Interim agreements*

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Abstract

Processes of bargaining are studied in which the players reach interim agreements that serve as status quo points for further bargaining. This is modeled in Nash's setup of bargaining problems, where the solution is a time parameterized path of interim agreements rather than a single point. We characterize path solutions for linear problems that satisfy the axioms of *restarting* and *covariance*, and show that if a Pareto efficient agreement is not reached immediately, then it is never reached in finite time. Adding the axioms of *individual rationality*, *relevance*, and *monotonicity*, we characterize the family of *continuous Raiffa solutions* and show that these solutions converge to a Pareto efficient agreement but never reach it in finite time. Finally, if a deadline is added to the bargaining problem, and the speed of bargaining is proportionally inverse to the deadline, then a Pareto efficient agreement is reached exactly at the deadline.

Keywords: Bargaining theory, Nash's solution, Raiffa's solution, interim agreements, deadline.

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1 Introduction

We study a frequently seen pattern in bargaining and negotiating situations in which the process proceeds through a succession of interim agreements. By an interim agreement we mean a fully implemented agreement that serves as the status quo for further bargaining. The examples below are well documented and indicate that processes of interim agreements tend not to be completed. We propose a formal framework that explains this fact and indicate how the addition of a deadline can mitigate this problem.

The first example concerns the ten international strategic arms control treaties between the US and the USSR that have stretched from SALT I, signed by President Nixon and Premier Brezhnev in 1972, to the New START Treaty announced by Presidents Obama and Medvedev in June of 2010. Other examples involve treaty "regimes", where an agreement is successively strengthened,

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such as the sequence of human rights treaties outlawing discrimination against women, torture, abuse of children, etc. The Arab-Israeli conflict has seen a succession of interim agreements, in what has come to be termed the “peace process.” Quandt (2005) explains the “process” terminology as “the gradual, step-by-step approach to resolving one of the world’s most difficult conflicts.” Some of these processes in which interim agreements are successively reached seem to be prolonged and almost everlasting.

1.1 The model

To study bargaining processes that proceed through interim agreements we use the framework of Nash (1950). In this framework a bargaining situation is described by a *bargaining problem* which consists of two elements. The first is a *feasibility set* given by the bargainers’ utilities of possible agreements and the second is a *status quo* point in the feasibility set, which prevails if no agreement is reached. We further assume, as in Nash’s framework and as in the standard in economic theory, that the utility functions are von Neumann-Morgenstern (vNM). Such functions are determined up to positive affine transformations.

Nash’s solution function and similar solution functions for bargaining problems assign to each bargaining problem a *single Pareto efficient point* that represents a final agreement. Here, in contrast, the solution specifies for each bargaining problem the agreement that holds at time t, for all $t \geq 0$, starting with the status quo point that holds at time 0. Thus, a solution assigns to each bargaining problem a time parameterized *path* of agreements in the feasibility set. For our first results the definition of a path does not assume anything about it, and in particular we do not assume continuity. Thus, the Nash bargaining solution is a special case of such a path, where at time 0 the interim agreement is the status quo point and for all $t > 0$ the agreement at t is the Nash solution of the problem.

1.2 The restarting and the covariance axioms

We start our study of path solutions for bargaining problems by assuming two simple axioms. The first axiom of *restarting* expresses the requirement that points on the solution path of a bargaining problem are interim agreements that serve as the status quo points for further bargaining. That is, if we restart the bargaining at an agreement point on the path that holds at time t, and we use this point as the status quo point for further bargaining, the resulting path coincides with the path of the original problem after time t. This requirement assumes implicitly that past experience is irrelevant for future bargaining. Of course, in some cases this assumption is not valid, but it can serve as a first approximation for the analysis of processes of interim agreements.

The second axiom of *covariance* is standard and is nothing more than the reflection of the assumption that the utilities are vNM. A bargaining problem is determined up to positive affine transformations of the players’ utility functions. When one problem is obtained from another by such transformations, then both are different descriptions of the same bargaining situation. Therefore, the solution of the transformed problem should be covariant with the solution of the original problem, that is, it should be obtained from the first solution by the very same transformations. The set of bargaining problems can be partitioned into equivalence classes of problems that can be transformed into each other.

We show that these two axioms are enough to significantly restrict the type of possible solutions for linear problems, namely problems the Pareto frontier of which is a hyperplane. Note, that the family of linear bargaining problems has two equivalence classes with respect to utility transformations. One consists of all problems in which the status quo point is on the Pareto frontier. We call these problems degenerate. The other class consists of the non-degenerate problems in which the status quo point is not on the Pareto frontier.

It turns out that any path solution that satisfies the axiom of covariance alone assigns to any degenerate problem the path which is the status quo at all times. The more interesting result concerns the class of non-degenerate problems. Here, the family of paths that are generated by a path solution that satisfies both axioms lies in one of three families of paths.

1. Paths that for all times $t > 0$ are constantly the same Pareto optimal point.

2. Paths that are linear in time and parallel to the Pareto frontier. The sum of utilities at each point of the path is constantly the sum of the utilities of the status quo point.

3. Paths that are exponential in time and get either away from the Pareto frontier or get closer to it and converge, only at infinity, to a Pareto efficient point.[1]

In the first type of solution, a final Pareto efficinet agreement is reached immediately. Thus, engaging in a process of interim agreements that does not reach a Pareto efficient agreement immediately is either of the second type or the third type. In both, a Pareto efficient agreement is never reached in finite time. In the second solution players never reach an interim agreement which is individually rational unless it is the status quo point. We now add the axiom of *individual rationality*, which requires that at some point in time, an individually rational agreement, which is not the status quo point, is reached. This axiom excludes the second solution. It also excludes the exponential solution that gets further away from the Pareto frontier. The only type of solution that remains is an exponential solution that converges at infinity to individually rational Pareto efficient agreement, but never reaches it in finite time.

[1]Note the fine distinction between being *close* to the Pareto frontier and being *closer* to it. Being close is measured by difference of utilities. But such differences are meaningless for vNM utility since it is invariant under multiplication. Being closer is measured by comparing two differences of utility, which is meaningful for vNM utility. Thus, at no point in time is it meaningful to say that the interim agreement is *close* to being Pareto optimal. However, we can say that an interim agreement is *closer* to being Pareto optimal than an interim agreement that preceded it.

The basic intuition of why a Pareto efficient agreement is not reached is that when an interim agreement is reached which is not Pareto efficient, then the remaining bargaining problem is equivalent to the original problem, as all non-degenerate linear problems are equivalent. But this rough intuition does not suffice for the full characterization of the three possible types of solutions. This is done by translating the two axioms into a functional equation that combines the two basic Cauchy functional equations.[2]

1.3 The continuous Raiffa solutions

The axioms of restating and covariance guarantee that a Pareto efficient agreement is not reached in finite time only for *linear* bargaining problems. We next show that a well known path solution for general bargaining problems also has this property. Raiffa (1953) described a solution for two-player bargaining problems generated by a differentiable path (see also Luce and Raiffa, 1957). The path, which starts at the status quo point, is directed at each point x on it to the ideal point generated by x, namely the pair of maximal payoffs that are individually rational with respect to x (see Figure 1). Diskin *et al.* (2010) have shown that this path can be presented for any number of players as a time parameterized path that is the solution of a differential equation. We generalize this solution to a family of solutions we call *continuous Raiffa solutions*. The solution studied in Diskin *et al.* (2010) is the symmetric solution in this family. Using the differential equations that describe them we are able to show that the paths of these solutions converge to Pareto efficient agreements, but never reach such an agreement in finite time.

