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Iterated Expectations and Common Priors

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A necessary and sufficient condition for the existence of a common prior for several players is given in terms of the players' present beliefs only. A common prior exists iff for each random variable it is common knowledge that all its iterated expectations converge to the same value; this value is its expectation with respect to the common prior. The proof is based on the presentation of type functions as Markov matrices. *Journal of Economic Literature* Classification Numbers: C70, D82. © 1998 Academic Press

1. INTRODUCTION

Ever since Harsanyi's (1967–1968) work on games with incomplete information, type spaces have been the most important tool for the study of such games. In most applications of type spaces to economics, it is assumed that players' beliefs can be derived from a common prior. Such a prior can be interpreted simply as the beliefs in a previous period. However, as we are interested in the players at the present time, it is desirable to express the assumption of a common prior in present time terms only. Thus, two question naturally arise:

1. How can we tell, by players' beliefs, that they have a common prior?

2. Can a common prior be expressed in terms of, or constructed from, the players' beliefs in a meaningful way?

Aumann (1976), in his agreement theorem, gave a necessary condition for the existence of a common prior in terms of present beliefs: if there is a common prior, then it is impossible to agree to disagree, i.e., to have common knowledge of differences in the beliefs of any given event. By extending the notion of disagreement to differences in the expectation of a general random variable, several authors (Morris, 1995; Feinberg, 1995; Bonanno and Nehring, 1996) were able to show that the impossibility of there being common knowledge of disagreement is not only a necessary,

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but also a sufficient condition for the existence of a common prior. The different proofs for the sufficiency of this condition use various derivatives of the separation theorem for convex sets. Samet (1998) showed how the condition follows directly from the simple observation that the set of priors of a player is the convex hull of his types.

The first question mentioned above was solved satisfactorily by this necessary and sufficient condition, but it gave no clue as to the second question: the fact that a disagreement cannot be common knowledge, which guarantees the existence of a common prior, tells us nothing about this common prior.

In this work a new necessary and sufficient condition for the existence of a common prior on a finite type space is given, in terms of present beliefs only. Unlike the previously known condition, *this* one also answers our second question: it provides a meaningful way to express the common prior in terms of present beliefs.

To understand the new condition, consider the following story. Suppose we ask Eve what return she expects on IBM stock. Being an expert economist, Eve has no problem providing us with an answer. Adam, when asked the same question, will also come up with an answer. Even if Eve and Adam attended the same school of economics, we should not be surprised to hear different answers, because since then they have been exposed to different sources of information.

Now, let us ask Eve what she thinks Adam's answer was. Eve scratches her head, and for good reason. She can think of many answers that Adam might have given. Being a Bayesian economist, she can compute, though, the expectation of the various answers and come up with Adam's expected answer. Likewise, Adam, will provide us with what he expects Eve's answer to be. Again, we do not anticipate that the answers at this stage will be the same.

We continue the process, moving back and forth between Eve and Adam, asking each to compute the expected value of the other's previous answer. Here is the good news: the two sequences of alternating expectations, the one that starts with Adam and the one that starts with Eve, converge. Moreover, the limits of these sequences are common knowledge to Adam and Eve.

The first question posed above is now answered by proving that *there exists a common prior for Adam and Eve*, *if and only if*, starting with any possible stock, *the sequences thus generated converge to the same limit*.

The second question is answered by showing that *if there is a common prior*, *then the common value of the limit of the sequences is the expected value of the stock with respect to the common prior*. Thus, the expected values of all possible stocks, with respect to the common prior, which fully describe the common prior, are given in terms of the limits of sequences, which are computed by the present beliefs of Eve and Adam.

Now we give a somewhat more formal description of our results, and show how they generalize to more than two players. Let $i_1, i_2, \ldots$ be an infinite sequence of names of players, such that each player's name is repeated infinitely many times. Let f be a random variable (i.e., a function on the state space). Player i_1 evaluates, in each state, the expected value of f. Denote this by $E_{i_1} f$. This evaluation itself is a random variable, and $E_{i_2} E_{i_1} f$ is its evaluated expected value for i_2, and so on. We call the sequence of the random variables thus obtained an *iterated expectation* of the random variable f. Different sequences give rise to different iterated expectations, but we prove:

MAIN RESULTS. *Each iterated expectation of a random variable converges and the value of its limit is common knowledge. Moreover, there exists a common prior if and only if for each random variable it is common knowledge that all its iterated expectations converge to the same value.*

The characterization of a common prior in this work is based on the stochastic nature of types and priors, rather than their convex structure, which, as we mentioned before, was used in the previous characterization. The following simple observation is the starting point.

