JOURNAL OF ECONOMIC THEORY 52, 190–207 (1990)

Ignoring Ignorance and Agreeing to Disagree*

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Received February 3, 1988; revised October 9, 1989

A model of information structure and common knowledge is presented which does not take states of the world as primitive. Rather, these states are constructed as sets of propositions, including propositions which describe knowledge. In this model information structure and measurability structure are endogenously defined in terms of the relation between the propositions. In particular, when agents are ignorant of their own ignorance, the information structure is not a partition of the state space. We show that Aumann's (*Ann. Statist.* **4** (1976), 1236–1239) famous result on the impossibility of agreeing to disagree, which was proved for partitions, can be extended to such information structures. *Journal of Economic Literature* Classification Numbers: 021, 026. © 1990 Academic Press, Inc.

1. INTRODUCTION

In his seminal paper, "Agreeing to Disagree," Aumann [1] has shown that agents who have the same prior distribution over the states of the world cannot agree to disagree. More precisely, if their posteriors for a certain event are common knowledge then these posteriors must coincide even though they are based on different information. The information that agents have in Aumann' model is given by partitions of the state space, one for each agent. With each state ω and agent i there is associated a set of states $P_i(\omega)$ that are indistinguishable by i from ω at ω. The family of sets $P_i(\omega)$ (where ω ranges over all states of the world) is assumed to be a partition.

One of the two main purposes of this paper is to show the Aumann's "no agreement" result can be extended to information structures (given by the family of sets $P_i(\omega)$) which are not partitions. The other purpose is more general. We present a refined model of knowlege and common knowledge that allows us to derive endogenously various information structures, measurability conditions, and some constraints on the information

* The author acknowledges helpful comments from Robert Aumann, Itzhak Gilboa, Ehud Kalai, Ehud Lehrer, Jean François Mertens, David Schmeidler, and Shmuel Zamir.

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0022-0531/90 $3.00

structure that are required for the "no agreement" theorem. To do so we introduce the knowledge of the agents as a formal component in the model. The objects that are known in our model are propositions, and for each proposition ϕ we assume the existence of a proposition $K_i\phi$ which is interpreted as "agent i knows that ϕ." We then define a state to be a full description of the world (where "world" can be narrowly interpreted as a game or a market environment) in terms of the propositions, including those which describe knowledge. Information structure and measurability structure of the space of states, as well as common knowledge, are naturally defined in terms of the relation between propositions.

There are three properties of knowledge in our model that together imply a partition of the states of the world into classes of indistinguishable states: (K1) when an agent knows a proposition he knows he does; (K2) any proposition known by an agent is true; (K3) when an agent does not know a proposition he knows he does not.

In our main theorem we show that the more general information structure which is implied by (K1) and (K2) is enough to guarantee Aumann's result. In other words, it is impossible to agree to disagree even when agents ignore their own ignorance. Moreover (K1) alone is enough if we make the following plausible assumption on the prior distribution. When we give up (K2) we allow for false propositions to be "known." "We" of course are external observers; the agent who knows a proposition considers that proposition to be true, and his prior distribution would assign probability zero to states in which this proposition is false. If this is indeed the case then (K1) alone suffices to prevent agreeing to disagree.

The basic features of this model—the construction of states as full, consistent descriptions of the world, and the relation between knowledge properties (like (K1)–(K3)) and information structure—are well known in modal and epistemic logic and constitute the backbone of these theories (Hintikka [8], Kripke [11]). It was Bacharach [3] who first introduced the appartus of epistemic logic in the context of Aumann's model. Unlike the modal logic model, his knowledge operators are applied to events in a σ-field rather than to propositions. He was able then to show that the partition assumed in Aumann [1] is derived from assumptions similar to (K1)–(K3). Bacharach also pointed out, following modal logic systems, the more general information structures that are generated when (K3) is omitted. Shin [13] pursued this line of research more vigorously. His model, like the one used in modal logic, begins with propositions and knowledge operators on them, which are used to construct states. Although he does not introduce measurable structure, he studies extensively the topology of various information structures and of common knowledge, starting with the natural topology, which is much the same we use in Section 3.

Two recent papers by Gilboa [6] and Kaneko [10] also make use of knowledge operators on propositions to study the ability to include a description of the "whole world" within a state, in the spirit of the informal discussion in Aumann [2].

Models, analogous to modal logic systems, have also been developed in computer science to study distributed systems. Halpern [7] is a survey of these works.

The structure of this paper is as follows. In Section 2 we define the states of the world as "lists" of true proposition. We then examine the implication of (K1)–(K3) on the information structure of the state space. In Section 3 we introduce the natural topology on state spaces starting with the simplest events "proposition ϕ is true" as a subbase. The results of this section are used later in Section 6 to show that all the events required for the study of common knowledge are measurable. In Section 4 we define what it is for a proposition to be common knowledge in a state. We show that if a proposition is common knowledge in a state then it is automatically true and common knowledge in a whole group of states. This provides a link between the defintions of common knowledge in terms of propositions and in terms of events. In Section 5 the notion of finitely generated knowledge is introduced. Informally it reflects the assumption that our (possibly infinite) knowledge is derived from finitely many propositions (a posteriori knowledge which may differ from state to state) by (a priori) deduction rules. This property of knowledge implies some restrictions on the information structure which are essential to derive the results of Section 6. For the special case of partitions, the required restriction is the countability of the partition, which is assumed in Aumann [1]. In Section 6 we prove that (K1) and (K2) are enough to guarantee the impossibility of agreeing to disagree, and show under what conditions (K1) alone suffices. In Section 7 we discuss various aspects of the model.

2. Propositions and States

Let Φ and I be two countable sets. We interpret elements of Φ as *propositions* describing a certain environment of interest. Alternatively one may think of Φ as a set of well formed formulas in some language. But since the structure of such a language plays no role in our study we prefer the less technical notion of propositions to describe the primitives of our theory. Elements of I are interpreted as *agents*. For each agent $i \in I$ there exists a mapping $K_i: \Phi \to \Phi$, where for each $\phi \in \Phi$ the proposition $K_i\phi$ is interpreted as saying "i knows ϕ." There exists also a mapping $\sim: \Phi \to \Phi$ where $\sim\phi$ is interpreted as "not ϕ" and such that for each ϕ, $\sim\phi \neq \phi$, and $\sim\sim\phi = \phi$.

