in all these states, but she can tell that all the states outside $\Pi(\omega)$ do not obtain, as the signals observed in these states are different from the one observed in ω. Of course, the signal is a metaphor for what the agent learns or knows.

Using the partition we can formally describe the agent's knowledge in terms of subsets of states. We say that the agent *knows a given subset of states E in state ω* if

[1] In similar structures that are used for the semantics of modal logic, states are referred to as *possible worlds*

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and only if $\Pi(\omega) \subseteq E$. This definition gives rise to a knowledge operator K on Ω, which associates with each subset of states E the subset $\mathrm{K}(E)$ of the states in which the agent knows E.

It is easy to verify that the knowledge operator K satisfies the following three properties: First, it preserves intersections. Second, when the agent knows E (in a given state ω), E must hold (in this state). Third, when the agent does not know E she knows that she does not know E. These three properties, when formulated as axioms in a formal logic, generate the modal logic S5, which served Hintikka (1962) to study epistemic logic—the logic of knowledge.

The converse also holds true: If an operator K on Ω is an S5 operator, that is, if it satisfies the three above-mentioned properties, then K is *partitional*, in the sense that it is defined by some partition Π of Ω (see, e.g., Geanakoplos 1994; Aumann 1999a).[2] Thus, Geanakoplos (1994) could state: "The partition approach to knowledge is completely equivalent to the knowledge operator approach satisfying S5."

The results presented here show that this equivalence is not complete. The properties of knowledge expressed in S5 do not in themselves imply partitions. It all depends on the type of structures to which the S5 axioms are applied. Geanakoplos (1994) and Aumann (1999a) chose to deal with structures for which S5 does imply partition. But for other, very reasonable structures, S5 does not imply partition.

More specifically, in the structure discussed above, the knowledge operator is defined on the power set of the state space. Here we consider structures of knowledge where the knowledge operator is defined on a given algebra of subsets of the state space, which is not necessarily the whole power set. For such structures, S5 knowledge operators may fail to be derived from partitions, as we demonstrate in Sect. 5.

The assumption that the knowledge operator is defined for each subset while being simple and convenient is undesirable in many cases. If we think of subsets of states as propositions that correspond to sentences in a finitery language, then we should confine ourselves to a countable number of subsets. Thus, if the state space is infinite, its power set does not fit our purposes as it is uncountable. Similarly, if we consider structures of knowledge and probabilistic beliefs, like the one in Aumann (1999b), we will want sometimes to restrict belief operators as well as knowledge operators to a σ-algebra of subsets which are not the whole power set.

We introduce the preliminaries of set algebras with an operator in the next section. in Sect. 3, we characterize S5 knowledge operators by a property of their ranges—the knowledge algebras. In Sect. 4 we formulate a necessary and sufficient condition for an S5 knowledge operator to be partitional and derive from it the known result that knowledge is partitional when the operator is defined on the power set or when the state space is finite. The results of these sections are used in Sect. 5 to construct two examples, one with a countable state space and the other with an uncountable one, with S5 knowledge operator that cannot be derived from any partition.

[2] The references are from the game theory literature because the question of deriving a partition from properties of an operator on a state space has not been addressed by students of modal logic. As we see in Sect. 6, the result in these references is stronger than the well-known result of modal logic that the accessibility relation in S5 models and frames must be an equivalence relation.

In the last section we examine partition structures from the perspective of modal logic. We conclude that the modal logic S5 characterizes partition structures when the semantics is confined to frames or general frames. However, when the semantics is that of set algebras with an operator, no logic can characterize partition structures.

2 Set algebras with S5 knowledge

A *set algebra* is a pair $(\Omega, \mathcal{A})$, where Ω is a non-empty set of *states* and $\mathcal{A}$ an algebra of subsets of Ω, namely, $\Omega \in \mathcal{A}$, and $\mathcal{A}$ is closed with respect to complements and intersections. The elements of $\mathcal{A}$ are called *events*. The complement of an event E is denoted by $\neg E$.

Definition 1 A *set algebra with an operator* (SAO) is a triplet $(\Omega, \mathcal{A}, \mathrm{K})$, where $(\Omega, \mathcal{A})$ is a set algebra, and K an operator $\mathrm{K}\colon \mathcal{A} \to \mathcal{A}$.