The continuous Raiffa solutions coincide with the exponential solutions on the linear bargaining problems. In particular, the continuous Raiffa solutions satisfy the axioms of restarting, covariance, and individual rationality. We show that the family of continuous Raiffa solutions is characterized by adding two more axioms. The first is the axiom of *relevance*, which requires that the solution for a bargaining problem depends only on the set of individually rational points in the feasible set. The second is the axiom of *monotonicity*, which requires that locally, at the status quo point, the agreement reached at time 0 is monotonic with respect to set inclusion of the feasible sets. This axiom, as opposed to all other axioms, assumes that the path is differentiable, and thus it eliminates the first type of solutions.

1.4 Deadlines

Next we show that it is possible to enable a process of interim agreement to end in finite time by specifying a deadline beyond which bargaining is impossible. Just specifying a deadline cannot help, as the players can bargain as if there is no deadline and just stop bargaining at the deadline. Of course, in this way a Pareto efficient agreement will not be reached. We need to add an axiom that relates

[2]For the solutions of the Cauchy functional equations see Aczél (1966).

the process to the deadline. This is done with the axiom of *acceleration*, which requires that the speed of bargaining be proportionally inverse to the deadline. More specifically, any agreement reached at time t when the deadline is T should be reached at time at when the deadline is aT. This axiom *alone* does not guarantee that a Pareto efficient agreement will hold at the deadline. Indeed, if the agreement reached at time t when the deadline is T is the agreement reached at time t/T when the deadline is 1, then the axiom is satisfied. However, when acceleration is added to the rest of the axioms, we can show that the process of interim agreement will reach the Pareto frontier and this will happen exactly at the deadline.

1.5 Related work

Economic theory does not shy away from processes that never terminate. Indeed, repeated games and stochastic games are respectable objects in this theory. As early as 1877 Walras suggested the tatonnement process, which gets closer and closer to market equilibrium, although not necessarily in finite time. Here we do not assume an infinite process, but rather derive it from two axioms, one of which, the restarting axiom, just expresses the fact that bargaining goes through interim agreements. Moreover, this axiom is consistent, for linear problems, with a trivial process in which final agreement is reached immediately. However, if agreement is not reached immediately, then this implies an infinite process.

The dynamic aspects of bargaining in Nash's setup have been dealt with in several works, although in none of them was time an explicit element of the theory. The first work on dynamic bargaining, using an axiom of step-by-step negotiation, was Kalai (1977). Later work emphasized axioms that involve the change of the status quo point while keeping the bargaining set fixed (Thomson (1987), Peters and van Damme (1993), Livne (1989), and Anbarci and Sun (2009)). The image of the Raiffa path for two players was axiomatize by Livne (1989) and Peters and van Damme (1993). This image is described by a differential equation that relates the change of utility of one player in terms of the utility of the other player. Thus, the temporal aspect of the path is not expressed is these two works. A time parameterized path of interim agreements is described in O'Neill *et al.* (2004), but bargaining is described there by a continuum of Pareto frontiers rather then one bargaining problem, as here.

The discrete Raiffa solution is characterized axiomatically in Anbarci and Sun (2009). In Diskin *et al.* (2010), a family of discrete generalized Raiffa solutions is axiomatized. Moreover, in this work the Raiffa time parameterized solution was introduced and has been shown to be the limit of the discrete solutions in this family.

2 Bargaining dynamics

2.1 Path solutions

We study the dynamics of bargaining using Nash's bargaining setup. We consider a finite set of bargainers N, and a family $\mathcal{B}$ of *bargaining problems* (S, d), where $S \subseteq \mathbb{R}^N$, called the *feasibility set*, consists of vectors of the vNM-utility of the bargainers from possible agreements between them, and $d \in S$ is called the *status quo* (or *disagreement*) point. To study the dynamics of reaching interim agreements, we study *path solutions* that assigns to each problem a time parameterized path rather than just a point as is the standard case in bargaining theory. Formally,

Definition 1. *A* path *for a problem* (S, d) *is a function* $\pi\colon [0, \infty) \to S$*, such that* $\pi(0) = d$*. A* path solution *on a family of bargaining problems* $\mathcal{B}$ *is a function* Π *on* $\mathcal{B}$ *that assigns to each problem* (S, d) *in* $\mathcal{B}$ *a path* $\Pi(S, d)$ *for this problem.*

We think of $\Pi(S, d)(t)$ as the interim agreement that holds at time t. Note that defining the path on the infinite time interval $[0, \infty)$ does not restrict in any way the interim agreements achieved. In particular, a path π can take a constant value for all $t \geq t_0$, which means that the agreement that holds at time t continues to hold for all times $t > t_0$.

2.2 Two basic axioms

For a path π, we think of $\pi(t)$ as an interim agreement that holds at time t and *serves as the status quo point for further bargaining.* Thus, suppose that starting at the status quo point d, the interim agreement d' holds at time t, and the interim agreement d'' holds at time $t+s$. If we think of d' as the status quo point for restarting the bargaining, then d'' is the interim agreement that holds at time s after restarting the bargaining from d'. This is the content of the first axiom.

Axiom 1. (Restarting)
For any problem (S, d), and $t, s \geq 0$:

$$\text{if } d' = \Pi(S,d)(t) \text{ and } d'' = \Pi(S,d)(t+s), \text{ then } d'' = \Pi(S,d')(s).$$

The covariance axiom presented next is required in bargaining theory where the utilities are vNM. Such utilities are determined up to positive affine transformations.[3] That is, if i's preferences are described by the utility function u_i, then they are also represented by any function $a_i u_i + b_i$ with $a_i > 0$. We call *utility transformation* any transformation $\tau\colon \mathbb{R}^N \to \mathbb{R}^N$, such that for each i and $x \in \mathbb{R}^N$, $\tau_i(x) = a_i x_i + b_i$ for some $a_i > 0$. Thus, for any utility transformation

[3]Utility functions that are defined up to a positive affine transformation are called *cardinal*. vNM utilities are the most notable example of cardinal utilities.

τ, the problems (S,d) and $(\tau(S),\tau(d))$ represent the same bargaining situation. We require that a path solution should be "independent" on the specific representation selected, namely it should be covariant with utility transformations. More specifically, if x is the agreement that holds at time t in the problem (S,d), then the agreement that holds at time t in $(\tau S,\tau d)$ should be τx.