Consider the set $\Sigma = \{0, 1\}^{\Phi}$. Each element of Σ can be thought of as an assignment of truth values to the propositions; 1 for "true" and 0 for "false". An element ω of Σ is called a *state of the world* (or a *state*)) if for each $\phi \in \omega$,

$$\omega(\phi) + \omega(\sim\phi) = 1,$$

i.e., if for each ϕ, ω assigns the value "true" to one and only one of the propositions ϕ and its negation $\sim\phi$. The set of states is denoted by Ω_0. The properties of the negation, $\sim$, guarantee that Ω_0 is not empty. (Even a weaker property of the negation suffices, namely: for each $n \geqslant 0$ and ϕ, $(\sim)^{2n+1}\phi \neq \phi$.) We identify the state ω with the set of propositions $\{\phi \mid \omega(\phi) = 1\}$. Thus we write $\phi \in \omega$ instead of $\omega(\phi) = 1$ and $\phi \notin \omega$ for $\omega(\phi) = 0$. We write $\Psi \subseteq \omega$ for a set of proposition Ψ if for each $\phi \in \Psi$, $\phi \in \omega$. The phrase "ϕ is true in state ω" is also used instead of $\omega(\phi) = 1$.

The *ken* of agent i in state ω (or the *informational content* of ω for i) is the set $K_i^-(\omega)$ of all propositions known by i in ω. That is,

$$K_i^-(\omega) = \{\phi \mid K_i\phi \in \omega\}.$$

We fix now a subset Ω of Ω_0. Define for each $i \in I$ a binary relation p_i on Ω by: $\omega' p_i \omega$ whenever $K_i^-(\omega) \subseteq \omega'$. We say in this case that ω' is *possible* in ω for i. This relation expresses the compatibility of the state ω' with the knowledge i has in ω; each proposition known by i in ω is true in ω' and thus the information i has in ω does not distinguish between ω and ω'. For each i and $\omega \in \Omega$ let $P_i(\omega)$ be the set of all the states which are possible in ω for i, i.e.,

$$P_i(\omega) = \{\omega' \mid \omega' p_i \omega\}.$$

For a set of states X, $P_i(X)$ is the set $\{P_i(\omega) \mid \omega \in X\}$.

Consider now the following three properties of knowledge in state ω.

(K1) For each $\phi \in \Phi$ and $i \in I$, if $K_i\phi \in \omega$ then $K_iK_i\phi \in \omega$.

(K2) For each $\phi \in \Phi$ and $i \in I$, if $K_i\phi \in \omega$ then $\phi \in \omega$.

(K3) For each $\phi \in \Phi$ and $i \in I$, if $\sim K_i\phi \in \omega$ then $K_i \sim K_i\phi \in \omega$.

Condition (K1) says that in state ω, if i knows ϕ he knows he does. (K2) says that every proposition known by i in ω is true in ω. (K3) means that if i does not know ϕ in ω then he knows he does not.

We denote by

Ω_1 the set of all states which satisfy (K1),

Ω_2 the set of all states which satisfy (K1) and (K2),

Ω_3 the set of all states which satisfy (K1), (K2), and (K3).

Clearly, $\Omega_3 \subseteq \Omega_2 \subseteq \Omega_1 \subseteq \Omega_0$.

The following theorem describes the relation p_i between states in terms of relation between the ken of i in different states.

THEOREM 1. *For each $i \in I$:*

(a) *For $\Omega \subseteq \Omega_2$, $\omega' p_i \omega$ iff $K_i^-(\omega') \supseteq K_i^-(\omega)$*

(b) *For $\Omega \subseteq \Omega_3$, $\omega' p_i \omega$ iff $K_i^-(\omega') = K_i^-(\omega)$.*

Proof. (a) Suppose $K_i^-(\omega') \supseteq K_i^-(\omega)$. By (K2), $\omega' \supseteq K_i^-(\omega')$ and therefore $\omega' p_i \omega$. Conversely, assume $\omega' p_i \omega$. If $\phi \in K_i^-(\omega)$ then by (K1), $K_i\phi \in K_i^-(\omega)$. Thus $K_i\phi \in \omega'$ and $\phi \in K_i^-(\omega')$.

(b) If $K_i^-(\omega') = K_i^-(\omega)$ then $\omega' p_i \omega$ by (a). Suppose $\omega' p_i \omega$ then by (a) $K_i^-(\omega') \supseteq K_i^-(\omega)$. Let $\phi \in K_i^-(\omega')$, i.e., $K_i\phi \in \omega'$. Suppose $\phi \notin K_i^-(\omega)$. Thus, $K_i\phi \notin \omega$ and $\sim K_i\phi \in \omega$ which by (K3) implies $K_i \sim K_i\phi \in \omega$ and therefore $\sim K_i\phi \in \omega'$, a contradiction. Q.E.D.

THEOREM 2. *For each $i \in I$,*

(a) *If $\Omega \subseteq \Omega_1$, p_i is transitive.*

(b) *If $\Omega \subseteq \Omega_2$, p_i is transitive and reflexive.*

(c) *If $\Omega \subseteq \Omega_3$, p_i is transitive, reflexive, and symmetric.*

Proof. (b) and (c) follow from parts (a) and (b) of Theorem 1, correspondingly. We omit the simple proof of (a). Q.E.D.

A proof of Theorem 2 in a slightly different setup is found in Hughes and Creswell [9].

Consider now the following three properties of Ω in terms of P_i.

(P1) For each $i \in I$ and $\omega \in \Omega$, if $\omega' \in P_i(\omega)$ then $P_i(\omega') \subseteq P_i(\omega)$.

(P2) For each $i \in I$ and $\omega \in \Omega$, $\omega \in P_i(\omega)$.

(P3) For each $i \in I$ and $\omega \in \Omega$, if $\omega' \in P_i(\omega)$ then $P_i(\omega') = P_i(\omega)$.

The following is an immediate corollary of Theorem 2.

