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Knowledge Spaces with Arbitrarily High Rank*

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The class of probabilistic belief spaces (Harsanyi, 1967–68, *Man. Sci.*, **14,** 159–182, 320–324, 486–502) contains a *universal space*, into which every other belief space can be mapped in a unique way by a belief morphism. We show that there is no analogous universal space in the class of knowledge spaces. To show this we define the *rank* of a knowledge space, which is the ordinality of the most complicated descriptions of knowledge in the space. We then show that a knowledge space can be mapped by a knowledge morphism only to spaces of higher or equal rank. We construct knowledge spaces for arbitrarily high rank, demonstrating that there is no universal space. *Journal of Economic Literature* Classification Numbers: D80, D82.

1. INTRODUCTION

Knowledge spaces have been a major tool for modeling interactive uncertainty in game theory and economics, ever since Aumann's seminal paper "Agreeing to Disagree" (Aumann, 1976). Player's knowledge is modeled, in a knowledge space, by ascribing to him a partition of the space. At each element of the space, called a *state*, he knows those events (i.e., subsets of the knowledge space) which contain the partition member that includes the state. To describe what the players know about a game, we associate with each state a *state of nature*, which specifies the value of the objective parameters of the game, like the payoffs or signals. In this way, it is possible to describe what each player knows, in each state,

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regarding nature, and given these descriptions, what each player knows about the knowledge of the others regarding nature. Continuing inductively, one can describe the mutual knowledge of the players of all orders.

In applications, finite knowledge spaces are often used, in which clearly not every conceivable description of mutual knowledge is represented. It would, therefore, be nice to know that there is a *universal* knowledge space, in which every state of mind of the players in whatever knowledge space (with the same states of nature) is represented. In other words, a universal space is one into which every knowledge space can be mapped by a *knowledge morphism*—a map that preserves both nature and the knowledge of the players. If such a big space exists, we could in principle always carry out the analysis in it, with no fear of neglecting any relevant state of affairs.

Type spaces, introduced by Harsanyi (1967–68) are another tool for the modeling of interactive uncertainty. Player's uncertainty in a state of such a space is represented by a σ-additive probability measure over the space (rather than an element of his partition in a knowledge space).

In the following section we develop a general approach to the definition of type spaces, of which both knowledge spaces and Harsanyi type spaces are special cases. This enables us to consider a general notion of universal spaces, and rigorously pose the question of the existence of a universal space for both families of models in a unified manner.

The existence of a universal space for Harsanyi type spaces was first proved by Mertens and Zamir (1985). They were followed by Brandenburger and Dekel (1993), Heifetz (1993), Mertens *et al.* (1994), and Heifetz and Samet (1996), who proved it for more diverse and general conditions.

In this work we show that *there is no universal space for knowledge spaces*. For this purpose we define the *rank* of a knowledge space to be the ordinal length of the maximal non-trivial description of mutual knowledge in the space. We then show that a knowledge space cannot be mapped by a knowledge morphism to a knowledge space of lower rank. In order to prove that there is no universal knowledge space, one to which all knowledge spaces can be mapped, it is enough, then, to show that there are knowledge spaces with arbitrarily high ranks. We do this in Section 3, where we show by an explicit construction that even with two states of nature and two players there are knowledge spaces of any ordinal rank. This fact, with a somewhat different setup and terminology was proved independently, in a non-constructive way, by Fagin (1994). Fagin *et al.* (1991) built a knowledge space with three players of order $\omega + 1$.

The lack of a universal space for knowledge spaces, in contrast to the existence of such a space for Harsanyi type spaces, can be traced down to a single factor: continuity. In Harsanyi type spaces the beliefs of the players are σ-additive, and therefore continuous on increasing and decreasing sequences of events. As a result, describing the beliefs of a player regarding all the finite-order beliefs of the other players dictates also his beliefs regarding limit events that involve all these finite orders together. Indeed, Heifetz and Samet (1996) showed that type spaces of non-additive beliefs have universal spaces when the belief functions are continuous. In this they demonstrated that continuity is what guarantees the existence of a universal space.