Axiom 2. (Covariance)
For any problem (S,d), utility transformation τ, and $t \geq 0$,

$$\Pi\big(\tau(S),\tau(d)\big)(t) = \tau\big(\Pi(S,d)(t)\big).$$

Observe that the relation between two problems of being obtained by a utility transformation is an equivalence relation.[4] Thus, by the axiom of covariance, Π is determined by fixing it on one problem in each equivalence class of this relation.

We now study the implication of these two axioms for linear bargaining problems.

3 Linear bargaining problems

For x and y in $\mathbb{R}^N$ we write $y \geqq x$ when for each $i \in N$, $y_i \geq x_i$; $y \geq x$ when $y \geqq x$ and $y \neq x$; and $y > x$ if for each $i \in N$, $y_i > x_i$.

A point $x \in S$ is *Pareto efficient in* S if there is no $y \in S$, such that $y \geq x$. We say that a problem (S,d) is *degenerate* if d is Pareto efficient in S. Otherwise it is *non-degenerate*. Observe that utility transformations preserve Pareto efficiency. That is, a point x is Pareto efficient in S if and only if it is Pareto efficient in τS. Therefore, utility transformations also preserve degeneracy and non-degeneracy.

A bargaining problem (S,d) is *linear* if there is $a > 0$ in $\mathbb{R}^N$, such that $S = \{x \in \mathbb{R}^N \mid ax \leq 1\}$. We denote by $\mathcal{B}_L$ the set of linear bargaining problems. Note that $\mathcal{B}_L$ consists of two equivalence classes with respect to utility transformation: the class of degenerate problems and the class of non-degenerate problems.

We characterize the paths assigned to linear bargaining problems by path solutions that satisfy the two axioms. First, we consider the simpler case of degenerate linear bargaining problems. In this case the path solution remains at the status quo point forever.

Proposition 1. *If Π is a path solution on $\mathcal{B}_L$ that satisfies the axiom of covariance, then for each degenerate problem (S,d), $\Pi(S,d)(t) = d$ for all $t \geq 0$.*

The next theorem describes the possible path solutions for non-degenerate problems. As all such problems can be transformed into the divide-the-dollar

[4] This is so because the set of utility transformations of each individual is a multiplicative group. Denote a transformation by the pair (a_i,b_i) that determines it. Then, $(a_i,b_i)(a'_i,b'_i) = (a_i a'_i, a_i b'_i + b_i)$, the unit is $(1,0)$, and the inverse is $(a_i,b_i)^{-1} = (a_i^{-1}, -b_i a_i^{-1})$.

problem, it is enough by the covariance axiom that we describe only the paths for this problem.

Theorem 1. *Let Π be a path solution on $\mathcal{B}_L$ that satisfies the axioms of restarting and covariance. Then the family of paths $\pi = \Pi(S, d)$, where (S, d) ranges over all non-degenerate linear problems, is a subset of one of the following three families of paths.*

- ***Constant Pareto paths**: For some Pareto efficient point p, $\pi(t) = p$ for all $t > 0$.*
- ***Linear paths**: The path π lies in the hyperplane that passes through d and parallel to the Pareto frontier. For each $t \geq 0$, $\pi(t) = \sigma(t) + d$ where σ is additive, that is, for each $t, s \geq 0$, $\sigma(t+s) = \sigma(t) + \sigma(s)$, and $\sum \sigma_i \equiv 0$.*
- ***Exponential paths**: There exist constants $\alpha \neq 0$ and a Pareto efficient point p such that $\pi(t) = p[1 - e^{\alpha t}]$.*

We discuss each of these families in detail.

Constant Pareto paths: In such a path a Pareto efficient agreement is reached in no time and it lasts forever. It is easy to see that the axiom of restarting holds. Indeed, when bargaining starts at time $t > 0$, the status quo is p, and as this point is Pareto efficient, it follows by Proposition 1 that the path remains at p for all $s \geq t$, which agrees with the path of the starting bargaining problem. Such a path solution can be considered a single point solution.

Linear paths: The only symmetric additive solution is the one in which for all i, $\sigma_i \equiv 0$, in which case the status quo point 0 lasts forever. In all other linear solutions some players increase their utility at the expense of others, as the sum of the utilities remains $\sum d_i$.

The obvious case where σ is additive is when $\sigma_i(t) = c_i t$ for each i. But there are additive functions that are not linear (see Aczél, 1966, Theorem 2, p. 35). The restriction $\sum \sigma_i \equiv 0$ does not prevent pathological non-linear additive paths. Indeed, choose any additive functions σ_i for $i = 1, \ldots, n-1$, and define $\sigma_n = -\sum_{i<n} \sigma_i$, then σ_n is additive as well and the sum of all σ_i is identically 0.

Exponential paths: When $\alpha > 0$ the sum of utilities $\sum \pi_i(t) = 1 - e^{\alpha t}$ is negative and moves away from the Pareto frontier of S. When $\alpha < 0$, $\pi(t)$ converges, as t goes to infinity, to the Pareto efficient point p, and players with $p_i > 0$ increase their utility with time.

In both the linear and the exponential paths the sum of the utilities of the players is less than 1 at all times. This leads to the following corollary:

Corollary 1. *Let Π be a path solution on $\mathcal{B}_L$ that satisfies the axioms of restarting and covariance. Then, if for some problem a non-Pareto efficient interim agreement is reached under Π at some $t > 0$, then in all non-degenerate problems, a Pareto efficient agreement is never reached under Π.*

Although Corollary 1 follows immediately from Theorem 1, we provide a direct proof of this corollary that does not rely on the characterization in Theorem 1. The rough intuition behind this result is as follows. All non-degenerate linear problems are equivalent. Thus, if an interim agreement holds which is not Pareto efficient, then the rest of the bargaining is similar to the original bargaining problem, and in a sense no progress has been achieved. This and Proposition 1 are translated into a proof of Corollary 1. However, this basic intuition does not bring us any closer to the specific characterization of Theorem 1. For this we need to construct a functional equation that is derived from the two axioms and solve it.

The following axiom is a weak requirement of individual rationality that singles out the solutions that converge to a Pareto agreement. The axiom requires that for non-degenerate problems the interim agreement at some point in time is individually rational. That is, at this time, some players gain while others do not lose, relative to the status quo point.

Axiom 3. (Individual rationality)
If (S, d) is non-degenerate then there exists $t > 0$ such that $\Pi(S, d) \geq d$.

Adding this axiom to the previous two axioms, eliminates all the additive solutions in Theorem 1. The constant solutions that survive are those for which $p \geq d$. The exponential paths that satisfy all three axioms must have $\alpha < 0$ and $p \geq d$. In particular, the solutions in Theorem 1 that satisfy the axiom of individual rationality are those that converge to an individually rational, Pareto efficient point. This is summarized in the following proposition.