Observation. *The type function of a player in a type space (namely, the function that associates with each state the player's probability measure over the type space at the state) is by definition the transition function of a Markov chain. The set of priors of a player is the set of invariant probability measures of this transition function.*

Consider, then, the Markov matrix M_i, the rows and columns of which are indexed by states, and each row of which is the type of player i in the state corresponding to the row. This matrix can be used in two ways. For any function f on the state space, written as a column vector, $M_i f$ is the vector of the expected values of f in each state. Another use, which is in a sense dual to the first, primal one, is to consider for any probability measure p on the state space, the probability measure pM_i, obtained from p after one transition of the Markov chain.

The primal use is of economic significance and importance. Players and traders choose their actions by comparing the expectation of certain functions, and this is what state spaces are used to describe. The dual use is not as natural for state spaces, since no stochastic process is going on and since we are not interested in general probability measures on the state space.

Nevertheless, the dual use of M_i is of importance, because the invariant probability measures of M_i are exactly the priors of player i. Thus, a common prior is a probability measure which is invariant for the matrices M_i for all players i.

The relation between the primal and dual aspects of the matrices M_i is what enables us to translate the question of existence of a common prior to a question concerning the primal use of these matrices, namely, the computation of expected values. Moreover, it also makes it possible to express a common prior in terms of expectations.

The stochastic analysis of type spaces is finer than the convex analysis used for the nonagreement condition. Thus, while both analyses can be used to characterize the existence of a common prior, it is only the stochastic approach that makes it possible to express the prior in terms of players' present belief, and also accounts for the uniqueness of a common prior, on elements of the meet, as we prove here. For further applications of stochastic analysis of type spaces, see Gaifman (1986), and Samet (1997).

We present type spaces, and state the main results in the next section. The interpretation of type functions as transition functions of Markov chains and all the proofs are given in Section 3.

2. TYPE SPACES, PRIORS, AND COMMON PRIORS

Let $I = \{1, \dots, n\}$ be a set of *players* and let Ω be a finite set of size m, the elements of which are called *states*. Subset of Ω are called *events*. For each $i \in I$, Π_i is a partition of Ω. For $\omega \in \Omega$ we denote by $\Pi_i(\omega)$ the element of Π_i containing ω. For each $i \in I$ and $\omega \in \Omega$, let $t_i(\omega)$ be a probability measure on Ω, such that:

(a) $t_i(\omega)(\Pi_i(\omega)) = 1$;

(b) for each $\omega' \in \Pi_i(\omega)$, $t_i(\omega') = t_i(\omega)$.

The function t_i is i's *type function*, and $t_i(\omega)$ is i's *type* at ω. The tuple

$$\langle I, \Omega, (\Pi_i, t_i)_{i \in I} \rangle$$

is called a *type space*.

The *meet* of $(\Pi_i)_{i \in I}$ is the partition Π of Ω which is the finest among all partitions that are coarser than Π_i for each $i \in I$. For an event A, the event that A is *common knowledge* is the union of all the elements of Π contained in A. We observe that any element P in the meet Π forms a type space when the partitions Π_i and the types t_i are restricted to it, and the meet of this space is $\{P\}$. We identify any probability measure on P with its natural extension to a probability measure over all of Ω.

We consider probability measures on Ω as row vectors in the m-dimensional space R^{Ω}. A *random variable* is a real-valued function on Ω, which we consider as a column vector in R^{Ω}. For a probability measure p and a random variable f on Ω, the expectation of f with respect to p is the product $pf = \sum_w p(\omega)f(\omega)$. For each player i and random variable f on Ω, i's *expectation* of f, denoted $E_i f$, is the random variable $(E_i f)(\omega) = t_i(\omega)f$.

A *prior* for player i is a probability measure p on Ω, such that for each state ω, if $p(\Pi_i(\omega)) > 0$, then i's type at ω is the conditional probability measure defined by p on $\Pi_i(\omega)$. That is, p is a prior for i if for each event A and state ω, $p(A|\Pi_i(\omega)) = t_i(\omega)(A)$ whenever the conditional probability measure is defined. The probability measure p is a *common prior* if it is a prior for each player i.