Knowledge, by contrast, does not have this continuity property. More specifically, one can not-know any of the events in an increasing sequence of events (that is, events that are less and less informative) and yet know (and thus fail not-to-know) the limit of this sequence. Our construction shows that this lack of continuity holds for all limit ordinals. This means that there are arbitrarily long descriptions of mutual knowledge, which preclude the existence of a universal knowledge space.

2. KNOWLEDGE SPACES AND THEIR RANK

Knowledge Spaces

Throughout this section we fix non-empty sets I and S. The elements of I are the *players* and the elements of S are the *states of nature*. We think of each element of S as being the specification of the possible values of the parameters that are relevant to the interaction between the players, e.g., payoff functions or strategy sets.

A *knowledge-space* on S is given by a triplet,

$$\langle \Omega, \Theta, (\Pi_i)_{i \in I} \rangle,$$

where Ω is a non-empty set whose elements are called *states of the world* and whose subsets are called *events*; Θ: $\Omega \to S$ specifies for each state of the world the state of nature that prevails there; and for each player $i \in I$, Π_i is a partition on Ω.

The partitions describe the players' knowledge. In state $\omega \in \Omega$ of the world player i considers as possible all the states in $\Pi_i(\omega)$—the element of Π_i, which contains ω. He cannot tell, though, which of them obtains. Thus, player i can tell at ω that event E holds if, and only if, E contains $\Pi_i(\omega)$. In such a case we say that *i knows E* at ω. The set of all states ω in which i knows E is *the event that i knows E*, and is denoted by $K_i(E)$. Thus,

$$K_i(E) = \{\omega \in \Omega | \Pi_i(\omega) \subseteq E\}.$$

The operators K_i: $2^{\Omega} \to 2^{\Omega}$ are called the *knowledge operators*.

Knowledge Morphisms

In order to make the modeling of interactive knowledge by knowledge spaces a complete theory, we have to be able to compare the objects of this theory, namely, knowledge spaces. Such comparisons can be done by mappings of spaces that preserve their structure as follows.

Let $\langle \Omega, \Theta, (\Pi_i)_{i \in I} \rangle$ and $\langle \Omega', \Theta', (\Pi'_i)_{i \in I} \rangle$ be two knowledge spaces on S. A function f: $\Omega \to \Omega'$ is a *knowledge morphism* if it satisfies the following two conditions:

(2.1) For each $\omega \in \Omega$, $\Theta(\omega) = \Theta'(f(\omega))$.

(2.2) For each $\omega \in \Omega$ and $i \in I$, $f(\Pi_i(\omega)) = \Pi'_i(f(\omega))$.

These conditions guarantee that f preserves the structure of the spaces. By (2.1), the same state of nature prevails in states that correspond by f. By (2.2) the partition structure of the first space is mapped onto that of the second. The following proposition expresses the preservation of the knowledge structure of (2.2) in terms of knowledge operators. The simple proof is omitted. We denote by K'_i the knowledge operators of the space $\langle \Omega', \Theta', (\Pi'_i)_{i \in I} \rangle$.

PROPOSITION 2.1. *The following condition is equivalent to* (2.2):

(2.3) *For each $i \in I$ and event $E \subseteq \Omega'$, $f^{-1}(K'_i(E)) = K_i(f^{-1}(E))$.*

Type Spaces

In order to see the analogy between knowledge spaces and (probabilistic) belief spaces, we now present knowledge spaces as type spaces.

We start with some simple definitions. Let X and Y be sets and f a given function f: $X \to Y$. We use f to define a function $\hat{f}$ that maps real valued set functions on X to real valued set functions on Y. For each real valued set function r on X and $E \subseteq Y$, $\hat{f}(r)(E) = r(f^{-1}(E))$.

A *type space* on S is a triple

$$\langle \Omega, \Theta, (t_i)_{i \in I} \rangle,$$

where Ω and Θ are defined as before, and for each i, t_i is a *type function* from Ω to the set of real valued functions on events. We call the real valued function $t_i(\omega)$ the *type* of i at ω. We describe important events in Ω in terms of the type functions as follows (see Monderer and Samet, 1989). For each real number p, event E in Ω, and player i,

$$B_i^p(E) = \{\omega | t_i(\omega)(E) \geq p\}.$$

Knowledge spaces can be easily described as type spaces where the types are restricted to a certain kind of real valued functions: With each event $P \subseteq \Omega$ we associate a 0−1 real valued set function δ_P, where, $\delta_P(E) = 1$