Proposition 2. *If a path solution Π on $\mathcal{B}_L$ satisfies Axioms 1, 2, and 3, then for each non-degenerate linear problem (S, d), $\pi = \Pi(S, d)$ is either a Pareto constant path defined by individually rational point p, or an exponential path with $\alpha < 0$. In either case the path converges to an individually rational, Pareto efficient agreement.*

4 The continuous Raiffa solutions

Our purpose now is to show that not reaching a Pareto efficient agreement in finite time can be extended to solutions of non-linear bargaining problems. A path solution for general two-player bargaining problems was sketched by Raiffa (1953). Diskin *et al.* (2010) described this solution, for any number of players, as a time parameterized path which is the solution of a differential equation. Here we consider a family of path solutions which we call *continuous Raiffa solutions.*[5] The Raiffa path solution studied in Diskin *et al.* (2010) is the symmetric solution in this family. The continuous Raiffa solutions for linear problems are the exponential path solutions studied in Section 3. We add two

[5]We refer to these solutions as continuous to distinguish them from the discrete Raiffa solutions which are characterized axiomatically in Diskin *et al.* (2010).

simple axioms to Axioms 1-3 that together characterize the family of continuous Raiffa solutions. Using the differential equations that describe the continuous Raiffa solutions we show that Pareto efficient agreement is never reached in finite time, but the path converges to such an agreement.

For the purpose of this section we consider the following set of bargaining problems $\mathcal{B}$. A pair (S, d) is in $\mathcal{B}$, if S is closed, convex, comprehensive,[6] and positively bounded.[7] In addition we require that all the boundary points of S are Pareto efficient. Note that $\mathcal{B}_L \subset \mathcal{B}$. We assume in this section a path solution Π such that for each (S, d), $\Pi(S, d)$ is a differentiable path.[8]

Axiom 4 requires that the only part of the bargaining problem which is relevant to the determination of the path is the set of individually rational outcomes.

Axiom 4. (Relevance)
If (S, d) and (T, d) are two problems in $\mathcal{B}$ such that $\{x \mid x \in S, x \geqq d\} = \{x \mid x \in T, x \geqq d\}$, then $\Pi(S, d) = \Pi(T, d)$.

A simple monotonicity axiom was introduced by Kalai (1977) for point-wise solutions. It requires that when the set of feasible agreements is enlarged, while the status quo point remains the same, the solution in the enlarged problem is at least as favorable to the players as the solution of the smaller problem.[9] For path solutions, the axiom cannot be stated so simply, because even if the status quo point at time $t = 0$ is common to both problems, at each other time $t > 0$, the status quo points, which are the points of the paths at time t, may be different in the two problems. Thus, we require local monotonicity only at time $t = 0$ at d. This is done using the differentiability of the paths.

Axiom 5. (Monotonicity)
If (S, d) and (T, d) are two problems in $\mathcal{B}$ such that $S \subseteq T$, then $\Pi'(T, d)(0) \geqq \Pi'(S, d)(0)$.

The path solutions that satisfy Axioms 1-5 are defined in terms of ideal points.[10] For each problem (S, d) in $\mathcal{B}$, $m_i(S, d)$ is the maximal utility that i can obtain while keeping the other players at their status quo utility levels. Thus, $m_i(S, d) = \max\{x_i \mid (x_i, d_{-i}) \in S\}$. The properties of S guarantee that this function is well defined, and obviously, $m(S, d) \geqq d$. By the comprehensiveness of S, $m_i(S, d)$ is the maximal utility of i at the individually rational points of (S, d). The *ideal point* for a bargaining problem (S, d) is the point $m(S, d) = (m_i(S, d))$.

We consider a time parameterized path π that starts at d, and is such that the rate of increase of utility of each player i at time t is proportional to her ideal

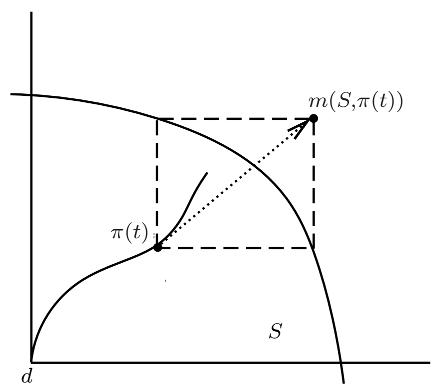
[6] That is, for each $x \in S$, $\{y \mid y \leqq x\} \subseteq S$.
[7] That is, there exists $b > 0$ in $\mathbb{R}^N$ and a constant α, such that $bx \leq \alpha$ for each $x \in S$.
[8] As the path is defined for $t \geq 0$, only right differentiability is required at 0.
[9] A more elaborate monotonicity axiom was introduced in Kalai and Smorodinsky (1975).
[10] Ideal points were used in Kalai and Smorodinsky (1975) to define the Kalai-Smorodinsky solution for bargaining problems.

utility gains in the bargaining problem $(S, \pi(t))$, that is, it is proportional to $m_i(S, \pi(t)) - \pi_i(t)$. Thus, we are considering a path π that solves the first-order autonomous differential equation

$$\pi_i'(t) = c_i\big[m_i(S, \pi(t)) - \pi_i(t)\big], \tag{1}$$

with the initial condition $\pi(0) = d$, for some $c \geq 0$ in $\mathbb{R}^N$. Figure 1 depicts such a path for two players where $c_1 = c_2 = 1$. For the symmetric case, where for all i, $c_i = 1$, Diskin *et al.* (2010) showed (in Theorem 5) that these differential equations have a unique solution and this solution converges to a Pareto efficient point. Their proof can be used verbatim for the general case presented here.

A *continuous Riaffa path solution*, Π^c, for $c \geq 0$ in $\mathbb{R}^N$, assigns to each problem (S, d) the unique path $\pi = \Pi^c(S, d)$ which solves the differential equation (1).



The derivative of a Raiffa path π at $\pi(t)$ is the dotted vector $m(S, \pi(t)) - \pi(t)$.

Figure 1: A Raiffa path: the symmetric case

Theorem 2. *A path solution satisfies Axioms 1-5 if and only if it is a continuous Raiffa path solution.*

We can now state the negative result concerning reaching Pareto efficient agreements.

Theorem 3. *Let Π^c be a continuous Raiffa solution. Then for each non-degenerate problem (S, d) in $\mathcal{B}$ and all $t \geq 0$, $\Pi^c(S, d)(t)$ is not Pareto efficient. However, the limit of $\Pi^c(S, d)(t)$ as $t \to \infty$ exists and it is Pareto efficient.*

Unlike Proposition 2, here it is impossible to reach a Pareto efficient agreement right away, as we assume that the path is differentiable and hence continuous.

5 Deadlines

We now examine the effect of deadlines on the dynamics of bargaining. For this purpose we assume that each bargaining problem is defined not only by the set of feasible agreement and a status quo point but also by a deadline. Formally, the set of bargaining problems is $\mathcal{B}^D = \mathcal{B} \times (0, \infty)$. Thus, each bargaining problem is a triple (S, d, T) where $(S, d) \in \mathcal{B}$, and $T > 0$ is a *deadline*. A *path* for a problem (S, d, T) is a differentiable function $\pi \colon [0, T] \to S$. A *path solution* Π assigns to each problem (S, d, T) a path, $\pi = \Pi(S, d, T)$, for this problem.