We say that $(\Omega, \mathcal{A}, \mathrm{K})$ is an S5 SAO, or equivalently that K is S5 *knowledge*, if K satisfies for each $E, F \in \mathcal{A}$ the following three relations called *axioms*:

K1.	$\mathrm{K}(E) \subseteq E$	truth
K2.	$\mathrm{K}(E) \cap \mathrm{K}(F) = \mathrm{K}(E \cap F)$	conjunction
K3.	$\neg\mathrm{K}(E) = \mathrm{K}(\neg\mathrm{K}(E))$	negative introspection

Axiom K3 implies:[3]

K4. $\mathrm{K}(E) = \mathrm{K}(\mathrm{K}(E))$ positive introspection

In addition, it follows from K2 that K is monotonic with respect to inclusion, that is, if $E \subseteq F$ then $\mathrm{K}(E) \subseteq \mathrm{K}(F)$.

We say that an SAO $(\Omega, \mathcal{A}, \mathrm{K})$ is *partitional*, or equivalently that K is partitional, if there exists a partition Π of Ω such that $\omega \in \mathrm{K}(E)$ iff $\Pi(\omega) \subseteq E$, where $\Pi(\omega)$ is the element of Π that contains ω. The partition Π is said to *generate* K. Note that we do not require that the elements of the partition Π be in $\mathcal{A}$.

It is straightforward to show,

Claim 1 *A partitional* SAO *is an S5 SAO.*

Here we study conditions on SAOs under which they are partitional. These conditions are used to construct examples of non-partitional SAOs.

[3] Indeed, taking complements in K3 we have $\mathrm{K}(E) = \neg\mathrm{K}(\neg\mathrm{K}(E))$. Applying K3 to the right-hand side we conclude $\mathrm{K}(E) = \mathrm{K}(\neg\mathrm{K}(\neg\mathrm{K}(E)))$. Replacing $\mathrm{K}(\neg\mathrm{K}(E))$, on the right-hand side, with $\neg\mathrm{K}(E)$ we get the desired equality.

3 Knowledge subalgebras

For a given algebra of sets $(\Omega, \mathcal{A})$, we characterize the subsets of $\mathcal{A}$ that can be obtained as the range of some S5 knowledge operator on $\mathcal{A}$. We show moreover that an S5 knowledge operator is determined by and expressed in terms of its range.[4]

We first note that this range must be an algebra.

Claim 2 *Let* $(\Omega, \mathcal{A}, \mathrm{K})$ *be an S5 SAO. Then, the range of* K, $\mathcal{A}_{\mathrm{K}} = \{\mathrm{K}(E) \mid E \in \mathcal{A}\}$, *is a subalgebra of* $\mathcal{A}$*, which we call the knowledge subalgebra. Moreover, the restriction of* K *to* $\mathcal{A}_{\mathrm{K}}$ *is the identity.*

Indeed, by K1, $\mathcal{A}_{\mathrm{K}}$ contains the empty set. By K2, it is closed under intersection, and finally, it is closed under complement by K3. By K4, K is the identity on $\mathcal{A}_{\mathrm{K}}$.

Next, we define the property of a subalgebra of $\mathcal{A}$ that characterizes it as an S5 knowledge algebra.

Definition 2 A subalgebra $\mathcal{A}' \subseteq \mathcal{A}$ has the *maximality property* (w.r.t $\mathcal{A}$) if for each $E \in \mathcal{A}$ the set of events $\mathcal{A}'_E = \{F \mid F \in \mathcal{A}', F \subseteq E\}$ has a maximal element with respect to inclusion.

Proposition 1 *Let* $(\Omega, \mathcal{A})$ *be a set algebra and* $\mathcal{A}'$ *be a subalgebra of* $\mathcal{A}$*. Then there exists an* S5 *knowledge operator* K *on* $\mathcal{A}$ *such that* $\mathcal{A}' = \mathcal{A}_{\mathrm{K}}$ *if and only if* $\mathcal{A}'$ *has the maximality property. In this case,* $\mathrm{K}(E)$ *is the maximal element in* $\mathcal{A}'_E$.

Proof Suppose that for some S5 knowledge operator on $\mathcal{A}$, $\mathcal{A}' = \mathcal{A}_{\mathrm{K}}$ and let $E \in \mathcal{A}$. Then $\mathrm{K}(E) \in \mathcal{A}'$, and by K1, $\mathrm{K}(E) \in \mathcal{A}'_E$. If $F \in \mathcal{A}'_E$ then by monotonicity $\mathrm{K}(F) \subseteq \mathrm{K}(E)$. Since K is the identity on $\mathcal{A}'$, $\mathrm{K}(F) = F$, and thus $F \subseteq \mathrm{K}(E)$. This shows that $\mathcal{A}'$ has the maximality property.