whenever $P \subseteq E$ and $\delta_P(E) = 0$ otherwise. The knowledge space $\langle \Omega, \Theta, (\Pi_i)_{i \in I} \rangle$ can be alternatively given by the triplet $\langle \Omega, \Theta, (t_i)_{i \in I} \rangle$, where for each state ω, $t_i(\omega)$ is the function $\delta_{\Pi_i(\omega)}$. As the type functions are 0–1 functions the only non-trivial events $B_i^p(E)$ are those for which $p = 1$. It is easy to see that $K_i(E) = B_i^1(E)$. Knowledge morphisms can be also expressed in terms of the type functions as follows. We denote by t_i' the type functions of the space $\langle \Omega', \Theta', (\Pi_i')_{i \in I} \rangle$.

PROPOSITION 2.2. *The following condition is equivalent to* (2.2):

(2.4) *For each $i \in I$ and $\omega \in \Omega$, $t_i'(f(\omega)) = \hat{f}(t_i(\omega))$.*

Representing knowledge spaces as type spaces reveals the common features of knowledge spaces and probabilistic belief spaces. To demonstrate this point we describe briefly the class of probabilistic belief spaces on the space S of states of nature.

A probabilistic belief space, or belief space for short, is a type space where the spaces S and Ω are compact topological spaces. Events in a space are elements of the Borel σ-field generated by the topology of the space. For each i and ω, $t_i(\omega)$ is a σ-additive probability measure on the Borel σ-field on Ω. The functions Θ and t_i are required to be continuous when the topology on the space of all probability measures on Ω is the topology of weak convergence. The continuity of t_i guarantees that the sets $B_i^p(E)$ are measurable, i.e., they are events. The event-to-event functions $B_i^p(\cdot)$ are called *belief operators*.

A continuous function f from a belief space Ω to a belief space Ω' is a *belief morphism* if it satisfies conditions (2.1) and (2.4). Thus, knowledge and belief morphisms are defined in exactly the same terms.

Universal Type Spaces

The use of type spaces in economic or game-theoretic models raises the question of the limitations of such spaces. Fixing one type space usually leaves out types which are not accounted for in the space. It is possible, and usually is the case, that there is a richer type space into which the first can be mapped by a type morphism. The question, then, is whether every type space is restrictive in this sense, or there exists a ‘‘biggest’’ type space that includes all possible types. This would be a type space to which all other type spaces (with the same space of nature states) can be mapped by a type morphism. This question can be posed, in precisely the same terms, for every class of type spaces and in particular for knowledge spaces and belief spaces.

Mertens and Zamir (1985) answered this question in the affirmative for belief spaces. They proved the existence of a universal belief space Ω^*, characterized by the property that for each belief space Ω there is a unique belief morphism from Ω to Ω^*. The same was proved for belief spaces with various topological properties other than compactness by Brandenburger and Dekel (1993), Heifetz (1993), and Mertens *et al.* (1994). Heifetz and Samet (1996) proved the theorem for the general measure-theoretic case. Epstein and Wang (1994) made a similar construction for more general types of beliefs.[1] Vassilakis (1992) investigated the question in a general category-theory framework.

In this work, we show that, unlike in belief spaces, there is no universal space for knowledge space. We prove:

THEOREM 2.3. *If there are at least two players in I and at least two states of nature in S, then there is no universal knowledge space Ω^* on S, i.e., a knowledge space with the property that, for each knowledge space Ω on S, there exists a knowledge morphism from Ω to Ω^*.*

To prove this theorem we employ the notion of the rank of a knowledge space.

The Rank of Knowledge Spaces

Knowledge spaces on S are designed for the purpose of expressing interactive knowledge concerning S. Thus, the events in Ω which are of interest are those that can be described in terms of states of nature and knowledge operators. The rank of a knowledge space is the ordinality of the non-trivial longest descriptions of such an event in the space.

To define rank formally, we use the following notation and terminology. For a partition P we denote by $\overline{P}$ the set of all arbitrary unions of elements of P. We say that the partition P is *generated* by a subset $\mathscr{F}$ of 2^{Ω}, if P is the coarsest partition such that $\overline{P}$ contains $\mathscr{F}$. Alternatively, P is the partition of Ω to equivalence classes, where two states in Ω are equivalent if they belong to the same sets in $\mathscr{F}$.