COROLLARY 1:

(a) *If $\Omega \subseteq \Omega_1$ then Ω satisfies* (P1).

(b) *If $\Omega \subseteq \Omega_2$ then Ω satisfies* (P1) *and* (P2).

(c) *If $\Omega \subseteq \Omega_3$ then Ω satisfies* (P1), (P2), *and* (P3). *In particular, $\{P_i(\omega) \mid \omega \in \Omega\}$ is a partition of Ω into equivalent classes with respect to equality of informational content.*

3. TOPOLOGY ON STATE SPACES

The family of sets $\{T_\phi, F_\phi \mid \phi \in \Phi\}$, where $T_\phi = \{\sigma \in \Sigma \mid \sigma(\phi) = 1\}$ and $F_\phi = \{\sigma \in \Sigma \mid \sigma(\phi) = 0\}$, is a subbase for the product topology on

$\Sigma = \{0, 1\}^{\Phi}$. T_ϕ and F_ϕ can be interpreted as the events "ϕ is true" and "ϕ is false" correspondingly. Convergence in this topology is pointwise; $\sigma_n \to \sigma$ iff for each ϕ, $\sigma_n(\phi) \to \sigma(\phi)$. The topology on Σ induces topology on each of the spaces Ω_i, $i = 0, ..., 3$. Moreover, as subsets of Σ:

LEMMA 1. *Each space Ω_i, $i = 0, ..., 3$, is closed in Σ and therefore compact.*

Proof. For a given ϕ the set $\{\sigma \mid \sigma(\phi) + \sigma(\sim\phi) = 1\}$ is closed. But

$$\Omega_0 = \bigcap_{\phi \in \Phi} \{\sigma \mid \sigma(\phi) + \sigma(\sim\phi) = 1\}$$

and thus Ω_0 is closed. Also

$$\Omega_1 = \Omega_0 \cap \left[\bigcap_{i \in I} \bigcap_{\phi \in \Phi} \{\sigma \mid \sigma(K_i K_i \phi) \geqslant \sigma(K_i \phi)\} \right].$$

and therefore Ω_1 is closed. The proofs for Ω_2 and Ω_3 are similar. Q.E.D.

We assume from now on that the space Ω is closed in Σ.

LEMMA 2. *If A is a closed subset of Ω then $P_i(A)$ is closed for each $i \in I$. In particular, for each $\omega \in \Omega$, $P_i(\omega)$ is closed.*

Proof. Suppose $\{\omega_n\} \subseteq P_i(A)$ and $\omega_n \to \omega$. There exists a sequence $\{\bar{\omega}_n\} \subseteq A$ such that for each n, $\omega_n \in P_i(\bar{\omega}_n)$. Since Ω is compact we may assume without loss of generality that $\bar{\omega}_n \to \bar{\omega}$, and since A is closed, $\bar{\omega} \in A$.

It is enough to show that $\omega \in P_i(\bar{\omega})$, i.e., that $K_i^-(\bar{\omega}) \subseteq \omega$. Indeed, suppose $K_i\phi \in \bar{\omega}$. Then for some N, $K_i\phi \in \bar{\omega}_n$ for all $n > N$. Thus $\phi \in \omega_n$ for $n > N$ and therefore $\phi \in \omega$. Q.E.D.

4. COMMON KNOWLEDGE

A proposition ϕ is *common knowledge* in ω if for each $n \geqslant 1$ and each sequence of agents, $i_1, ..., i_n$, $K_{i_1} \cdots K_{i_n}\phi \in \omega$.

The state ω' is *commonly possible* in ω if there exists $n \geqslant 1$ and a sequence of agents $i_1, ..., i_n$ such that $\omega' \in (P_{i_1}(P_{i_2}(\cdots P_{i_n}(\omega) \cdots))$. The set of all states which are commonly possible in ω is denoted by $P(\omega)$, i.e.,

$$P(\omega) = \bigcup (P_{i_1}(P_{i_2}(\cdots P_{i_n}(\omega) \cdots)),$$

where the union ranges over all finite sequences of agents.

Common knowledge and common possibility are related as follows.

THEOREM 3. *If ϕ is common knowledge in ω_0 then ϕ is true in every state which is commonly possible in ω_0. Moreover, ϕ is common knowledge in each such state.*

Proof. Let ω' be commonly possible in ω_0. Then there exists $n \geqslant 1$, a sequence $i_1, ..., i_n$ and states $\omega_1, ..., \omega_n$ such that $\omega_j p_{i_j} \omega_{j-1}$ for $j = 1, ..., n$ and $\omega_n = \omega'$. If ϕ is common knowledge in ω_0 then $K_{i_1} \cdots K_{i_n} \phi \in \omega_0$. It follows immediately by induction on j that $K_{i_j} \cdots K_{i_n} \phi \in \omega_{j-1}$ and thus $K_{i_n} \phi \in \omega_{n-1}$ which implies $\phi \in \omega_n = \omega'$.

To show that ϕ is common knowledge in ω', we observe that for each $n \geqslant 1$ and sequence $i_1, ..., i_n$, $K_{i_1} \cdots K_{i_n} \phi$ is also common knowledge in ω and therefore, by the first part of this theorem, true in ω'. Q.E.D.

The relation between $P(\omega)$ and $P_i(\omega)$ is given in the next lemma. The simple proof is omitted.

LEMMA 3. *For each $i \in I$ and $\omega \in \Omega$, $P_i(P(\omega)) \subseteq P(\omega)$. Moreover, if $\Omega \subseteq \Omega_2$ then $P_i(P(\omega)) = P(\omega)$. If $\Omega \subseteq \Omega_3$ then $P(\omega)$ is the minimal element of the join of the partitions $\{P_i\}_{i \in I}$ which contains ω.*

We recall that the join of the partitions $\{P_i\}_{i \in I}$ is the finest partition of Ω which is coarser than each P_i. In Aumann's model where common knowledge is an attribute of events, an event is common knowledge at ω if it contains this minimal element of the joint, $P(\omega)$.

We end this section with a topological property of $P(\omega)$.