In the sequel we assume that for each i and ω, $t_i(\omega)(\{\omega\}) > 0$. Our results do not hold without this positivity assumption, but similar results can be proved for the general case. To formulate such results, the notion of common knowledge, which plays a central role here, should be replaced by common 1-belief (as defined in Monderer and Samet (1989)). In particular, the role played by the elements of the meet is played, in the general case, by events E which are minimal nonempty events for which E is the common 1-belief in E. Under the positivity assumption, the elements of the meet are those minimal events. We make the positivity assumption in order to simplify the formulation and the proofs of the main results.

We present here necessary and sufficient conditions for the existence of a common prior which result from the stochastic nature of type spaces explored in the next section.

First, we show in the following proposition that the question of existence of a common prior on Ω can be reduced to the question of the existence of common priors on the elements of the meet.

PROPOSITION 1. *For each $P \in \Pi$ there exists at most one common prior on P. The set of common priors on Ω is the convex hull of the common priors on the elements P in Π.*

We need the following definitions. We call a sequence $s = (i_1, i_2, \ldots)$, of elements of I, an *I-sequence* if for each player i, $i = i_k$ for infinitely many ks. The *iterated expectation* of a random variable f with respect to the I-sequence s is the sequence of random variables $(E_{i_k} \cdots E_{i_1} f)_{k=1}^{\infty}$.

PROPOSITION 2. *For each random variable f on Ω and I-sequence s, the limit of the iterated expectation of f with respect to s exists and its value is common knowledge in each state; that is, it is constant on each element P in Π.*

In view of Proposition 1, there exists a common prior on Ω iff there exists a common prior on at least one of the elements of the meet. Thus it is enough to characterize the existence of a common prior for the case that the meet consists of only one element.

THEOREM 1. *Suppose* $\Pi = \{\Omega\}$. *Then there exists a common prior iff for each random variable f it is common knowledge in each state that the iterated expectations of f, with respect to all I-sequences s, converge to the same limit. Moreover, if p is the common prior, then this limit is pf.*

Two remarks are in order. First, note that each random variable is a linear combination of the random variables $\chi_{\{\omega\}}$—the characteristic functions of single states. Whereas the expectation operators E_i are also linear, Theorem 1 holds true if the random variables f, in this theorem, are restricted to the functions $\chi_{\{\omega\}}$, or more generally to characteristic functions of events, χ_E.

Second, we observe that Theorem 1 generalizes Aumann's agreement theorem as follows. Suppose the event $(E_1 \chi_E = \alpha) \cap (E_2 \chi_E = \beta)$ is common knowledge, i.e., $E_1 \chi_E$ and $E_2 \chi_E$ have fixed values α and β, correspondingly, on Ω. Then, obviously, all iterated expectations of χ_E for I-sequences that start with 1 are constantly α, while for those sequences that start with 2, they are constantly β. If there is common prior, then, by Theorem 1, $\alpha = \beta$.

3. TYPES AS THE TRANSITION FUNCTIONS OF MARKOV CHAINS

For a given type space $\langle I, \Omega, (\Pi_i, t_i)_{i \in I} \rangle$ define for each player i a matrix M_i in R^{Ω^2}, by $M_i(\omega, \omega') = t_i(\omega)(\{\omega'\})$. Then M_i is a Markov matrix representing the transition function t_i. The following proposition follows directly from the definitions of type spaces and of transition functions of Markov chains.

PROPOSITION 3.

(a) *Each element of the partition* Π_i *is an irreducible class of* M_i.

(b) *A probability measure p on* Ω *is a prior for i iff it is a invariant probability measure for* M_i; *that is,* $pM_i = p$.

(c) *For each random variable f,* $M_i f = E_i f$.