We associate with a given knowledge space Ω a partition P^{α} of Ω, for each ordinal α. The elements of $\overline{P}^{\alpha}$ are called *α-order events*. We start with the partition of Ω which is defined in terms of states of nature. Then each partition P^{α} refines the previously defined partitions by adding to them events obtained by applying knowledge operators to previously defined events. Formally, the partitions P^{α} are defined as follows. P^0 is the partition of Ω to the sets $\Theta^{-1}(s)$ for $s \in S$. The partition P^{α} is generated by the sets of the previously defined partitions and by events expressing knowledge of lower order events. Namely, P^{α} is generated by the sets in $\bigcup_{\beta < \alpha} P^{\beta}$ and the sets of the form $K_i(E)$, where $E \in \bigcup_{\beta < \alpha} \overline{P}^{\beta}$, i.e., E is a β-order event for some $\beta < \alpha$. Clearly, for $\alpha < \beta$,

[1] Their result is even more general and allows for some kind of preferences (over acts) from which beliefs cannot be explicitly isolated.

$\bar{P}^{\alpha} \subseteq \bar{P}^{\beta}$. But the cardinality of a sequence of strictly refining partitions cannot exceed the cardinality of Ω. Therefore, we must have, for some α, $P^{\alpha} = P^{\alpha+1}$, or equivalently, $P^{\alpha} = P^{\beta}$ for all $\beta > \alpha$. The minimal α that satisfies this is called the *rank* of Ω.

PROPOSITION 2.4. *If there is a knowledge morphism from Ω to Ω', then the rank of Ω does not exceed that of Ω'.*

Proof. Let f be the knowledge morphism from Ω to Ω', and denote by P^{α} and P'^{α} the partitions associated with Ω and Ω', respectively. Denote by K_i and K_i' the respective knowledge operators. We show that, for each α,

$$f^{-1}(P'^{\alpha}) = P^{\alpha}. \tag{2.5}$$

Therefore, if $\bar{P}'^{\alpha} = \bar{P}'^{(\alpha+1)}$ then $\bar{P}^{\alpha} = \bar{P}^{(\alpha+1)}$, which proves the proposition.

We proceed by induction on α. By (2.1), $f^{-1}(\Theta'^{-1}(s)) = \Theta^{-1}(s)$ for each $s \in S$, and therefore (2.5) holds for $\alpha = 0$. Suppose it holds for all $\beta < \alpha$. We show that f^{-1} maps the set of generators of P'^{α} onto the set of generators of P^{α}, which proves (2.5). By the induction hypothesis, $f^{-1}(\bigcup_{\beta<\alpha} P'^{\beta}) = \bigcup_{\beta<\alpha} P^{\beta}$. Also if $F \in \bigcup_{\beta<\alpha} \bar{P}'^{\beta}$, then by (2.3), $f^{-1}(K_i'(F)) = K_i(f^{-1}(F))$, where by the induction hypothesis $f^{-1}(F) \in \bigcup_{\beta<\alpha} \bar{P}^{\beta}$. This shows that f^{-1} maps the set of generators into the set of generators. To see that the map is onto we note that by the induction hypothesis for each $E \in \bigcup_{\beta<\alpha} \bar{P}^{\beta}$ there exists $F \in \bigcup_{\beta<\alpha} \bar{P}'^{\beta}$ such that $f^{-1}(F) = E$. Q.E.D.

In the next section we prove:

THEOREM 2.5. *If there are at least two players in I, and at least two states of nature in S, then for each ordinal α there exists a knowledge space W^{α} on S with rank α. Moreover, the cardinality of W^{α}, for an infinite ordinal α, does not exceed that of α.*

Proposition 2.4 and Theorem 2.5 are used now to prove Theorem 2.3.

Proof of Theorem 2.3. If $|I| \geq 2$ and $|S| \geq 2$ then, by Proposition 2.5, for any state space Ω^* on S there is a space Ω on S with rank higher than that of Ω^*. By Proposition 2.4, there is no knowledge morphism from Ω to Ω^*. Q.E.D.