We think of $\pi(t)$ as an interim agreement that holds at time t, and serves as the status quo point for further bargaining that has a deadline at $T - t$. The axiom of restarting should be restated in the new context, taking into account the change of the deadline during the bargaining process.

Axiom 1'. (Restarting)
For all $(S, d, T) \in \mathcal{B}^D$, and $t, s \geq 0$ such that $t + s \leq T$:

$$\text{if } d' = \Pi(S, d, T)(t), \text{ and } d'' = \Pi(S, d, T)(t + s),$$
$$\text{then } d'' = \Pi(S, d', T - t)(s).$$

The formulation of all other axioms for $\mathcal{B}^D$ is the same as for $\mathcal{B}$ except that a deadline is added to each bargaining problem. In an axiom where more than one bargaining problem is involved, the same deadline is added to all the problems. In this section we use the same enumeration for the amended axioms as in the previous sections.

Any path solution Π on $\mathcal{B}$ induces a solution $\hat{\Pi}$ on $\mathcal{B}^D$ by disregarding the deadline. That is, for each $t \in [0, T]$, $\hat{\Pi}(S, d, T)(t) = \Pi(S, d)(t)$. Moreover, $\hat{\Pi}$ satisfies the axioms 1',2-5. Thus, in order for the deadline to have some impact on the path solution we require that the speed of the bargaining process depends on the deadline. We quantify this dependence by requiring that the speed be proportional to the deadline. Thus, for example, if we divide the deadline by 2, the speed of the process should be twice as fast. In other words, an agreement achieved at time t when the deadline is T, will be reached at time $t/2$ when the deadline is $T/2$.

Axiom 6. (acceleration)
For each $a > 0$, and $t \geq 0$: $\Pi(S, d, T)(t) = \Pi(S, d, aT)(at)$.

Note that the acceleration axiom *alone* does not imply that bargaining will end in finite time, let alone at or before the deadline. Indeed, consider the path solution $\Pi(S, d, T)(t) = \Pi^c(S, d)(t/T)$. In this solution, for all deadlines T, the agreements in the time interval $[0, T]$ are the agreements on the path $\Pi^c(S, d)$ in the interval $[0, 1]$. As $\Pi^c(S, d)$ never reaches the Pareto frontier of S, by Theorem 3, $\Pi(S, d, T)$ does not reach it either. However, this solution obviously satisfies the acceleration axiom.

When the acceleration axiom is added to the rest, deadlines have a significant effect. The interim agreement points along the path for a given problem (S, d, T) are the same as the interim agreements achieved along the Raiffa path for (S, d).

Theorem 4. *A solution path* Π *on* $\mathcal{B}^D$ *satisfies axioms 1',2-6 if and only if there exists* $c \geq 0$ *in* R^N *such that for each problem* (S, d, T) *and* $t \in [0, T)$*,*

$$\Pi(S, d, T)(t) = \Pi^c(S, d)\big(\ln(1 - t/T)^{-1}\big). \tag{2}$$

The function $\ln(1 - t/T)^{-1}$ maps the time interval $[0, T)$ onto the time interval $[0, \infty)$. Thus, $\Pi(S, d, T)$ squeezes a generalized Raiffa path of (S, d) in the time interval $[0, \infty)$ into the time interval $[0, T)$. Moreover, by Theorems 3 and 2, $\Pi^{\mathrm{R}}(S, d)$ never reaches the Pareto frontier for non-degenerate (S, d), and $\lim_{t\to\infty} \Pi^{\mathrm{R}}(S, d)(t)$ is Pareto efficient. As Π is continuous, it follows that it reaches a Pareto agreement at time T but not before that. This is summarized in the following theorem.

Theorem 5. *If a solution path* Π *on* $\mathcal{B}^D$ *satisfies axioms 1',2-6 then for non-degenerate* (S, d)*,* $\Pi(S, d, T)$ *reaches the Pareto frontier at time* T*, but not earlier.*

We now describe the solutions of $(S_0, 0, T)$, where $S_0 = \{x \mid \sum x_i \leq 1\}$ (known also as the divide-the-dollar problem). We note that $m_i(S_0, x) - x_i = 1 - \sum_i x_i$. Thus the differential equation (1) is $\pi_i'(t) = c_i(1 - \sum_i \pi_i(t))$ with $\pi(0) = 0$. The solution of this equation is $p_i[1 - e^{-\alpha t}]$, where $\alpha = \sum_j c_j$ and $p_i = c_i/\alpha$. The path π is the exponential path described in Theorem 1. Thus, by (2), for a path solution Π that satisfies 1',2-6, for each i, $\pi_i = \Pi_i(S_0, 0, T)$ is given by:

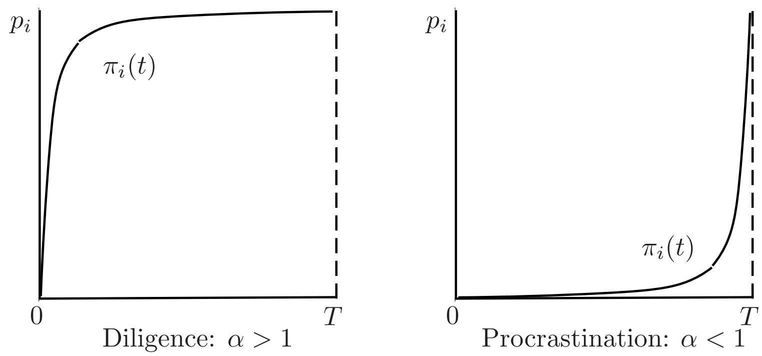
$$\pi_i(t) = p_i[1 - (1 - \frac{t}{T})^{\alpha}]. \tag{3}$$

Indeed, to derive this result for the divide-the-dollar problem, only Axioms 1',2,3 and 6 are required, as is shown in the proof of Theorem 4.

The derivative π_i' is increasing when $\alpha < 1$ and decreasing when $\alpha > 1$. Values of $\alpha < 1$ signify procrastination. The process is slow in the beginning but becomes hectic before the deadline as the speed tends to infinity towards the completion of the bargaining. Values of $\alpha > 1$ mark diligence. Most of the agreement is achieved early in the process which slows down to a complete halt at time T, where $\pi_i'(T) = 0$. For $\alpha = 1$, π is obviously linear (see Figure 2).

6 Proofs

Proof of Proposition 1. Let Π be a path solution that satisfies the axiom of covariance. Consider a linear problem $(S, 0)$ where 0 is Pareto efficient in S. Then, $\Pi(2S, 0)(t) = 2\Pi(S, 0)(t)$ for all t. But $2S = S$, and therefore, $\Pi(S, 0)(t) = 2\Pi(S, 0)(t)$. This implies that $\Pi(S, 0)(t) = 0$ for all $t \geq 0$. Let (S, d)



A player's utility as a function of time in the divide-the-dollar problem with deadline T, when Axioms 1',2,3,6 are satisfied, for two path solutions that differ in the parameter α.