For any permutation σ *of I, we write,*

$$M_\sigma = M_{\sigma(1)} \cdots M_{\sigma(n)}.$$

PROPOSITION 4. *For any permutation* σ *of I, the meet,* Π, *is a partition of* Ω *into irreducible, aperiodic, classes of* M_σ. *Thus, the restriction of* M_σ *to any* $P \in \Pi$ *is ergodic and therefore has a unique invariant probability measure* p_σ^P *on P.*

Proof. Let $\omega, \omega' \in P \in \Pi_{\sigma(i)}$. Then

$$M_\sigma(\omega, \omega') \geq M_{\sigma(1)}(\omega, \omega) \cdots M_{\sigma(i-1)}(\omega, \omega) M_{\sigma(i)}(\omega, \omega')$$
$$\times M_{\sigma(i+1)}(\omega', \omega') M_{\sigma(n)}(\omega', \omega') > 0.$$

Therefore, any two states in the same element of a partition of a player communicate. Hence, if ω is in an equivalence class of states, then $\Pi_i(\omega)$, for each i, is a subset of this class. This means that each class is a union of elements of Π. Also, for each $P \in \Pi$, the probability of $\omega \in P$ staying in P under M_σ is 1, and therefore P is an irreducible equivalence class. The Markovian matrix M_σ is irreducible aperiodic since for each σ, $M_\sigma(\omega, \omega) > 0$. ■

PROPOSITION 5. *The following conditions are equivalent.*

(1) *p is a common prior on Ω.*

(2) *p is an invariant probability measure of the Markov matrix M_i for each $i \in I$.*

(3) *p is an invariant probability measure of the Markov matrix M_σ for each permutation σ.*

Proof. Clearly (1) and (2) are equivalent by Proposition 3(b), and (2) implies (3). Suppose (3) is true and let p be the invariant probability measure in (3). Thus,

$$pM_1 M_2 \cdots M_n = p.$$

Multiplying this equality from the right by M_1 yields

$$pM_1 M_2 \cdots M_n M_1 = pM_1.$$

Therefore, pM_1 is an invariant probability measure of $M_2 \cdots M_n M_1$. However, by (3), p is an invariant probability measure of this Markov matrix, and by Proposition 4, $M_2 \cdots M_n M_1$ has a unique invariant probability measure on each element $P \in \Pi$. Thus, $pM_1 = p$ and, similarly, $pM_i = p$ for each $i \in I$. ■

Proof of Proposition 1. By Proposition 5, if p is a common prior on P, then it is an invariant probability measure of the restriction of M_σ to P, which is unique by Proposition 4. Let p be a common prior and denote by p^P the conditional probability measure of p to $P \in \Pi$, when it exists. Then, clearly, p is a convex combination of the measures p^P. It is enough, now, to show that each p^P is an invariant probability measure on P. Indeed, by Proposition 3(b), p is an invariant probability measure of M_i for each i and, therefore, by Proposition 3(a), p^P is an invariant probability measure of the restriction of M_i to P, for each i. Hence, by Proposition 5, p^P is a common prior on P. ■

To prove Proposition 2 and Theorem 1, we first prove a variant of these claims. Let σ be a permutation of I. Denote by E_σ the operator, which is defined for each f by

$$E_\sigma f = E_{\sigma(1)} \cdots E_{\sigma(n)} f.$$

The *iterated expectation* of a random variable f with respect to σ in the sequence $(E_\sigma^k f)_{k=1}^\infty$.

Proposition 2′ and Theorem 1′, which follow, correspond to Proposition 2 and Theorem 1, but they are formulated in terms of iterated expectation with respect to permutation rather than I-sequences.

PROPOSITION 2′. *For each random variable f on* Ω *and permutation* σ, *the limit of the iterated expectation of f with respect to* σ *exists and is measurable with respect to* Π, *i.e., it is constant on each element P in* Π.

THEOREM 1′. *Suppose* $\Pi = \{\Omega\}$. *Then there exists a common prior iff for each random variable f, the iterated expectations of f, with respect to all permutations* σ, *converge to the same limit. Moreover, if p is the common prior, then this limit is pf.*

Note that the iterated expectation of f with respect to a permutation σ is a subsequence of the iterated expectation of f with respect to the I-sequence,

$$\sigma(1), \ldots, \sigma(n), \sigma(1), \ldots, \sigma(n), \ldots,$$

and, therefore, the claim of Proposition 2′ is weaker than that of Proposition 2. In Theorem 1′, the claim concerning the sufficiency of the condition for the existence of a common prior is stronger than the corresponding claim in Theorem 1, while the claim concerning its necessity is weaker.