Conversely, suppose that $\mathcal{A}'$ satisfies the maximality property. Let K be the operator such that for each $E \in \mathcal{A}$, $\mathrm{K}(E)$ is the maximal element in $\mathcal{A}'_E$. Obviously K1 holds by definition. It can be easily verified that if A and B are the maximal elements in $\mathcal{A}'_E$ and $\mathcal{A}'_F$ correspondingly, then $A \cap B$ is the maximal element in $\mathcal{A}'_{E \cap F}$. This shows that $K2$ holds. Since by definition K is the identity on $\mathcal{A}'$ and the algebra is closed under complements, K3 follows. □

4 Partitional set algebras with an operator

Each S5 SAO $(\Omega, \mathcal{A}, \mathrm{K})$ is naturally associated with the partition of Ω into sets of states which are included in the same events of the knowledge subalgebra $\mathcal{A}_K$. We show that if knowledge is partitional, it is this partition that defines the knowledge operator. The necessary and sufficient condition for an S5-SAO to be partitional is formulated in terms of this partition.

Formally, the set of all the events that describe knowledge at ω is called the *ken* at ω and is denoted by $\mathrm{Ken}(\omega)$. Thus, $\mathrm{Ken}(\omega) = \{E \mid \omega \in E \in \mathcal{A}_{\mathrm{K}}\}$. Note, that as $\mathcal{A}_K$ is

4 I thank Hannu Salonen for indicating that the results in this section have already been proved in Halmos (1962, pp. 44–45) for Boolean algebras with an operator.

an algebra, for each event E, either $E \in \mathrm{Ken}(\omega)$ or $\neg E \in \mathrm{Ken}(\omega)$. The ken-partition of Ω, Π_{Ken}, partitions Ω into sets of states with the same ken.

Proposition 2

$$\Pi_{\mathrm{Ken}}(\omega) = \cap_{E \in \mathrm{Ken}(\omega)} E$$

Proof If $w' \in \Pi_{\mathrm{Ken}}(\omega)$ then $\Pi_{\mathrm{Ken}}(\omega') = \Pi_{\mathrm{Ken}}(\omega)$. Thus, $\omega' \in \cap_{E \in \mathrm{Ken}(\omega')} E = \cap_{E \in \mathrm{Ken}(\omega)} E$. Conversely, suppose $\omega' \in \cap_{E \in \mathrm{Ken}(\omega)} E$, then $\mathrm{Ken}(\omega) \subseteq \mathrm{Ken}(\omega')$. Suppose $E \in \mathrm{Ken}(\omega')$, but $E \notin \mathrm{Ken}(\omega)$. Then $\omega' \in E$, and $\neg E \in \mathrm{Ken}(\omega)$. Therefore $\neg E \in \mathrm{Ken}(\omega')$, and $\omega' \in \neg E$, which is a contradiction. Thus, $\mathrm{Ken}(\omega) = \mathrm{Ken}(\omega')$, and therefore $\omega' \in \Pi_{\mathrm{Ken}}(\omega)$. □

Proposition 3 *If* $(\Omega, \mathcal{A}, \mathrm{K})$ *is partitional then* K *is generated by the ken-partition* Π_{Ken} *and this partition is weakly coarser then any partition that generates* K.

Proof If Π generates K, then for any $F \in \mathcal{A}_{\mathrm{K}}$, $\omega \in F$ iff $\Pi(\omega) \subseteq F$. Hence by Proposition 2, $\Pi(\omega) \subseteq \Pi_{\mathrm{Ken}}(\omega)$. This shows that $\Pi_{\mathrm{Ken}}(\omega)$ is weakly coarser than $\Pi(\omega)$. It also shows that if $\Pi_{\mathrm{Ken}}(\omega) \subseteq F$ then $\omega \in \mathrm{K}(F)$. To see the converse, assume $\omega \in \mathrm{K}(F)$. Then $\mathrm{K}(F) \in \mathcal{A}_{\mathrm{K}}$ and again, by Proposition 2 and the monotonicity of K, $\Pi_{\mathrm{Ken}}(\omega) \subseteq \mathrm{K}(F) \subseteq F$. □

The next theorem provides a necessary and sufficient condition for an SAO to be partitional.

Theorem 1 *An* S5 SAO $(\Omega, \mathcal{A}, \mathrm{K})$ *is partitional if and only if for each* ω *and* $A \in \mathcal{A}$ *the following condition holds:*

If for each $E \in \mathrm{Ken}(\omega)$, $E \cap A \neq \emptyset$, *then* $\Pi_{\mathrm{Ken}}(\omega) \cap A \neq \emptyset$.

Proof Suppose K is partitional and $\Pi_{\mathrm{Ken}}(\omega) \cap A = \emptyset$. Then $\Pi_{\mathrm{Ken}}(\omega) \subseteq \neg A$ and hence by Proposition 3, $\mathrm{K}(\neg A) \in \mathrm{Ken}(\omega)$. By K1, $\mathrm{K}(\neg A) \cap A = \emptyset$. Thus for $E = \mathrm{K}(\neg A)$ in $\mathrm{Ken}(\omega)$, $E \cap A = \emptyset$.