LEMMA 4. *For each $\omega \in \Omega$, $P(\omega)$ is a countable union of closed sets.*

Proof. The proof follows from the definition of $P(\omega)$, Lemma 2, and the countability of I. Q.E.D.

5. FINITELY GENERATED KNOWLEDGE

We say that the set of propositions Ψ *logically implies* a proposition ϕ, if ϕ is true whenever all the propositions in Ψ are, i.e., if for every state ω, $\Psi \subseteq \omega$ implies that $\phi \in \omega$. The set Ψ *informationally implies* ϕ for agent i if $K_i \Psi$ logically implies $K_i \phi$ (where $K_i \Psi = \{K_i \psi \mid \psi \in \Psi\}$), i.e., if whenever i knows all the propositions in Ψ he knows also ϕ. A set of propositions Ψ *generates* i's ken in ω if $\Psi \subseteq K_i^-(\omega)$ and Ψ informationally implies each ϕ in $K_i^-(\omega)$. It is easy to check that Ψ generates i's ken iff $\Psi \subseteq K_i^-(\omega)$ and for each ω', if $\Psi \subseteq K_i^-(\omega')$ then $K_i^-(\omega) \subseteq K_i^-(\omega')$. That is Ψ is part of what i knows in ω and whenever he knows this part he knows everything else he knows in ω.

To illustrate this notion, consider a state space $\Omega \subseteq \Omega_1$. Suppose that for some i and ϕ,

$$K_i^-(\omega) = \{\phi, K_i\phi, K_iK_i\phi, ...\}.$$

It follows by (K1) that $\{\phi\}$ generates i's ken in ω. The knowledge of the propositions $K_i\phi$, $K_iK_i\phi$, ... is acquired by i "effortlessly"; it is implied by general, a priori knowledge rules which are state independent. Of course (K1)–(K3) are very basic rules of knowledge and in general Ω can be constructed with many other such rules. As opposed to these propositions, ϕ could be a piece of information that was gained by experience (a posteriori knowledge) and was not derived from previously known propositions. It is natural to assume that this type of information, which requires some "effort" to gain, is of finite size. This leads us to the following definition. Knowledge is *finitely generated* in Ω if for each agent i and state ω, $K_i^-(\omega)$ is generated by a finite set of propositions.

Consider now the equivalence relation $\approx_i$ defined on Ω by equality of kens, i.e., $\omega \approx_i \omega'$ iff $K_i^-(\omega) = K_i^-(\omega')$. Let Π_i be the partition of Ω to equivalence classes with respect to $\approx_i$.

LEMMA 5. *If $\Omega \subseteq \Omega_2$ then the following five conditions are equivalent*:

(1) $P_i(\omega) = P_i(\omega')$.

(2) $\omega \in P_i(\omega')$ *and* $\omega' \in P_i(\omega)$.

(3) $K_i^-(\omega) = K_i^-(\omega')$.

(4) $\omega \approx_i \omega'$.

(5) $A(\omega) = A(\omega')$, *where* $A(\omega)$ *and* $A(\omega')$ *are the elements of* Π_i *which contain* ω *and* ω' *correspondingly.*

Proof. (1) and (2) are equivalent by Corollary 1(b). The equivalence of (2) and (3) follows from Theorem 1(a). The equivalence of (3), (4), and (5) follows from the definition of $\approx_i$ and Π_i. Q.E.D.

Let $\Delta_i = \{P_i(\omega) \mid \omega \in \Omega\}$. By Corollary 1(c) when $\Omega \subseteq \Omega_3$, $\Delta_i = \Pi_i$ but when $\Omega \subseteq \Omega_2$ these sets are not necessarily the same. The following theorem relates Δ_i to Π_i and to finitely generated knowledge.

THEOREM 4. *Suppose $\Omega \subseteq \Omega_2$ and knowledge is finitely generated in Ω. Then for each $i \in I$, the sets Δ_i and Π_i are countable. Moreover, the σ-fields $\mathfrak{F}(\Delta_i)$ and $\mathfrak{F}(\Pi_i)$ generated by these two sets, correspondingly, coincide.*

Proof. Let A be an element of Π_i. If $\omega, \omega' \in A$ *then* $K_i^-(\omega) = K_i^-(\omega')$ and thus ω and ω' have the same finite generators. Let $\Psi: \Pi_i \to 2^{\Phi}$ be a map which assigns to each A in Π_i a finite subset of Φ, $\Psi(A)$, which generates $K_i^-(\omega)$ for each $\omega \in A$. Since Φ is countable it has countably many finite subsets and therefore Ψ has countably many values. Thus in order to show that Π_i is countable it is enough to prove that Ψ is one-to-one. To see this, suppose $\Psi(A) = \Psi(A')$ and let $\omega \in A$ and $\omega' \in A'$. $\Psi(A)$ and $\Psi(A')$ are finite generators of $K_i^-(\omega)$ and $K_i^-(\omega')$, respectively. Hence $\Psi(A) = \Psi(A') \subseteq K_i^-(\omega')$ and therefore, since $\Psi(A)$ generates $K_i^-(\omega)$, $K_i^-(\omega') \supseteq K_i^-(\omega)$. By symmetry we conclude that $K_i^-(\omega) = K_i^-(\omega')$ and hence $A = A'$.

To see that Δ_i is countable we observe that by Lemma 5, $P_i(\omega) = P_i(\omega')$ iff $A(\omega) = A(\omega')$ and therefore there exists a one-to-one correspondence between Δ_i and Π_i.

We prove now that $\Pi_i \subseteq \mathfrak{F}(\Delta_i)$ and $\Delta_i \subseteq \mathfrak{F}(\Pi_i)$ which show that $\mathfrak{F}(\Delta_i) = \mathfrak{F}(\Pi_i)$.