Proof of Proposition 2′. By Proposition 3(c), $E_\sigma^k f = M_\sigma^k f$ for each f and k. Whereas, by Proposition 4, M_σ is ergodic on P, $\lim_{k\to\infty} M_\sigma^k f$ is constantly $p_\sigma^P f$ over P. ■

Proof of Theorem 1′. As in the proof of Proposition 2′,

$$\lim_{k\to\infty} E_\sigma^k f = \lim_{k\to\infty} M_\sigma^k f$$

and the limit is constantly $p_\sigma f$ on Ω, where p_σ is the unique invariant probability measure of M_σ on Ω. Thus, for each f, the limits for all σ are the same iff for each f, $p_\sigma f$ are the same for all σ, which is true iff there is a probability measure p such $p_\sigma = p$ for all σ. This amounts, by the equivalence of (1) and (3) in Proposition 5, to saying that p is a common prior. ■

We turn now to prove Proposition 2 and Theorem 1. We first prove Proposition 6, which generalizes a theorem concerning the convergence of the powers of an ergodic stochastic matrix to the case in which different matrices are multiplied. We then prove Lemma 1, which shows that the conditions of Proposition 6 hold in our case.

We say that a Markov matrix A is *bounded by* ε if all its positive entries are bounded from below by ε; that is, if for each row r and column c, either $A(r,c) = 0$ or $A(r,c) > \varepsilon$. We say that A is *positive* if all its entries are positive.

PROPOSITION 6. *Let $A_1, A_2, \ldots, A_k, \ldots$ be a sequence of Markov matrices of the same dimension. Denote $A^{(k)} = A_k A_{k-1} \cdots A_1$. Let $1 = k_1 < k_2 < \cdots < k_l < \cdots$ be an increasing sequence of indices and denote by B_l, for $l = 1, \ldots, \infty$, the block $B_l = A_{k_{l+1}-1} \cdots A_{k_l}$. If there exists $\varepsilon > 0$ such that B_l is positive and bounded by ε, for each l, then there exists a matrix A, all the rows of which are identical, such that $\lim_{k\to\infty} A^{(k)} \to A$. Moreover, if there exists a probability measure p which is invariant for A_k for all k, then all the rows of A are p.*

Proof. It is enough to show that for each column vector x, $A^{(k)}x$ converges to a vector all the components of which are identical. Indeed, if we prove this, then substituting unit vectors for x shows the existence of the limit matrix A with the desired property.

For a vector x write max x for the maximal coordinate x_i and min x for the minimal one. If A is a Markov matrix and $y = Ax$, then $\max y \le \max x$ and $\min y \ge \min x$. Moreover, if A is positive and bounded by $\varepsilon > 0$, then $\max y \le \varepsilon \min x + (1-\varepsilon)\max x$ and $\min y \ge \varepsilon \max x + (1-\varepsilon)\min x$, and, therefore, $\max y - \min y \le (1-2\varepsilon)(\max x - \min x)$.

Thus, if $y^{(k)} = A^{(k)}x$, then $\max y^{(k)}$ is a decreasing sequence and $\min y^{(k)}$ is an increasing one. We need to show that $\max y^{(k)} - \min y^{(k)} \to 0$. This is indeed true, as for each l, $y^{(k_{l+1}-1)} = B_l \cdots B_1 x$, and, therefore, $\max y^{(k_{l+1}-1)} - \min y^{(k_{l+1}-1)} \le (1-2\varepsilon)^l(\max x - \min x)$.

If p is an invariant probability measure of each A_i, then $pA^{(k)} = p$ for each k and therefore $pA = p$. Whereas all the rows of A are identical, pA is a row of A. ■

For two nonnegative matrices of the same order, A and B, we write $A \succeq B$, if each entry which is positive in B is positive also in A.

LEMMA 1. *Let $B = A_r \cdots A_1$, where A_i, for $i = 1, \ldots, r$, is one of the matrices $M_1, \ldots, M_n$, and let $1 \le j_1 < \cdots < j_k \le r$. Then, $B \succeq A_{j_k} A_{j_{k-1}} \cdots A_{j_1}$.*

Proof. Let B' be a matrix obtained from B by substituting A' for A_i, where A' is a nonnegative matrix such that $A_i \succeq A'$. Then obviously, $B \succeq B'$. Whereas for $l = 1, \ldots, n$, $M_l \succeq I$, where I is the unit matrix, we can substitute I for all the matrices A_i, in B, with $i \notin \{j_1, \ldots, j_k\}$, to obtain the desired relation. ■

Define $\varepsilon_0 = \min_{i,\,\omega} M_i(\omega, \omega)$. Then $\varepsilon_0 > 0$ and all the matrices M_i are bounded by ε_0.