Conversely, suppose the condition holds. We need to show that $\omega \in \mathrm{K}(A)$ iff $\Pi_{\mathrm{Ken}}(\omega) \subseteq A$. The "only if" direction is the same as the proof in Proposition 3. For the other direction, suppose $\Pi_{\mathrm{Ken}}(\omega) \subseteq A$. Since $\Pi_{\mathrm{Ken}}(\omega) \cap \neg A = \emptyset$ it follows by the condition that there exists E such that $\mathrm{K}(E) \in \mathrm{Ken}(\omega)$ and $\mathrm{K}(E) \cap \neg A = \emptyset$, that is, $\mathrm{K}(E) \subseteq A$. Thus, by K4 and the monotonicity of K, $\omega \in \mathrm{K}(E) = \mathrm{K}(\mathrm{K}(E)) \subseteq \mathrm{K}(A)$. □

The following theorem is a corollary of the characterization of partitional SOA s.

Theorem 2 *If* $(\Omega, \mathcal{A}, \mathrm{K})$ *is an* S5 SAO *and* $\Pi_{\mathrm{Ken}} \subseteq \mathcal{A}$ *then* $(\Omega, \mathcal{A}, \mathrm{K})$ *is partitional.*

Proof Suppose $\Pi_{\mathrm{Ken}} \subseteq \mathcal{A}$ and let $M = \Pi_{\mathrm{Ken}}(\omega)$. Then $\neg M = \cup_{E \in \mathrm{Ken}(\omega)} \neg E$. Since K is the identity on $\mathcal{A}_{\mathrm{K}}$, and it is monotonic, it follows that for each $E \in \mathrm{Ken}(\omega)$, $\neg E = \mathrm{K}(\neg E) \subseteq \mathrm{K}(\neg M)$. Thus, $\neg M \subseteq \mathrm{K}(\neg M)$. This, with K1, implies $\neg M = \mathrm{K}(\neg M)$, which means that $\neg M \in \mathcal{A}_{\mathrm{K}}$ and hence $M \in \mathcal{A}_{\mathrm{K}}$. If for each $E \in \mathrm{Ken}(\omega)$, $E \cap A \neq \emptyset$, then in particular $M \cap A \neq \emptyset$, which means that the condition in Theorem 1 holds. □

Obviously $\Pi_{\text{Ken}} \subseteq \mathcal{A}$ holds when $\mathcal{A} = 2^{\Omega}$. It also holds when $\mathcal{A}$ is finite, since $\text{Ken}(\omega)$ is closed under finite intersection. Hence the following corollary of Theorem 2.

Corollary 1 *If* $(\Omega, \mathcal{A}, \mathrm{K})$ *is an* S5 SAO *and either* $\mathcal{A}$ *is finite or* $\mathcal{A} = 2^{\Omega}$*, then it is partitional.*

Corollary 2 *If* $(\Omega, \mathcal{A}, \mathrm{K})$ *is an* S5 SAO*, then* K *can be extended to an* S5 *knowledge operator on* 2^{Ω} *iff* $(\Omega, \mathcal{A}, \mathrm{K})$ *is partitional.*

Indeed, if $(\Omega, \mathcal{A}, \mathrm{K})$ is partitional then the partition that generates K generates an S5 knowledge operator on 2^{Ω} which extends K. Conversely, if K can be extended to an S5 operator on 2^{Ω}, then this extension is partitional by Corollary 1 and the same partition also generates K on $\mathcal{A}$.

The condition in Theorem 2 is sufficient for an SAO to be partitional but not necessary. We show this in two simple examples that will be used in the sequel. We note first that,

Proposition 4 *If* K *is an identity operator on* $\mathcal{A}$ *then* $(\Omega, \mathcal{A}, \mathrm{K})$ *is partitional.*

Proof It is easy to see that if K is the identity operator, it satisfies K1–K3. Also, in this case, $\text{Ken}(\omega)$ is the set of all events in $\mathcal{A}$ that contain ω, and therefore $\Pi_{\text{Ken}}(\omega) = \cap_{\omega \in E} E$. Thus, $\omega' \in \Pi_{\text{Ken}(\omega)}$ iff ω' and ω belong to the same events in $\mathcal{A}$. Hence, for each E, $\omega \in \mathrm{K}(E) = E$ iff $\Pi_{\text{Ken}(\omega)} \subseteq E$. □

In the following two examples we consider algebras $(\Omega, \mathcal{A}_0, \mathrm{K})$ where K is the identity on $\mathcal{A}_0$ and $\mathcal{A}_0$ separates points. This last condition means that for each two states $x \neq y$ in Ω there exists $E \in \mathcal{A}_0$ such that $x \in E$ and $y \in \neg E$. In this case $\Pi_{\text{Ken}}(\omega)$ is the partition of Ω into singletons. In our examples, $\mathcal{A}_0$ does not contain singletons. Hence the elements of the partition that defines the identity knowledge operator are not in $\mathcal{A}_0$. This, with Proposition 4, shows that the condition in Theorem 2 is not necessary. The state space is uncountable in the first example and countable in the second.