3. A CONSTRUCTION OF KNOWLEDGE SPACES WITH ARBITRARY RANK

We prove Theorem 2.5 by constructing the spaces W^{α}. The construction is carried out for two states of nature and two players, but can be easily extended to more states of nature and more players; extra states are simply ignored and extra players are assigned trivial partitions. This construction constitutes a positive reply to a question that Fagin (1994) posed as an open problem.

The set of states of nature, in our construction, consists of the two results of tossing a coin, H and T. The set of players is $I = \{1, 2\}$, where we adopt the notational convention that j is the opponent of player i. We construct for every ordinal α a knowledge space

$$\langle W^{\alpha}, \Theta, (\Pi_1, \Pi_2) \rangle,$$

on $\{H, T\}$, of rank α.

For the construction of W^{α} we need the following definition. A *consciousness record of length* α (or a *record of length* α, for short) is a sequence $r = (r^{\beta})_{\beta < \alpha}$ such that:

(3.1) $r \in \{S, D\}^{\alpha}$, i.e., r is a sequence of the letters "S" and "D."

(3.2) For each limit ordinal $\lambda < \alpha$, there exists an ordinal $\gamma < \lambda$ such that $r^{\beta} = D$ for all ordinals β that satisfy $\gamma \leq \beta < \gamma$. That is, for each limit ordinal λ there is a smaller ordinal from which on there are only Ds in the sequence up to (not including) λ.

We define the space W^{α} by

$$W^{\alpha} = \{(w_0, w_1, w_2) \mid w_0 \in \{H, T\}, w_1 \text{ and } w_2 \text{ are records of length } \alpha\}.$$

The records w_1 and w_2 are called the records of players 1 and 2, respectively. The function Θ: $W^{\alpha} \to \{H, T\}$ is defined by $\Theta(w) = w_0$. Note that by condition (3.2) in the definition of a record, the cardinality of W^{α}, for infinite α, is the same as the cardinality of α.

The formal description of the players' partitions, which we give later, is motivated by the following "story." The letters S and D stand for "sober" and "drunk." The players are hopelessly addicted to alcohol. No matter how much they try to avoid it, they finally fail and give up as embodied by condition (3.2) in the definition of a record.

A player's consciousness record tells what he knows about nature and the other player's record, as follows. Sobriety of a player i at 0 ($w_i^0 = S$) means that he knows whether the state of nature is H or T. When he is drunk at 0 he cannot distinguish between them. Being sober at 1 ($w_i^1 = S$) means that the player knows the record of the other player at 0. That is, he can tell whether his opponent is sober or drunk at 0. Thus, i can tell whether j knows the state of nature or not. This makes sense even if he himself cannot tell the true state of nature, in case he is drunk at 0. Similar meaning is given to sobriety in all non-limit ordinals (which are the ordinals of the form $\beta + 1$): $w_i^{\beta+1} = S$ means that i can tell whether w_j^{β} is S or D, while $w_i^{\beta+1} = D$ means that i cannot tell it.

The meaning of sobriety for limit ordinals must be different, as such ordinals do not have immediate predecessor. To explain the meaning of being sober at limit ordinals we introduce the following definition.

By condition (3.2) in the definition of a record, for each record r and a limit ordinal λ there exists a minimal ordinal, smaller than λ, from which on the elements of r are constantly D up to (not including) λ. We denote this ordinal by $m^\lambda(r)$. The *λ-parity* of the record r is the parity[2] of $m^\lambda(r)$.

Being sober at a limit ordinal λ means that the player knows the λ-parity of his opponent's record. The crucial point is that i's record up to (not including) the limit ordinal λ never enables him to perceive whether the other player is λ-even or λ-odd. This is so because i is always drunk from $m^\lambda(w_i)$ on, and therefore he cannot exclude the possibility that the other player stood the temptation longer than he did, and fell drunk (up to λ) only at some later ordinal $\gamma > m^\lambda(w_i)$, where γ may be even as well as odd. Therefore, becoming sober again in ordinal λ enables the player to exclude some records of the other player that he cannot exclude when he is drunk there. The state of consciousness of the players at limit ordinals λ determines, therefore, if they can resolve this uncertainty concerning previous ordinals. The state of consciousness in λ becomes itself the subject for uncertainty in later stages and so on.

To complete the informal description of players' information in W^α we add that each player always knows his own record.