Figure 2: Diligence and procrastination

be a problem with d being Pareto efficient. The transformations $x_i \to x_i - d_i$ transform (S, d) into a linear problem $(\hat{S}, 0)$ where 0 is Pareto efficient in $\hat{S}$. By the covariance axiom $0 = \Pi(\hat{S}, 0)(t) = \Pi(S, d)(t) - d$. Thus, $\Pi(S, d)(t) = d$ for all $t \geq 0$. ∎

Proof of Theorem 1. It is easy to see that if for some non-degenerate linear problem (S_0, d), $\Pi(S_0, d)$ belongs to one of the three families, then by the axiom of covariance for all non-degenerate linear problems (S, d), $\Pi(S, d)$ belongs to the same family. Thus, it is enough to check that when Π satisfies Axioms 1 and 2, then $\pi = \Pi(S_0, 0)$ is in one of the three families, where $S_0 = \{x \mid \sum x_i \leq 1\}$.

For $t \geq 0$, let $d = \pi(t)$. The utility transformations $x_i \to [1 - \sum_j \pi_j(t)]x_i + \pi_i(t)$ transform the problem $(S_0, 0)$ into the problem (S_0, d). Hence, by Axiom 2, $\pi = \Pi(S_0, 0)$ is transformed by these transformations into $\Pi(S_0, d)$. That is, for each i and $s \geq 0$, $\pi_i(s)$ is transformed into $\Pi_i(S_0, d)(s)$. Thus, $\Pi_i(S_0, d)(s) = [1 - \sum_j \pi_j(t)]\pi_i(s) + \pi_i(t) = \pi_i(t) + \pi_i(s) - [\sum_j \pi_j(t)]\pi_i(s)$. By Axiom 1, $\Pi_i(S_0, d)(s) = \pi_i(t + s)$. We conclude that for each non-negative t and s and each i

$$\pi_i(t + s) = \pi_i(t) + \pi_i(s) - \big[\sum_j \pi_j(t)\big]\pi_i(s) \tag{4}$$

We show that the solutions to these functional equations must be in one of the three families in the theorem. Obviously, when $\sum_j \pi_i \equiv 0$, then by (4) for each i, π_i is additive and π belongs to the additive family of solutions.

We assume now that for some $s_0 > 0$, $\sum_j \pi_i(s_0) \neq 0$. We note that by exchanging t and s in (4) we conclude that $\big[\sum_j \pi_j(t)\big]\pi_i(s) = \big[\sum_j \pi_j(s)\big]\pi_i(t)$, and in particular for $s = s_0$,

$$\text{(5)} \qquad \big[\sum_j \pi_j(t)\big]\pi_i(s_0) = \big[\sum_j \pi_j(s_0)\big]\pi_i(t)$$

Denoting $c_i = \pi_i(s_0)/\sum_j \pi_j(s_0)$, we have $\sum_i c_i = 1$, and we can rewrite (5) as:

$$\text{(6)} \qquad \big[\sum_j \pi_j(t)\big]c_i = \pi_i(t),$$

Note, that $c_i = 0$ if and only if $\pi_i \equiv 0$. Indeed, if $c_i = 0$ then by (6) $\pi_i \equiv 0$. Conversely, if $\pi_i \equiv 0$, then in particular $\pi_i(s_0) = 0$, and thus by definition, $c_i = 0$. We conclude that if $\pi_i \not\equiv 0$ then $c_i \neq 0$ and we can rewrite (4), in this case, as,

$$\text{(7)} \qquad \pi_i(t+s) = \pi_i(t) + \pi_i(s) - c_i^{-1}\pi_i(t)\pi_i(s)$$

For $\pi_i \not\equiv 0$ define a function f_i on nonnegative real numbers t by $f_i(t) = 1 - c_i^{-1}\pi_i(t)$. It follows easily from (7) that f_i satisfies the following Cauchy functional equation:

$$\text{(8)} \qquad f_i(t+s) = f_i(t)f_i(s).$$

Observe that as $\pi(t) \in S$ for each $t \geq 0$, then $\sum_j \pi_j(t) \leq 1$ for each $t \geq 0$. Hence, f_i is bounded from below, as by (6), $f_i(t) = 1 - c_i^{-1}\pi_i(t) = 1 - \sum_j \pi_j(t) \geq 0$. This implies that (8) can have only the following solutions: either $f_i \equiv 0$, or $f_i(0) = 1$ and $f_i(t) = 0$ for $t > 0$, or $f_i(t) = e^{\alpha_i t}$ for some real number α_i (see Aczél, 1966, Theorem 1, pp. 38-39). The first of these solution means that $\pi_i \equiv c_i$. This is impossible for $c_i \neq 0$, as $\pi_i(0) = 0$. Substituting the other two solutions into the definition of f_i, we conclude that for each i such that $\pi_i \not\equiv 0$, one of the following two cases holds:

1. $\pi_i(t) = c_i$ for all $t > 0$;
2. $\pi_i(t) = c_i[1 - e^{\alpha_i t}]$.

Observe that case 1 and 2 hold trivially when $\pi_i \equiv 0$, since in this case $c_i = 0$. We now show that if case 1 holds for one i with $\pi_i \not\equiv 0$, then it holds for all players. Suppose that for some i_0, $\pi_{i_0}(t) = c_{i_0}$ for $t > 0$. Then, by (6), $\sum_j \pi_j(t) = 1$ for $t > 0$. But this implies, again by (6), that for each i, $\pi_i(t) = c_i$ for $t > 0$. Thus if case 1 holds for one i it holds for all. As $\sum c_i = 1$, c is Pareto efficient, and π is a constant Pareto solution.

Next, we show that if case 2 holds for i with $\pi_i \not\equiv 0$ it holds for all players, and moreover, all the α_i's are the same. Obviously, $\alpha_i \neq 0$ because otherwise $\pi_i \equiv 0$. Suppose that for some j with $\pi_j \not\equiv 0$ the equality in case 2 does not hold. Then the equation in case 1 must hold, and as we have shown it follows that this equation holds for all players. In particular, for i, $\pi_i(t) = c_i$ for all $t \geq 0$. Thus, $c_i = c_i[1 - e^{\alpha_i t}]$ for all $t > 0$. But this is impossible, since the right hand side is not constant as $\alpha_i \neq 0$. Thus, if case 2 holds for i with $\pi_i \not\equiv 0$ it holds for all players.