Example 1 Let Ω be the set of infinite 0-1 sequences, $\{0, 1\}^{\mathbb{N}}$, and $\mathcal{A}_0$ the algebra generated by finite cylinders. Formally, for each $x \in \Omega$ and a finite set $I \subseteq \mathbb{N}$, the *finite cylinder* (*cylinder* for short) $C = C(x, I)$ is defined by $C = \{y \mid y_i = x_i, \forall i \in I\}$. For all cylinders C and D, $C \cap D$ is a cylinder, and $\neg C$ is a finite union of cylinders or the empty set. The non-empty elements in $\mathcal{A}_0$ consist of all finite unions of cylinders. It is straightforward that $\mathcal{A}_0$ separates points and does not include singletons.

Example 2 Let Ω be the set of integers $\mathbb{Z}$. An *arithmetic sequence* (*sequence* for short) is a set of the form $S = \{a + nd \mid n \in \mathbb{Z}\}$ for $a, d \in \mathbb{Z}$ and $d \neq 0$. Note that for sequences S and T, $S \cap T$ is either a sequence or empty, and $\neg S$ is either a finite union of sequences or the empty set (in case $d = 1$). Let $\mathcal{A}_0$ be the algebra generated by all sequences. Its non-empty elements are all the finite unions of sequences. It is easy to see that $\mathcal{A}_0$ separates points and does not include singletons.

The necessary and sufficient condition for an SAO to be partitional, in Theorem 1, is formulated in terms of events *and* states. It is impossible to formulate such a condition in terms of the algebra of events only. To see this we define what it means for SAOs to have the same algebraic structure. Two SAOs, $(\Omega, \mathcal{A}, \mathrm{K})$ and $(\Omega', \mathcal{A}', \mathrm{K}')$ are *algebraically isomorphic* if the Boolean algebras with operators $(\mathcal{A}, \mathrm{K})$ and $(\mathcal{A}', \mathrm{K}')$ are isomorphic. That is, if there exists a map $f : \mathcal{A} \to \mathcal{A}'$ which is one-to-one and onto, such that for each E and F in $\mathcal{A}$, $f(E \cap F) = f(E) \cap f(F)$, $f(\neg E) = \neg f(E)$, and $f(\mathrm{K}(E)) = \mathrm{K}'(f(E))$.

Proposition 5 *Every* S5 SAO *has an algebraically isomorphic partitional* SAO.

Thus, the algebraic structure itself does not suffice to determine whether an SAO is partitional or not. It is the specific structure of the state space that determines it. We prove this proposition at the end of Sect. 6

5 Non-partitional S5 SAOs

Using the previous two examples we construct two S5 SAOs that are not partitional.

Example 3 Consider the state space Ω and the algebra $\mathcal{A}_0$ generated by the cylinders from Example 1. Let A be the set of all points x such that for some n, $x_i = 0$ for all $i > n$. Then A is countable and therefore does not contain events in $\mathcal{A}_0$ since they are uncountable. Also, $\neg A$ does not contain any event in $\mathcal{A}_0$ because the cylinder determined by x and I includes the state y such that $y_i = x_i$ for each $i \in I$, and $y_i = 0$ for $i \notin I$, and $y \in A$.

Let $\mathcal{A}$ be the algebra generated by the algebra $\mathcal{A}_0$ and A.

Proposition 6 *The algebra $\mathcal{A}_0$ has the maximality property with respect to $\mathcal{A}$.*

Proof Let $X \in \mathcal{A}$. It is easy to see that $X = (E \cap A) \cup (F \cap \neg A)$ for some E and F in $\mathcal{A}_0$. Denote $G = E \cap F$. Then X is the union of three disjoint sets: $X = G \cup ((E \setminus G) \cap A) \cup ((F \setminus G) \cap \neg A)$. The set G is the maximal element among the subsets in $\mathcal{A}_0$ that are contained in X. Indeed, suppose $H \in \mathcal{A}_0$ and $H \subset X$. Consider the set $H \cap (E \setminus G)$. Obviously it is disjoint from G. It is also disjoint from $(F \setminus G) \cap \neg A$, since $E \setminus G$ is disjoint from $F \setminus G$. Therefore, it is a subset of the third set in the decomposition of X, namely, $H \cap (E \setminus G) \subseteq (E \setminus G) \cap A \subseteq A$. But $H \cap (E \setminus G) \in \mathcal{A}_0$, and hence $H \cap (E \setminus G) = \emptyset$. Similarly, $H \cap (F \setminus G) = \emptyset$ and therefore $H \subseteq G$. $\square$