LEMMA 2. *Let $B = M_{i_1} \cdots M_{i_1}$ and $C = M_i B$, and suppose B is bounded by ε. Then*:

(*a*) $C \geq B$;

(*b*) *if C has no positive entries other than those of B, then C is also bounded by ε*;

(*c*) *in any case, C is bounded by $\varepsilon_0 \varepsilon$.*

Proof. Part (a) follows from Lemma 1. To prove (b), suppose the assumption of (b) holds and $C(\omega, \omega') > 0$. Then $B(\omega, \omega') > 0$. However, for any $\overline{\omega} \in \Pi_i(\omega)$, the row $M_i(\omega, \cdot)$ is the same as the row $M_i(\overline{\omega}, \cdot)$. Therefore, for each such $\overline{\omega}$, $C(\overline{\omega}, \omega') = C(\omega, \omega') > 0$ and hence, by the assumption in (b), $B(\overline{\omega}, \omega') > 0$. Now, by the definition of C and M_i,

$$C(\omega, \omega') = \sum_{\overline{\omega}:\ \overline{\omega} \in \Pi_i(\omega)} M_i(\omega, \overline{\omega}) B(\overline{\omega}, \omega'),$$

but because $B(\overline{\omega}, \omega')$ is positive, it is bounded from below by ε, and hence

$$C(\omega, \omega') \geq \sum_{\overline{\omega}:\ \overline{\omega} \in \Pi_i(\omega)} M_i(\omega, \overline{\omega}) \varepsilon = \varepsilon.$$

Obviously, (c) is true because each positive entry of C is the sum of products of positive entries from M_i and B, where each of these products is bounded from below by $\varepsilon_0 \varepsilon$. ■

Proof of Proposition 2. Let $s = (i_1, i_2, \ldots)$ be an I-sequence. Fix $P \in \Pi$. For each k, define A_k to be the restriction of M_{i_k} to P. We show that the sequence $A^{(k)}$ (in the notation of Proposition 6) converges. For each permutation σ of I, M_σ is ergodic on P and therefore there is a whole number ν_σ such that for all $\mu \geq \nu_\sigma$, the restriction of M_σ^μ to P is positive. Let $\nu = \max_\sigma \nu_\sigma$. Whereas there are finitely many permutations of I, there must be a permutation σ and blocks $B_l = A_{k_{l+1}-1} \cdots A_{k_l}$ such that for each block B_l, there are indices $j_1, \ldots, j_{n\nu}$ which satisfy $k_l \leq j_1 < j_2 < \cdots < j_{n\nu} \leq k_{l+1} - 1$ and such that

$$j_{n\nu}, j_{n\nu-1}, \ldots, j_1 = \sigma(1), \ldots, \sigma(n), \sigma(1), \ldots, \sigma(n), \ldots, \sigma(1), \ldots, \sigma(n).$$

By Lemma 1, $B_l \succeq A_{j_{n\nu}} \dots A_{j_1}$. But $A_{j_{n\nu}} \dots A_{j_1}$ is the restriction of M_σ^ν to P, and therefore B_l is positive on P. Now, for $k_l \le r \le k_{l+1} - 1$, let $B_l^{(r)} = A_r \cdots A_{k_l}$. By Lemma 2(a), $B_l^{(r)} \succeq B_l^{(r-1)}$, for each $r > k_l$. If the positive entries in $B_l^{(r)}$ are the same as in $B_l^{(r-1)}$, then by Lemma 2(b) both matrices have the same bound. If there are more positive entries in $B_l^{(r)}$, then by Lemma 2(c) its bound is ε_0 times the bound of $B_l^{(r-1)}$. But the set of positive entries can be enlarged for at most m^2 indices r. Thus, all blocks B_l are uniformly bounded by $\varepsilon_0^{m^2}$.

Thus, by Proposition 6, $A^{(k)}$ converges to a matrix A, the rows of which are identical. Obviously, for each f and ω, $E_{i_k} \cdots E_{i_1} f = A^{(k)} f$ on P and the latter converges to the constant function pf, where p is a row in A. ■

Proof of Theorem 1. That the convergence in the theorem implies the existence of a common prior follows from Theorem 1′. If there is a common prior p, then convergence holds by Proposition 2. ■

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