We show that all the α_i's are the same. Consider j for which $\pi_j \not\equiv 0$. Then for all $t > 0$, $\pi_j(t)/\pi_i(t) = (c_j/c_i)([1 - e^{\alpha_j t}]/[1 - e^{\alpha_i t}])$. But by (6), $\pi_j(t)/\pi_i(t) = (c_j/c_i)$ for all $t > 0$. Thus, $[1 - e^{\alpha_j t}]/[1 - e^{\alpha_i t}] = 1$ for all $t > 0$. This implies that $e^{\alpha_i t} = e^{\alpha_j t}$ for all $t > 0$, which implies that $\alpha_i = \alpha_j$. Denote by α the constant which is the same for all i such that $\pi_i \not\equiv 0$. Then, in case 2, for all i, $\pi_i(t) = c_i[1 - e^{\alpha t}]$, which shows that π is an exponential path. ∎

Proof of Corollary 1. Let Π be a path solution that satisfies Axioms 1 and 2. Suppose that for some $t' > 0$ and a problem (S, d), $d' = \Pi(S, d)(t')$ is not a Pareto efficient point of S. By Proposition 1, (S, d) must be non-degenerate. Thus, (S, d) can be transformed into (S, d') by some utility transformation τ. Hence, by Axiom 2, $\Pi(S, d') = \tau(\Pi(S, d))$. As τ preserves Pareto efficiency, $\Pi(S, d')(t')$ is not Parto efficient. By Axiom 1, $\Pi(S, d)(2t') = \Pi(S, d')(t')$. Thus, $\Pi(S, d)(2t')$ is not Pareto efficient. By induction, $\Pi(S, d)(2^k t')$ is not Pareto efficient for all whole numbers k.

Suppose now that for some $\hat{t}$, $\hat{d} = \Pi(S, d)(\hat{t})$ is Pareto efficient. Then, by Axiom 1, for all $t > \hat{t}$, $\Pi(S, d)(t) = \Pi(S, \hat{d})(t - \hat{t})$. The right hand side is Pareto efficient by Proposition 1, thus we conclude that after time $\hat{t}$ all interim agreements are Pareto efficient, contrary to what we have shown.

As all non-degenerate problems are obtained by a utility transformation of (S, d), and since utility transformation preserves Pareto efficiency, it follows by Axiom 2 that in all such problems a Pareto efficient agreement is never reached. ∎

Proof of Theorem 2. We omit the straightforward proof that Π^c satisfies Axioms 1-5. Suppose Π is a differentiable path solution that satisfies Axioms 1-5. We first show that there exists $c \geq 0$ in R^N such that $\Pi = \Pi^c$ on $\mathcal{B}_L$. As the solution of (1) is covariant under utility transformations, it is enough to show that for the divide-the-dollar problem $(S_0, 0)$, $\Pi(S_0, 0) = \Pi^c(S_0, 0)$.

By Proposition 2 there exist $(p_i) \geq 0$ with $\sum_i p_i = 1$ and $\alpha > 0$ such that $\Pi(S_0, 0) = p_i(1 - e^{-\alpha t})$. Let $c_i = p_i \alpha$. Observe that for the divide-the-dollar problem, $m_i(S_0, \pi(t)) = 1 - \sum_{j \neq i} \pi_j(t)$. Thus, (1) in this case is

$$\pi'_i(t) = c_i[1 - \sum_j \pi_j(t)] \tag{9}$$

The unique solution of (9) is $(c_i/\sum_j c_j)[1 - e^{-\sum_j c_j t}] = p_i(1 - e^{-\alpha t})$.

For a general problem (S, d) we fix i and construct two problems, (S^-, d) and (S^+, d), with the following properties:

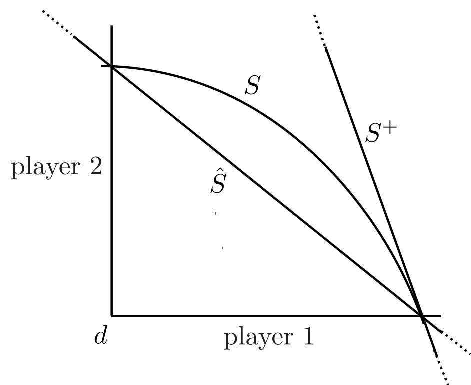
1. $S^- \subseteq S \subseteq S^+$;

2. (S^+, d) is a linear problem;

3. (S^-, d) agrees with a linear problem on the set of individually rational points;

4. $(\Pi_i^c)'(S^-, d)(0) = (\Pi_i^c)'(S^+, d)(0)$.

The details of the construction of S^+ and S^- are in the proof of Theorem 1 in Diskin *et al.* (2010) (See Figure 3).



The problems (S^+, d) and (S^-, d) are linear. The bargaining problem (S^-, d) is defined by $S^- = S \cap \hat{S}$.

Figure 3: The problems (S^+, d) and (S^-, d) for $i = 1$

Write, $\pi^+ = \Pi(S^+, d)$, $\pi^- = \Pi(S^-, d)$, $\pi = \Pi(S, d)$. By 1 and monotonicity, $(\pi_i^-)'(0) \leq (\pi_i)'(0) \leq (\pi_u^+)'(0)$. As we have shown, $\Pi = \Pi^c$ for linear problems, and therefore by 2, $(\pi_i^+)'(0) = (\Pi_i^c)'(S^+, d)(0)$. By 3 and the relevance axiom $(\pi_i^-)'(0) = (\Pi_i^c)'(S^-, d)(0)$. Using 4 we conclude that $(\Pi_i^c)'(S^-, d)(0) = \pi_i'(0) = (\Pi_i^c)'(S^+, d)(0)$. But, since Π^c satisfies the monotonicity axiom it follows by 1 and 4 that $(\Pi_i^c)'(S^-, d)(0) = \Pi_i'(S, d)(0) = (\Pi_i^c)'(S^+, d)(0)$. Thus, $(\pi_i)'(0) = (\Pi_i^c)'(S, d)(0)$.

For $t > 0$, let $\hat{d} = \Pi(S, d)(t)$. By the restarting axiom, for each $s \geq 0$, $\Pi(S, d)(t + s) = \Pi(S, \hat{d})(s)$. Hence, $\Pi'(S, d)(t) = \Pi'(S, \hat{d})(0) = (\Pi^c)'(S, \hat{d})(0)$. But, $(\Pi_i^c)'(S, \hat{d})(0) = c_i[m_i(S, \hat{d}) - \hat{d}_i] = c_i[m_i(S, \Pi(S, d)(t)) - \Pi_i(S, d)(t)]$. Therefore, $\Pi(S, d)$ satisfies equation (1). ∎

Proof of Theorem 3. We first show that for any (S, d), $\Pi^c(S, d)$ cannot reach the Pareto frontier before time $\alpha = (\sum_i c_i)^{-1}$. We then show, that if for some (S, d), $\Pi^c(S, d)$ reaches the Pareto frontier in finite time, we can construct another problem where it is reached at $\alpha/2$. This contradiction proves that no path can reach the Pareto frontier in finite time.