$\Pi_i \subseteq \mathfrak{F}(\Delta_i)$: To see this choose an element $A(\omega)$ in Π_i. Define $B(\omega) = P_i(\omega) \setminus \bigcup P_i(\omega')$, where the union ranges over all ω' in $P_i(\omega)$ such that $P_i(\omega) \supset P_i(\omega')$ (where '$\supset$' means strict inclusion). We show that $A(\omega) = B(\omega)$. Indeed if $\omega' \in B(\omega)$ then it must be the case that $P_i(\omega') = P_i(\omega)$ and therefore by the equivalence of (1) and (4) in Lemma 5, $\omega' \in A(\omega)$. Thus $B(\omega) \subseteq A(\omega)$. Conversely, if $\omega' \in A(\omega)$ then $P_i(\omega') = P_i(\omega)$. Clearly ω' is not an element of any $P_i(\omega'')$ which satisfy $P_i(\omega) \supset P_i(\omega'')$ (because otherwise, by (P2), $P_i(\omega') \subseteq P_i(\omega'') \neq P_i(\omega)$). Hence $\omega' \in B(\omega)$ which shows that $A(\omega) \subseteq B(\omega)$. We conclude that $A(\omega) = B(\omega)$. The fact that $B(\omega) \in \mathfrak{F}(\Delta_i)$ follows from the countability of Δ_i.

$\Delta_i \subseteq \mathfrak{F}(\Pi_i)$: By Theorem 1, if $\omega' \in P_i(\omega)$ then $A(\omega') \subseteq P_i(\omega)$. Thus $P_i(\omega) = \bigcup A(\omega')$ where the union ranges over all ω' in $P_i(\omega)$. The latter union is in $\mathfrak{F}(\Pi_i)$ due to the countability of Π_i. Q.E.D.

Another useful implication of finitely generated knowledge, which we use in the sequel, is the following.

THEOREM 5. *Suppose $\Omega \subseteq \Omega_2$ and knowledge is finitely generated in Ω. Let $\{\omega_n\}$ be a sequence in Ω such that $P_i(\omega_{n+1}) \subseteq P_i(\omega_n)$ for $n \geqslant 1$. Then, for large enough n and m, $P_i(\omega_n) = P_i(\omega_m)$.*

Proof. Since Ω is compact $\{\omega_n\}$ has a converging subsequence. In order to show that a decreasing sequence $\{P_i(\omega_n)\}_{n \geqslant 1}$ is constant for large enough n, it is enough to prove that a subsequence of it has this property. Thus we may assume without loss of generality that $\{\omega_n\}$ converges to some ω in Ω.

We prove first that

$$(1) \quad K_i^-(\omega) = \bigcup_n K_i^-(\omega_n).$$

Note that for $m>n$, $\omega_m \in P_i(\omega_m) \subseteq P_i(\omega_n)$ and thus $\{\omega_m\}_{m>n} \subseteq P_i(\omega_n)$. Since, by Lemma 2, $P_i(\omega_n)$ is closed, it follows that $\omega \in P_i(\omega_n)$ and this is for all n. Therefore by Theorem 1, $K_i^-(\omega) \supseteq K_i^-(\omega_n)$ for each n, i.e., $K_i^-(\omega) \supseteq \bigcup_n K_i^-(\omega_n)$. Conversely, suppose $\phi \notin \bigcup_n K_i^-(\omega_n)$; then for each n, $K_i(\phi) \notin \omega_n$ and hence $\sim K_i(\phi) \in \omega_n$. Since convergence in Ω is pointwise it follows that $\sim K_i(\phi) \in \omega$ and therefore $\phi \notin K_i^-(\omega)$. Thus (1) is established. Note also that

(2) the sequence $\{K_i^-(\omega_n)\}_{n \geqslant 1}$ is increasing,

since $\omega_{n+1} \in P_i(\omega_n)$ for $n \geqslant 1$.

Now let Ψ be a finite generator of $K_i^-(\omega)$. Then in particular $\Psi \subseteq K_i^-(\omega)$ and therefore, since Ψ is finite, by (1) and (2), $\Psi \subseteq K_i^-(\omega_n)$ for big enough n. Therefore, since knowledge is finitely generated, $K_i^-(\omega_n) \supseteq K_i^-(\omega)$, which proves, by (1) and (2), that for big enough n, $K_i^-(\omega) = K_i^-(\omega_n)$. By Lemma 5, this implies that $P_i(\omega_n)$ is constant for big enough n. Q.E.D.

Let us finally look at two simple conditions each of which implies that knowledge is finitely generated.

THEOREM 6. *Knowledge is finitely generated in each of the following cases*:

(a) *Ω is finite.*

(b) *For each $i \in I$, Δ_i is finite.*

Proof. Clearly (a) implies (b) and therefore it suffices to prove for case (b). Suppose Δ_i is finite and let $\omega \in \Omega$. Let $P_i(\omega_1), \ldots, P_i(\omega_m)$ be all the elements of Δ_i which satisfy $P_i(\omega_j) \setminus P_i(\omega) \neq \varnothing$, for $j = 1, \ldots, m$. It must be the case then that $K_i^-(\omega) \setminus K_i^-(\omega_j) \neq \varnothing$ for $j = 1, \ldots, m$ (since otherwise $K_i^-(\omega) \subseteq K_i^-(\omega_j)$ and by Theorem 1(a) and Corollary 1(b), $P_i(\omega_j) \subseteq P_i(\omega)$). Now choose for each $j = 1, \ldots, m$, $\psi_j \in K_i^-(\omega) \setminus K_i^-(\omega_j)$ and let $\Psi = (\psi_1, \ldots, \psi_m\}$. We claim that Ψ is a finite generator of $K_i^-(\omega)$. Indeed, suppose $\Psi \subseteq K_i^-(\omega')$. Then for all $j = 1, \ldots, m$, $P_i(\omega') \neq P_i(\omega_j)$ because if for some j, $P_i(\omega') = P_i(\omega_j)$, then $\psi_j \in K_i^-(\omega') = K_i^-(\omega_j)$ which contradicts the choice of ψ_j. Since $P_i(\omega')$ is not one of the sets $P_i(\omega_j)$ it must satisfy $P_i(\omega') \subseteq P_i(\omega)$ which implies $K_i^-(\omega') \supseteq K_i^-(\omega)$. This proves that Ψ generates $K_i^-(\omega)$. Q.E.D.