By Proposition 1, there exists an S5 knowledge operator K such that $\mathcal{A}_{\mathrm{K}} = \mathcal{A}_0$. The operator K is the identity on $\mathcal{A}_0$ and therefore, as in Example 1, for each $x \in \Omega$, $\Pi_{\mathrm{Ken}}(x) = \{x\}$. Thus, if $(\Omega, \mathcal{A}, \mathrm{K})$ is partitional, then by Proposition 3 the partition generating K is the partition into singletons. But this partition does not generate K since for $x \in A$, $\Pi_{\mathrm{Ken}}(x) \subseteq E$ and therefore x should be in $\mathrm{K}(A)$. This is impossible because by the definition of K, $\mathrm{K}(A) = \emptyset$.

Obviously, the condition in Theorem 1 must fail. Indeed, consider a state $x \in \neg A$. Then $\Pi_{\mathrm{Ken}}(x) \cap A = \emptyset$. But for each $E \in \mathcal{A}$, $\mathrm{K}(E) \in \mathcal{A}_0$, and as $\neg A$ does not contain any element of $\mathcal{A}_0$, $\mathrm{K}(E) \cap A \neq \emptyset$.

Example 4 Consider the state space Ω and the algebra $\mathcal{A}_0$ generated by the arithmetic sequences from Example 2. Let A be the set of all positive integers. Then, neither A nor $\neg A$ contains any arithmetic sequence. Let $\mathcal{A}$ be the field generated by the field $\mathcal{A}_0$ and A. The proof that $\mathcal{A}_0$ defines an S5 knowledge operator which is not partitional can be taken verbatim from the previous example.

6 Characterizing partitional structures syntactically

6.1 Three types of semantics

The question whether the semantic feature of partition can be defined syntactically depends on the type of semantics we adopt. We examine three types of semantics for $\mathcal{L}_k$—the modal propositional language with modality k. We show that the question is answered in the affirmative for the semantics of frames and of general frames, and in the negative for the semantics of SAOs.

We assume in this section that the reader is familiar with modal logic. For details consult Blackburn et al. (2001) and Kracht (1999).

6.1.1 SAO semantics

A *model* for $\mathcal{L}_k$ *based* on an SAO $(\Omega, \mathcal{A}, \mathrm{K})$ is a tuple $(\Omega, \mathcal{A}, \mathrm{K}, V)$, where V is a *valuation* function that assigns to each primitive formula in $\mathcal{L}_k$ an event in $\mathcal{A}$. The function V is extended inductively to a *meaning* function $[\![\cdot]\!]\colon \mathcal{L}_k \to \mathcal{A}$, as follows. For each primitive formula p, $[\![p]\!] = V(p)$. For all formulas φ and ψ, $[\![\sim \varphi]\!] = \neg[\![\varphi]\!]$, $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$, and $[\![k\varphi]\!] = \mathrm{K}([\![\varphi]\!])$.

SAO semantics, as such, is not studied in the modal logic literature. However, more general semantics plays an important role: the semantics in which the structures used as bases for models are Boolean algebras with operators. SAOs are special cases where the Boolean algebra is that of subsets. For economic theory and game theory SAOs are more attractive than general Boolean algebras because states play an important role in these theories both conceptually and practically.

6.1.2 Relational semantics

The first semantics for propositional modal languages was proposed by Kripke and is given in terms of frames and Kripke models. We show that these can be viewed as a special case of SAOs and the models based on them. To achieve this we rephrase the standard definitions of a Kripke model, using SAO terminology. A *frame* is a pair (Ω, R) where R is a binary relation on Ω called the *accessibility relation*. A *Kripke model* based on a frame (Ω, R) is a tuple (Ω, R, V) where V is a *valuation* function that assigns a subset of Ω to each primitive formula in $\mathcal{L}_k$. The operator K *associated* with R is defined such that for each ω and subset E, $\omega \in \mathrm{K}(E)$ if and only if $\{\omega' \mid (\omega, \omega') \in R\} \subseteq E$. The SAO *associated* with (Ω, R) is $(\Omega, \mathcal{A}, \mathrm{K})$, where $\mathcal{A} = 2^{\Omega}$. Similarly, the model $(\Omega, \mathcal{A}, \mathrm{K}, V)$ based on this SAO is *associated* with the Kripke model (Ω, R, V). The meaning function of formulas in a Kripke model (Ω, R, V) is defined as the meaning function in the associated model $(\Omega, \mathcal{A}, \mathrm{K}, V)$.