This informal description of agents' knowledge in W^α results in the following definition of the players' partitions. For a state $w = (w_0, w_1, w_2)$ in W^α the element of i's partition, $\Pi_i(w)$, which contains it, is defined by

$$\Pi_i(w) = \Big\{ \big(\overline{w}_0, \overline{w}_1, \overline{w}_2\big) \in W_\alpha : \\
\begin{array}{ll}
\text{(i)} & \overline{w}_i = w_i \\
\text{(ii)} & w_i^0 = S \;\Rightarrow\; \overline{w}_0 = w_0 \\
\text{(iii)} & \text{for all ordinals } \beta \text{ such that } \beta + 1 < \alpha: \\
& w_i^{\beta+1} = S \;\Rightarrow\; \overline{w}_j^{\,\beta} = w_j^\beta \\
\text{(iv)} & \text{for every limit ordinal } \lambda < \alpha: \\
& w_i^\lambda = S \;\Rightarrow\; \overline{w}_j \text{ and } w_j \text{ have the same } \lambda\text{-parity} \Big\}.
\end{array}$$

It is easy to verify that the sets $\Pi_i(w)$ form a partition of W^α, and that $\Pi_i(w)$ is indeed the element of that partition that contains w.

[2] An infinite ordinal β is said to be odd or even according to whether the unique finite n such that $\beta = \lambda + n$, where λ is a limit ordinal, is odd or even.

We show in the following theorem that the intuitive explanation we gave to sobriety and drunkness holds formally, in terms of the knowledge operators K_i that are induced by the partitions.

With some abuse of notation, we denote by H the event that the state of nature is H, that is, $H = \{w | w_0 = H\}$. We denote by T the complement of H. For every $\beta < \alpha$ let $S_i^\beta = \{w | w_i^\beta = S\}$ be the event that i is sober in ordinal β, and let D_i^β be its complement—the event that he is drunk in β. Finally, let E_i^λ and O_i^λ be the complementary events that player i's λ-parity is even and odd, respectively, i.e., $E_i^\lambda = \{w | w_i \text{ is } \lambda\text{-even}\}$ and O_i^λ is its complement.

THEOREM 3.1.[3] *In W^α, for each player $i \in I$,*

(3.3) $S_i^0 = K_i(H) \cup K_i(T)$,

(3.4) $S_i^{\beta+1} = K_i(S_j^\beta) \cup K_i(D_j^\beta)$ *for all ordinals β with $\beta + 1 < \alpha$,*

(3.5) $S_i^\lambda = K_i(E_j^\lambda) \cup K_i(O_j^\lambda)$ *for all limit ordinals $\lambda < \alpha$.*

To prove Theorem 3.1 we use Lemma 3.2 below. We first introduce some notation. For a state w, player i, and a letter L from $\{S, D\}$, denote by

$$\left(w | w_i^\beta \to L\right)$$

the state obtained from w by substituting L for w_i^β. Similarly, the state obtained from w by substituting in it the state of nature L (from $\{H, T\}$) is denoted by

$$\left(w | w_0 \to L\right).$$

For a state w, an ordinal $\beta < \alpha$, and player i, we denote by $w_i^{<\beta}$ the β-initial segment of w_i, i.e., the sequence $(w_i^\gamma)_{\gamma < \beta}$, and by $w^{<\beta}$ the triplet $(w_0, w_1^{<\beta}, w_2^{<\beta})$. Note that $w^{<\beta} \in W^\beta$. By $w_i^{\geq \beta}$ we denote the sequence $(w_i^\gamma)_{\gamma \geq \beta}$. Thus $(v^{<\beta}, w^{\geq \beta}) \in W^\alpha$ is the concatenation of the initial segment of v with the terminal segment of w.