6. AGREEING TO DISAGREE

Let $\mathfrak{B}$ be the Borel σ-fields on Ω and let μ be a probability measure on $(\Omega, \mathfrak{B})$. The measure μ is interpreted as a prior distribution on Ω which is common to all agents. In particular for events of the form $T_\phi = \{\omega \mid \phi \in \omega\}$,

$\mu(T_\phi)$ is obviously interpreted as the prior probability that the proposition ϕ is true.

LEMMA 6. *For each $i \in I$ and $\omega \in \Omega$, $P_i(\omega)$ and $P(\omega)$ are measurable.*

Proof. This lemma follows from Lemmas 2 and 4. Q.E.D.

Assume now that knowledge in Ω is finitely generated. Fix a proposition ϕ in Φ. For each $\omega \in \Omega$ and $i \in I$ such that $\mu(P_i(\omega)) > 0$ denote by $q_{i,\omega}$ the posterior probability of ϕ given the knowledge of agent i in ω; that is,

$$q_{i,\omega} = \mu(T_\phi \mid P_i(\omega)) = \mu(T_\phi \cap P_i(\omega))/\mu(P_i(\omega)).$$

Let $Q_i = \{q_{i,\omega} \mid \mu(P_i(\omega)) > 0\}$. By Theorem 4, Q_i is countable. We assume now that for each $q \in Q_i$ there exists a proposition $\phi_i(q)$ which is interpreted as saying that the posterior of ϕ for i is q. The propositions that correspond to different q's are different; i.e., if $q \neq q'$ then $\phi_i(q) \neq \phi_i(q')$. We denote by Ψ_i the countable set of all such propositions; i.e., $\Psi_i = \{\phi_i(q) \mid q \in Q_i\}$. We assume that for each state ω and agent i, at most one proposition from Ψ_i is true in ω, namely the one that describes the posterior of ϕ in ω for i. We call this condition *regularity*. Formally regularity requires that

For each i and ω, if $\mu(P_i(\omega)) > 0$ then $\Psi_i \cap \omega = \phi_i(q_{i,\omega})$,

and if $\mu(P_i(\omega)) = 0$ then $\Psi_i \cap \omega = \emptyset$.

We say that in Ω it is *impossible to agree to disagree* if for each $\omega \in \Omega$ the following holds:

If for each $i \in I$, $\phi_i(q_{i,\omega})$ is common knowledge in ω

then for each $j, k \in I$, $q_{j,\omega} = q_{k,\omega}$.

THEOREM 7. *If $\Omega \subseteq \Omega_2$ then it is impossible to agree to disagree in Ω.*

Proof. Suppose $\phi_i(q_{i,\omega})$ is common knowledge in ω for all i. Then by Theorem 3, for each agent i, $\phi_i(q_{i,\omega}) \in \omega'$ for each $\omega' \in P(\omega)$. By the regularity codition it follows that $\phi_i(q_{i,\omega}) = \phi_i(q_{i,\omega'})$ and thus $q_{i,\omega} = q_{i,\omega'}$, i.e., i has the same posterior of ϕ in all the states of $P(\omega)$. Let us denote this common posterior by q_i. We will show that

$$(1) \quad \mu(T_\phi \mid P(\omega)) = q_i.$$

Since the left-hand side of this equality is independent of i this proves that the posteriors of all the agents are the same.

By Lemma 3, $P(\omega) = \bigcup_{\omega' \in P(\omega)} P_i(\omega')$, and therefore by Theorem 4, there is a subset Γ_i of Π_i such that $P(\omega)=\bigcup_{B\in\Gamma_i} B$. Since this is a countable union of disjoint sets it is suffices to show, in order to prove (1), that

(2) For each $B\in\Gamma_i$, either $\mu(B)=0$ or $\mu(T_\phi \mid B)=q_i$.

Let $\Gamma_i'=\{B \mid B\in\Gamma_i,\ B$ satisfies (2)$\}$. Proving (2) is equivalent to proving

(3) $\Gamma_i'=\Gamma_i$.

Suppose to the contrary that (3) does not hold, i.e., $\Gamma_i\supset\Gamma_i'$. Denote $G=\bigcup_{B\in\Gamma_i'} B$. By our assumption $P(\omega)\supset G$, and we can choose a state ω_0 which satisfies

(4) $\omega_0\in P(\omega)\setminus G$.

Then, since $\omega_0\in P_i(\omega_0)$ it follows that

(5) $\omega_0\in P_i(\omega_0)\setminus G$.

We claim that ω_0 can be chosen such that also

(6) For each $\omega'\in P_i(\omega_0)\setminus G$; $\quad P_i(\omega')\setminus G=P_i(\omega_0)\setminus G$.

Indeed suppose to the contrary that each ω_0 which satisfies (4) and (5) does not satisfy (6). We construct by induction, under this assumption, a sequence $\{\omega_n\}$ such that for all $n\geqslant 1$,

(7) $\omega_n\in P(\omega)$,

(8) $\omega_n\in P_i(\omega_n)\setminus G$

(9) $P_i(\omega_n)\setminus G\supset P_i(\omega_{n+1})\setminus G$.

For ω_1 select any state which satisfies (4) and (5). Suppose $\omega_1, ..., \omega_n$ were selected. Then by (7) and (8), ω_n satisfies (4) and (5) as ω_0, and therefore by our assumption does not satisfy (6), which means that for some $\omega_{n+1}\in P_i(\omega_n)\setminus G$, $P_i(\omega_n)\setminus G\neq P_i(\omega_{n+1})\setminus G$. Since $P_i(\omega_{n+1})\subseteq P_i(\omega_n)$ it must be the case that (9) is satisfies. This completes the construction of the sequence. But (9) implies that for each $n\geqslant 1$, $P_i(\omega_n)\supset P_i(\omega_{n+1})$, which contradicts Theorem 5.

Now let ω_0 satisfy (4), (5), and (6). We prove that

(10) $P_i(\omega_0)\setminus G\in\Pi_i$.

First, for each $\omega'\in P_i(\omega_0)\setminus G$ it follows by (6) that $P_i(\omega')\setminus G=P_i(\omega_0)\setminus G$ and thus by (5) $\omega_0\in P_i(\omega')$. Hence $\omega'\approx_i\omega_0$ by the equivalence of (2) and (4) in Lemma 5. Conversely, suppose that $\omega'\approx_i\omega_0$; then $\omega'\in P_i(\omega_0)$.