6.1.3 General relational semantics

A *general frame* is a tuple $(\Omega, \mathcal{A}, R)$, where $\mathcal{A}$ is an algebra of subsets of Ω and R a binary relation on Ω. In addition it is required that the operator K *associated* with R maps $\mathcal{A}$ into $\mathcal{A}$. The SAO *associated* with $(\Omega, \mathcal{A}, R)$ is $(\Omega, \mathcal{A}, \mathrm{K})$. A model based on the general frame $(\Omega, \mathcal{A}, R)$ is a tuple $(\Omega, \mathcal{A}, R, V)$ where the valuation function V assigns an event in $\mathcal{A}$ to each primitive formula. The model *associated* with $(\Omega, \mathcal{A}, R, V)$ is $(\Omega, \mathcal{A}, \mathrm{K}, V)$. The meaning function of formulas in a model $(\Omega, \mathcal{A}, R, V)$ is defined as the meaning function in the associated model $(\Omega, \mathcal{A}, \mathrm{K}, V)$.

Obviously, a frame is, in particular, a general frame, and each general frame can be identified with the associated SAO. We refer to frames, general frames, and SAOs as *structures*.

A formula φ is *valid* in any of the above-mentioned models if $[\![\varphi]\!] = \Omega$. A formula φ is *valid* in a structure if it is valid in each model based on the structure.

A logic Λ in the propositional modal language is *sound* for a structure if all the theorems of Λ are valid in the strucutre. A logic Λ is *sound* for a family of structures $\mathcal{F}$ if it is sound for each member of $\mathcal{F}$.

6.2 Characterizing partitional structures

Consider a family of structures $\mathcal{F}$ and let $\mathcal{F}' \subseteq \mathcal{F}$. We say that a logic Λ *characterizes* $\mathcal{F}'$ in $\mathcal{F}$ if $\mathcal{F}'$ consists of all the members of $\mathcal{F}$ for which Λ is sound. We are interested in characterizing families of partitional structures. For this we consider the logic S5 which is derived from the conjunction axiom $k(\varphi \wedge \psi) \leftrightarrow (k\varphi \wedge k\psi)$,[5] the truth axiom $k\varphi \rightarrow \varphi$, and the negative introspection axiom $\sim k\varphi \leftrightarrow k \sim k\varphi$.

The following is a well known result of modal logic.

Proposition 7 *The logic* S5 *characterizes the family of frames with an equivalence accessibility relation in the family of all frames.*

The following claim can be easily proved.

Claim 3 *An* SAO $(\Omega, \mathcal{A}, \mathrm{K})$ *with* $\mathcal{A} = 2^{\Omega}$ *is associated with a frame* (Ω, R) *with an equivalence relation* R *if and only if* $(\Omega, \mathcal{A}, \mathrm{K})$ *is partitional.*

In light of this claim, Proposition 7 can be equivalently stated as follows.

Proposition 7' *Let* $\mathcal{F}$ *be the family of* SAOs $(\Omega, \mathcal{A}, \mathrm{K})$ *with* $\mathcal{A} = 2^{\Omega}$ *which are associated with frames, and* $\mathcal{F}'$ *the family of the partitional* SAOs *in* $\mathcal{F}$*. Then, the logic* S5 *characterizes* $\mathcal{F}'$ *in* $\mathcal{F}$*.*

[5] This axiom is equivalent to axiom T: $k(\varphi \rightarrow \psi) \rightarrow (k\varphi \rightarrow k\psi)$.

A stronger result than Proposition 7 is known in the game theory literature that characterizes partitional structures in a larger family of SAOs (e.g. Geanakoplos 1994; Aumann 1999a, and Corollary 1 above).

Proposition 8 *Let $\mathcal{F}$ be the family of* SAOs $(\Omega, \mathcal{A}, \mathrm{K})$ *with* $\mathcal{A} = 2^{\Omega}$, *and* $\mathcal{F}'$ *the family of the partitional* SAOs *in* $\mathcal{F}$. *Then, the logic* S5 *characterizes* $\mathcal{F}'$ *in* $\mathcal{F}$.

We cannot state a result stronger than Proposition 7 by changing frames to general frames. The logic S5 is sound for general frames whose accessibility relation is not necessarily an equivalence relation. Indeed, S5 is sound for any general frame $(\Omega, \mathcal{A}, R)$ with $\mathcal{A} = \{\Omega, \emptyset\}$, independently of the relation R. Yet, we can strengthen Proposition 7' by changing frames to general frames.