LEMMA 3.2. *Let $v, w \in W^\alpha$, where for some $\gamma < \alpha$ we have $v^{<\gamma+1} \in \Pi_i(w^{<\gamma+1})$ in the space $W^{\gamma+1}$. Then, in W^α there exists a state $u \in \Pi^i(\omega)$ with the same γ-initial segment as that of v, i.e., $u^{<\gamma} = v^{<\gamma}$.*

Proof. The state

$$\bar{u} = \left(v^{<\gamma}, w^{\geq \gamma}\right)$$

is "almost" in $\Pi_i(w)$; it satisfies conditions (i), (ii), and (iii) in the definition of $\Pi_i(w)$, as may be easily verified. The only problem may arise with

[3] In Hart *et al.* (1996) the "Knowing Whether" operator J on events E in a partition space is defined by $J(E) = K(E) \cup K(E^c)$. Thus, Theorem 3.2 asserts that being Sober at a given stage means to know whether an appropriate event takes place. A variant of W^ω (for the first infinite ordinal ω) was the main tool in the above paper.

condition (iv): If there is a limit ordinal $\gamma < \lambda < \alpha$ in which player i is sober in w (i.e., $w_i^\lambda = S$), and the λ-parity of the other player j in $\bar{u}$ (i.e., the λ-parity of $\bar{u}_j$) is different than the λ-parity of w_j, then player i excludes $\bar{u}$, that is $\bar{u} \notin \Pi^i(w)$.

By changing one coordinate in $\bar{u}$ we overcome this difficulty. Let λ be the minimal limit ordinal exceeding γ. Choose an ordinal δ which satisfies

$$\max\left(m^\lambda(w_i), m^\lambda(w_j)\right) < \delta < \lambda$$

such that the parity of $\delta + 1$ is the same as the λ-parity of w_j. Define

$$u = \left(\bar{u}|w_j^\delta \to S\right) = \left((v^{<\gamma}, w^{\geq\gamma})|w_j^\delta \to S\right).$$

Observe, that $u_i^{\delta+1} = w_i^{\delta+1} = D$, so u continues to satisfy condition (iii) in the definition of $\Pi_i(w)$. Furthermore, $m^\lambda(u_j) = \delta + 1$ and thus the λ-parity of u_j is, by the definition of δ, the same as that of w_j. Since λ is the smallest limit ordinal exceeding γ, the λ'-parity of u_j is the same as that of w_j for all limit ordinals $\gamma < \lambda' < \alpha$. We conclude that u satisfies also condition (iv) in the definition of $\Pi_i(w)$, as required. Q.E.D.

Proof of Theorem 3.1. The inclusions of the left-hand sides of (3.3), (3.4), and (3.5) in the corresponding right-hand sides follow directly from articles (ii), (iii), and (iv), respectively, in the definition of Π_i. It remains to prove the reverse inclusions.

To prove it in (3.3), observe that if $w \notin S_i^0$, that is, $w \in D_i^0$, then

$$(w|w_0 \to H), (w|w_0 \to T) \in \Pi_i(w),$$

and hence $w \notin K_i(H) \cup K_i(T)$.

We now prove the inclusion of the right-hand side of (3.4) in its left-hand side. Suppose $w \notin S_i^{\beta+1}$. Define the states $s, d \in W^\alpha$ by

$$s = \left(w|w_j^\beta \to S\right)$$

$$d = \left(w|w_j^\beta \to D\right).$$

Observe that in $W^{\beta+2}$, both $s^{<\beta+2}, d^{<\beta+2} \in \Pi_i(w^{<\beta+2})$. By Lemma 3.2 there exist states $s', d' \in \Pi_i(w)$ with the same $(\beta+1)$-initial segment as s, d, respectively. In particular, $s_j'^\beta = S$ and $d_j'^\beta = D$. Consequently, $w \notin K_i(S_j^\beta) \cup K_i(D_j^\beta)$, as required.

The proof of the inclusion from right to left in (3.5) is similar. Suppose $w \notin S_\lambda^i$. Choose $\beta < \lambda$ that has the same parity as the λ-parity of w_j and such that

$$\beta \geq \max\left(m^\lambda(w_i), m^\lambda(w_j)\right).$$

From β on, up to (not including) λ, both players are constantly drunk in w. Define the state $v \in W^{\alpha}$ by

$$v = \left(w | w_j^{\beta} \to S\right).$$

As $m^{\lambda}(v) = \beta + 1$ it follows that w_j and v_j have opposite λ-parities. However, since $w_i^{\lambda} = D$, in $W^{\lambda+1}$, both

$$w^{<\lambda+1}, v^{<\lambda+1} \in \Pi_i\left(w^{<\lambda+1}\right).$$

By Lemma 3.2 there exists $u \in \Pi_i(w)$ with the same λ-initial segment as that of v. In particular, the λ-parity of u^j is opposite to that of w_j. Since $w \in \Pi_i(w)$ as well, we conclude that $w \notin K_i(E_j^{\lambda}) \cup K_i(O_j^{\lambda})$, as required.