Moreover, both states belong to the same element of Π_i and since $\omega_0 \notin G$ also $\omega' \notin G$. Thus $\omega' \in P_i(\omega_0)\setminus G$. This completes the proof of (10).

We note that $P_i(\omega_0)\setminus G \subseteq P(\omega)$ and therefore by (10)

$$(11)\quad P_i(\omega_0)\setminus G \in \Gamma_i.$$

But, since $P_i(\omega_0)\setminus G$ is disjoint from G,

$$(12)\quad P_i(\omega_0)\setminus G \notin \Gamma_i'.$$

We show now that

$$(13)\quad \text{either } \mu(P_i(\omega_0)\setminus G)=0 \text{ or } \mu(T_\phi \mid P_i(\omega_0)\setminus G)=q_i.$$

(13) and (11) contradict (12), and this completes the proof. To prove (13) we note that by (4) $\omega_0 \in P(\omega)$ and thus

$$(14)\quad \mu(T_\phi \mid P_i(\omega_0))=q_i.$$

Also $P_i(\omega_0)\cap G \subseteq P_i(\omega_0)$. This and (14) imply that to prove (13) it is enough to show that

$$(15)\quad \text{either } \mu(P_i(\omega_0)\cap G)=0 \text{ or } \mu(T_\phi \mid P_i(\omega_0)\cap G)=q_i.$$

For this we recall that for each ω', $\omega' \in A(\omega') \subseteq P_i(\omega')$, where $A(\omega')$ is the element in Π_i which contains ω', and therefore

$$(16)\quad P_i(\omega_0)\cap G = \bigcup_{\omega' \in P_i(\omega_0)\cap G} A(\omega').$$

But each $A(\omega')$ in the latter union is in Γ_i' which proves (15). Q.E.D.

It is possible to extend Theorem 7 to $\Omega \subseteq \Omega_1$ provided that we restrict Ω and the prior distribution μ as follows. We say that μ is *consistent* with Ω if $\mu(\Omega\setminus\Omega_2)=0$. We note that for each $\omega \in \Omega\setminus\Omega_2$ there exist an agent i and a proposition ϕ such that i knows ϕ, but ϕ is not true in ω. Clearly this implies that ω is impossible for i in ω. The consistency of μ guarantees that the prior distribution reflects this impossibility.

A state ω is a *dead end* in Ω if for some $i \in I$, $P_i(\omega)=\varnothing$. Note that if $\Omega \subseteq \Omega_2$ then Ω does not have dead ends.

THEOREM 8. *If $\Omega \subseteq \Omega_1$, μ is consistent with Ω, and Ω does not have dead ends then it is impossible to agree to disagree in Ω.*

Proof. We give an outline of the proof. Since Ω does not have dead ends, $P(\omega) \neq \varnothing$ for each ω. Consider the state space $\Omega' = \Omega \cap \Omega_2$. Define $P_i'(\omega)=P_i(\omega)\cap\Omega_2$ and $P'(\omega)=P(\omega)\cap\Omega_2$. $P_i'(\omega)$ and $P'(\omega)$ are the sets of possible and commonly possible states for i in ω, correspondingly, in Ω'. The consistency of μ enables us to carry the same proof as in Theorem 7, with respect to Ω'. Q.E.D.

7. Discussion

Common Knowledge and Epistemic Logic

The basic features of the model presented in section 2 are common in the literature of formal model logic and epistemic logic. (See Hintikka [8] for a philosophical analysis of the model and Hughes and Creswell [9] for the mathematical development of the theory.) It is worth noting though that for our purposes we do not need the full body of these theories. First of all, unlike formal modal logic systems we do not start with a language, but rather with a set of propositions the structure of which is irrelevant to us. Also the epistemic operators K_i are functions from the set of propositions into itself rather than letters of an alphabet. As a result these operators may have properties that the corresponding operators of epistemic logic cannot have. For example, K_i in this work is not necessarily one-to-one, while it is always the case that $K_i\phi \neq K_i\psi$ for any $\phi \neq \psi$ when K_i is an epistemic logic operator. Thus in our model the same proposition may express simultaneously that i knows two different propositions. Moreover, we do not restrict the relation between K_i's of different agents. It is possible to have a proposition ϕ such that $K_i\phi = K_j\phi$ for two distinct agents i and j. It may also be the case that for each i and ϕ,

$$(*) \quad K_iK_i\phi = K_i\phi.$$

This means that knowing ϕ and knowing that ϕ is known are the same. When this is the case, requirement (K1) is automatically satisfied in each state of the world and $\Omega_1 = \Omega_0$.

In our model it may be possible to verify that a certain proposition is common knowledge without resorting to infinite application of K_i's. This may be the case for example if the only source of knowledge in our model is the newspaper and the propositions $K_i\phi$ for all i are the same proposition: "ϕ is in the newspaper." In this case it is enough that all agents simply know ϕ in ω, in order for ϕ to be common knowledge in ω. Such models formalize ideas of Lewis [12] and Clarck and Marshall [4] which try to eliminate infinite processes of verifying common knowledge.

Another feature of the theory of common knowledge here is the lack of any "logical" restriction on knowledge. Beyond (K1)–(K3) there is no required relation between knowledge and propositional calculus or other logical structure. Unlike epistemic logic we do not require that agents have any deductive tools and our agents do not necessarily know all tautologies. In short the whole theory is indifferent to logic. Bacharach [3], on the contrary, assumes in his model that agents' knowledge follows some logical rules. We will comment on this in the next paragraph.