We call a problem (S,d) *normalized* if $d=0$ and $m_i(S,0)=1$ for each i. Let $(S,0)$ be a normalized problem and let $\pi=\Pi^c(S,0)$. Then, for each i and t, $\pi_i(t)=c_i\int_0^t m_i(S,\pi(\tau))-\pi_i(\tau)d\tau$. By the definition of m, the right hand side of 1 is nonnegative and therefore $\pi\geq 0$. By the comprehensiveness of S, $m(S,\pi(t))\leq m(S,0)$. Thus, for each i and t, $m_i(S,\pi(\tau))-\pi_i(\tau)\leq m_i(S,0)=1$. Hence, $\pi_i(t)\leq c_i t$, and thus, for each $t>0$,

$$t\geq \alpha\sum_i \pi_i(t). \tag{10}$$

Suppose that for some T, $\pi(T)$ is Pareto efficient. Note that $m(S,\pi(T))=\pi(T)$. Else, for some i, $m_i(S,\pi(T))>\pi_i(T)$, contrary to $\pi(T)$ being Pareto efficient. By the convexity of S, the simplex—the convex hull of the unit vectors in R^N—is contained in S. This shows that $\sum_i \pi_i(T)\geq 1$, because otherwise, $m(S,\pi(T))\neq\pi(T)$. This with (10) implies that $T\geq\alpha$. Since every non-degenerate problem (S,d) can be transformed into a normalized problem, it follows that for any problem (S,d), if $\Pi^c(S,d)(T)$ is Pareto efficient then $T\geq\alpha$.

Suppose now that for a problem (S,d), $\Pi^c(S,d)(T)$ is on the Pareto frontier. Let $t=T-\alpha/2$. Since $T\geq\alpha$, $t>0$. Let $d'=\Pi^c(S,d)(t)$. By the axiom of restarting, $\Pi^c(S,d')(\alpha/2)=\Pi^c(S,d)(t+\alpha/2)=\Pi^c(S,d)(T)$. Thus, $\Pi^c(S,d')$ reaches a Pareto efficient point at time $\alpha/2$, which is less then α, contrary to what we have proved. This shows that $\Pi^c(S,d)$ can never reach a Pareto frontier in finite timet. ∎

Proof of Theorem 4. It is simple to verify that a solution Π defined by (2) satisfies Axioms 1', 2-6. We prove that any solution that satisfies these axiom is of the form in (2). We first fix T and prove it for $\pi=\Pi(S_0,0,T)$. For this we need to show that (3) holds. We show it using all the axioms but 5 and 6.

By the restarting axiom

$$\Pi_i(S_0,\pi(t),T-t)(\hat{t})=\Pi_i(S_0,0,T)(t+\hat{t})=\pi_i(t+\hat{t}). \tag{11}$$

By the accelerating axiom,

$$\Pi_i(S_0,\pi(t),T-t)(\hat{t})=\Pi_i(S_0,\pi(t),T)(\hat{t}T/(T-t)). \tag{12}$$

The transformations $x_i\rightarrow[1-\sum_j\pi_j(t)]x_i+\pi_i(t)$ transform the problem $(S_0,0,T)$ into the problem $(S_0,\pi(t),T)$. Thus, by covariance,

$$\Pi_i(S_0,\pi(t),T)(\hat{t}T/(T-t))=[1-\sum_j\pi_j(t)]\pi_i(\hat{t}T/(T-t))+\pi_i(t). \tag{13}$$

From the last three equations we conclude:

$$\pi_i(t+\hat{t})-\pi_i(t)=[1-\sum_j\pi_j(t)]\pi_i(\hat{t}T/(T-t)). \tag{14}$$

Dividing the last equation by $\hat{t}$ and taking the limit as $\hat{t}\rightarrow 0$, remembering that since $\pi_i(0) = 0$, $\lim_{\hat{t}\to 0} \pi_i(\hat{t}T/(T-t))/\hat{t} = [T/(T-t)]\pi_i'(0)$, we conclude that

$$\pi_i'(t) = [1 - \sum_j \pi_j(t)][T/(T-t)]\pi_i'(0). \tag{15}$$

The general solution $f\colon [0,T) \to R^N$ of the first-order linear differential equation $f_i'(t) = [1 - \sum_j f_j(t)][T/(T-t)]c_i$ is $f_i(t) = a_i + kc_i(T-t)^{\sum_j c_j T}$, where $\sum_j a_j = 1$. With the initial condition $f(0) = 0$, $a_i = c_i/\sum_j c_j$, and $k = -(1/\sum_j c_j)T^{-\sum_j c_j T}$, which gives the unique solution $f_i(t) = [c_i/\sum_j c_j][1 - (1-t/T)^{\sum_j c_j T}]$. For this solution $f_i'(0) = c_i$. Thus, equation (15) has a family of solutions determined by an arbitrary $c = (c_i)$ for $f'(0)$. By the axiom of individual rationality, $c \geq 0$. Setting $\beta = \sum_j c_j$, and $p_i = c_i/\beta$, we conclude that if Π satisfies Axioms 1',2,3, and 4, then $\pi = \Pi(S_0, 0, T)$ satisfies $\pi(t) = p_i[1 - (1-t/T)^{\beta T}]$, for some $p \geq 0$ such that $\sum_j p_j = 1$ and $\beta > 0$.

However, this computation has been carried out for a given deadline T. Thus, the constants p and β depend on T: $p = p(T)$ and $\beta = \beta(T)$. By the accelerating axiom, for each $a > 0$, $\Pi(S_0, 0, T)(t) = \Pi(S_0, 0, aT)(at)$. Thus, $p_i(T)[1-(1-t/T)^{\beta(T)T}] = p_i(aT)[1-(1-t/T)^{\beta(aT)aT}]$. Letting $t = T$ we have $p_i(T) = p_i(aT)$. As this holds for each $a > 0$ we conclude that $p(T)$ is a constant that does not depend on T. Thus, $(1-t/T)^{\beta(T)T} = (1-t/T)^{\beta(aT)aT}$. Taking a logarithm, we have $\beta(T)T = \beta(aT)aT$. Thus, $\beta(T)T$ is a constant $\alpha > 0$ which is independent of T. Therefore, there are constants $p \geq 0$ such that $\sum_j p_j = 1$ and $\alpha > 0$, which are independent on T and for each T, $\Pi(S_0, 0, T)(t) = p_i[1 - (1-t/T)^{\alpha}]$.

We now observe that as $(\ln(1-t/T)^{-1})' = (T-t)^{-1}$, (2) holds if and only if $\pi = \Pi(S, d, T)$ satisfies the differential equation

$$\pi_i'(t) = c_i(T-t)^{-1}[m_i(S, \pi(t)) - \pi_i(t)] \tag{16}$$

with the initial condition $\pi(0) = d$. Since we have shown that (2) holds for $\pi = \Pi(S_0, 0, T)$ it follows that π satisfies (16). As (16) is covariant with utility transformation, it holds for all linear problems. Using the same construction as in the proof of Theorem 2 we show with the monotonicity and relevance axioms that for every problem (S, d, T), $\pi = \Pi(S, d, T)$ satisfies (16), which implies that it satisfies (2). ∎

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