Q.E.D.

Proof of Theorem 2.5. Consider the partitions P^{β} associated with W^{α} as described in the previous section. By definition, P^0 is the partition of W_{α} into the two sets H and T. We prove that, for all β with $1 \leq \beta \leq \alpha$, the partition P^{β} satisfies for each $w \in W^{\alpha}$, $P^{\beta}(w) = \{\overline{w} \mid \overline{w}^{<\beta} = w^{<\beta}\}$. That is, P^{β} separates the states of W^{α} by the state of nature and the coordinates of both records for ordinals that are smaller than β. This implies that, for all $\beta' < \beta \leq \alpha$, $P^{\beta'} \neq P^{\beta}$ and $P^{\alpha}(w) = \{w\}$ which shows that α is the rank of W^{α}.

The proof is by induction on β. For $\beta = 1$ we have, by Theorem 3.1, $S_i^0 = K_i(H) \cup K_i(D)$. Thus P^1 is generated by H, T (which are the elements of P^0), and S_i^0 for $i = 1, 2$, which is what we need to show.

Suppose the claim was proved for all ordinals smaller than β. If β is a limit ordinal, then the claim is true for β, since in this case P^{β} is generated by $\bigcup_{\gamma < \beta} P^{\gamma}$, and the latter consists, by the induction hypothesis, of all sets of the form $\{\overline{w} | \overline{w}^{<\gamma} = w^{<\gamma}\}$ for all $\gamma < \beta$.

If β is not a limit ordinal, then $\beta = \gamma + 1$, where $1 < \gamma + 1 < \alpha$. Now, either $\gamma = \delta + 1$, and then, by Theorem 3.1, $S_i^{\gamma} = K_i(S_j^{\delta}) \cup K_i(D_j^{\delta})$, or else γ is a limit ordinal, in which case, by Theorem 3.1, $S_i^{\gamma} = K_i(E_j^{\gamma}) \cup K_i(O_j^{\gamma})$. In either case the set S_i^{γ} is obtained by applying knowledge operators to sets which are, by our induction hypothesis, γ-events. Therefore the sets S_i^{γ}, for both players, are $(\gamma+1)$-events.

By the induction hypothesis it follows that $P^{\gamma+1}$ separates the coordinates of levels smaller than $\gamma + 1$. It remains to prove the converse, namely, that if $v, w \in W_{\alpha}$ have the same $(\gamma+1)$-initial segment, i.e.,

$$v^{<\gamma+1} = w^{<\gamma+1} \tag{3.6}$$

then v and w are not separated by $P^{\gamma+1}$.

For this we have to show that events obtained by applying knowledge operators to γ-order events, do not separate v and w. Let G be a γ-order event and suppose that $w \in K_i(G)$. We need to show that $v \in K_i(G)$. Equivalently we are assuming that

$$\Pi_i(w) \subseteq G, \tag{3.7}$$

and we want to show that

$$\Pi_i(v) \subseteq G. \tag{3.8}$$

Suppose then that

$$\bar{v} \in \Pi_i(v). \tag{3.9}$$

We have to prove that $\bar{v} \in G$. Now, by (3.9) and (3.6) we have, in $W^{\gamma+1}$,

$$\bar{v}^{<\gamma+1} \in \Pi_i(v^{<\gamma+1}) = \Pi_i(w^{<\gamma+1}).$$

Therefore, by Lemma 3.2 there exists a state

$$u \in \Pi_i(w) \tag{3.10}$$

with the same γ-initial segment as that of $\bar{v}$, that is,

$$u^{<\gamma} = \bar{v}^{<\gamma}. \tag{3.11}$$

By (3.10) and (3.7),

$$u \in G. \tag{3.12}$$

Since G is a γ-order event, the induction hypothesis says that G does not separate states that have the same γ-initial segment. Therefore, by (3.11) and (3.12) we conclude that $\bar{v} \in G$, which proves (3.8), as required.
Q.E.D.

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