Proposition 9 *Let $\mathcal{F}$ be the family of* SAOs *that are associated with general frames, and* $\mathcal{F}'$ *the family of all partitional* SAOs. *Then,* $\mathcal{F}' \subseteq \mathcal{F}$, *and the logic* S5 *characterizes* $\mathcal{F}'$ *in* $\mathcal{F}$.

Proof Suppose that $(\Omega, \mathcal{A}, \mathrm{K})$ is partitional and K is generated by the partition Π. Define the relation R by $(\omega, \omega') \in R$ iff $\omega' \in \Pi(\omega)$. Since K is partitional, $\omega \in \mathrm{K}(E)$ iff $\Pi(\omega) \subseteq E$. But $\Pi(\omega) = \{\omega' \mid (\omega, \omega') \in R\}$, and therefore K is associated with R. This shows that $\mathcal{F}' \subseteq \mathcal{F}$.

Obviously S5 is sound for each partitional SAO. Conversely, suppose that S5 is sound for $(\Omega, \mathcal{A}, \mathrm{K})$ which is associated with a general frame $(\Omega, \mathcal{A}, R)$. We need to show that it is partitional. We observe, first, that $(\Omega, \mathcal{A}, \mathrm{K})$ must be an S5 SAO. If it were not, then one of the axioms K1–K3 would fail. In this case, it is easy to construct a valuation V such that the meaning of one of the axioms of the logic S5 fails to hold true in all states.

We show that K is generated by Π_{Ken}, that is, for each ω and $F \in \mathcal{A}$, $\omega \in \mathrm{K}(F)$ if and only if $\Pi_{\mathrm{Ken}}(\omega) \subseteq F$. One direction trivially holds: if $\omega \in \mathrm{K}(F)$ then $\mathrm{K}(F) \in \mathrm{Ken}(\omega)$. Therefore $\Pi_{\mathrm{Ken}}(\omega) \subseteq \mathrm{K}(F) \subseteq F$. For the converse, note that by definition, $\omega \in \mathrm{K}(F)$ iff $\{\omega' \mid (\omega, \omega') \in R\} \subseteq F$. Now, for each $E \in \mathrm{Ken}(\omega)$, $\omega \in E = \mathrm{K}(E)$. Thus, $\{\omega' \mid (\omega, \omega') \in R\} \subseteq E$, and by Proposition 2 $\{\omega' \mid (\omega, \omega') \in R\} \subseteq \Pi_{\mathrm{Ken}}(\omega)$. Thus if $\Pi_{\mathrm{Ken}}(\omega) \subseteq F$ then $\{\omega' \mid (\omega, \omega') \in R\} \subseteq F$ and hence, $\omega \in \mathrm{K}(F)$. □

Finally, it follows from what we have shown in the previous section that the family of partitional structures is not characterized by S5 in the wider family of all SAOs, and moreover it is not characterized there by any logic.

Proposition 10 *There is no logic that characterizes the partitional* SAOs *in the family of all* SAOs.

Proof Suppose that a logic Λ characterizes the partitional SAOs in the family of all SAOs. Then, the theorems of Λ are valid in all the partitional SAOs. In particular they are valid in all the frames. Hence, by the completeness theorem for S5 the theorems of Λ are theorems of S5. Since Λ is not sound for any SAO which is not partitional, the logic S5, which is at least as strong, is not sound for it a fortiori. But it is easy to see that S5 is sound for any S5 SAO, and the examples in Sect. 5 are of S5 SAOs which are not partitional. □

Proof of Proposition 5 By the Jónsson-Tarski theorem, any Boolean algebra with an operator is isomorphic to a Boolean algebra of subsets with an operator which is associated with a binary relation on the state space. In particular, for each SOA $(\Omega, \mathcal{A}, \mathrm{K})$, the Boolean algebra with operator $(\mathcal{A}, \mathrm{K})$ is isomorphic to a Boolean algebra of sets with operator $(\mathcal{A}', \mathrm{K}')$ which is part of an SAO $(\Omega', \mathcal{A}', \mathrm{K}')$ such that K' is associated with a binary relation R on Ω'. Thus, $(\Omega', \mathcal{A}', \mathrm{K}')$ is associated with a general frame. If $(\Omega, \mathcal{A}, \mathrm{K})$ is an S5-SAO then $(\Omega', \mathcal{A}', \mathrm{K}')$ is also an S5-SAO, because the axioms K1–K3 are formulated purely in terms of events and the two SAOs are isomorphic. By Proposition 9 an S5 SAO which is associated with a general frame is partitional. □

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