Alternative Approaches

In our model each proposition ϕ is associated with an event T_ϕ, the set of all states in which ϕ is true. In models which start with states as primitives, we can think of events as representing propositions. The formal equivalence of "event E is true in state ω" is simply $\omega \in E$. Knowledge operators map events to events; the event $K_i E$ is the event "i knows E" and $\omega \in K_i E$ means "i knows E in ω". The properties (K1)–(K3) can be easily translated as well. The possibility relation between states is similarly defined. ω' is possible for i in ω, if each event which is known to i in ω (i.e., $\omega \in K_i E$) is true in ω' (i.e., $\omega' \in E$). Thus the set of states which are possible to i in ω is $P_i(\omega) = \bigcap_{\omega \in K_i E} E$. (Appropriate conditions should be specified to guarantee that this, possibly uncountable, intersection is an event.) It can be easily shown that under the assumptions (K1)–(K3) the possibility relation generates a partition of the state space. Bacharach [3] uses an event-based model but his definition of the possibility relation differs from the one we mentioned here. As a result he needs an additional property, (K4), to guarantee a partition, namely for each i and events $E_1, E_2, ..$; $K_i(E_1 \cap E_2 \cap \cdots) = K_i(E_1) \cap K_i(E_2) \cap \cdots$. This is interpreted as saying that knowing a conjunction is equivalent to knowing each conjugant. Such a requirement is not needed, either in a proposition-based model, or in an event-based model if the possibility relation is defined as it is in this work.

Another possible approach is to discard the knowledge operators altogether and use the possibility relation on states as the primitive notion. One then replaces the properties (K1)–(K3) by properties analogous to (P1)–(P3).

For obvious reasons we preferred the proposition-based model. The natural topology defined on the state space, when we start from propositions, enables us to derive the measurability of all the needed sets rather than to assume it. More importantly, the notion of finitely generated knowledge and the properties it implies in Theorems 4 and 5, which are crucial to the proof of the main result, Theorem 7, are most naturally set in this model and would have required ad hoc assumptions in any of the other models.

Updating and Learning

Aumann's "Agreeing to Disagree" also has dynamic variations in which agents interchange information until their posteriors become common knowlege at which point they must coincide (see, e.g., Geanakoplos and Polemarchakis [5] and Bacharach [3]). In these dynamic models knowledge increases in each step and as a result the partitions are refined. When one tries to apply such a procedure in our model one faces a difficulty. Suppose $\Omega \subseteq \Omega_3$ and i does not know ϕ in ω, i.e., $\sim K_i\phi \in \omega$. If i gains some new information and he knows ϕ then the state of the world is no longer ω. Changes in i's knowledge result in a change in the state of the world. But the partition of Ω cannot change at all. It depends on relations between the states, which are fixed. Even worse, information cannot increase. Suppose we are now in a new state ω' where $K_i\phi \in \omega'$. Unfortunately, by moving to ω', i lost some knowledge; he knew $\sim K_i\phi$ in ω while of course he does not know it in ω' since it is not true there.

To resolve this apparent difficulty one has to introduce time into the model. A state of the world should be a description of the whole history of the world. In particular knowledge is now time dependent. Formally we have for each t, $t = 1, 2, \ldots$, epistemic operators $K_{i,t}$ which are interpreted as "i knows at time t that..." Correspondingly we have for each period t and state ω kens $K_{i,t}^{-}(\omega)$ and sets of possible states for i, $P_{i,t}(\omega)$. Properties (K1) and (K2) should be applied to each $K_{i,t}^{-}$. Ω_1, Ω_2, and Ω_3 are defined mutatis mutandis. We add also a new requirement. For each agent i, proposition ϕ, time t, and state ω,

(K0) If $K_{i,t}\phi \in \omega$ then $K_{i,t+1}\phi \in \omega$.

This simply says that agents do not forget what they knew. (K0) guarantees that for each t, i, and ω, $K_{i,t}^{-}(\omega) \subseteq K_{i,t+1}^{-}(\omega)$; i.e., knowledge does not decrease. As a result of the growing knowledge, information structure is refined in time, that is, for each i, ω, and t, $P_{i,t+1}(\omega) \subseteq P_{i,t}(\omega)$. In particular for $\Omega \subseteq \Omega_3$ the partitions of the agents are refined.

The use of this model for dynamic processes of information exchange makes possible careful examination of the conditions under which the exchange leads to, or ends in common knowledge. Results similar to those of Bacharach [3] can be obtained for information structures which are not partitions.

Why Ω_2?

Both (K1) and (K3) require infinite application of the operators K_i. When the agents are bounded in their ability to process information one can expect that these two assumptions may fail to hold. We bring now an argument, based on such bounded ability, that supports rejection of (K3) while it still enables the acceptance of (K1).

Suppose we have a measure of complexity on Φ, comp: $\Phi \to R^+$ such that $\{\text{comp}(\phi) \mid \phi \in \Phi\}$ is unbounded. We assume that for each i and ϕ

$$\text{(C1)} \quad \text{comp}(K_i\phi) \geqslant \text{comp}(\phi),$$

and

$$\text{(C2)} \quad \text{comp}(\sim\phi) \geqslant \text{comp}(\phi).$$

Assume further that

$$\text{(C3)} \quad \text{comp}(K_iK_i\phi) = \text{comp}(K_i\phi).$$

(An extreme case of this is when $K_iK_i\phi = K_i\phi$.) If knowledge in our model involves the ability to produce the known proposition or to use it in a deductive process then it is natural to assume that knowledge of an agent in a given state is bounded by complexity. Formally this means that for each i and ω there exists a bound $M_{i,\omega}$ such that for each $\phi \in K_i^-(\omega)$, $\text{comp}(\phi) \leqslant M_{i,\omega}$. Under this assumption Ω cannot satisfy (K3). Indeed for ϕ with $\text{comp}(\phi) > M_{i,\omega}$, $\text{comp}(\sim K_i\phi) \geqslant \text{comp}(\phi) > M_{i,\omega}$ by (C1) and (C2) and therefore $\sim K_i\phi \notin K_i^-(\omega)$. On the other hand, by (C3), (K1) can be satisfied notwithstanding the bounded complexity. Note that it is not the resemblance of (C3) to (K1) that gives (K1) the advantage over (K3). Indeed since the previous result depends only on (C1) and (C2) it still hols even if we add the assumption

$$\text{(C4)} \quad \text{comp}(K_i \sim K_i\phi) = \text{comp}(\sim K_i\phi)$$

(or even the stronger assumption $\text{comp}(K_i \sim K_i\phi) = \text{comp}(\phi)$), which corresponds to (K3) in the same way (C3) corresponds to (K1).

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