An axiomatic approach to the allocation of a fixed cost through prices

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We study the allocation of fixed costs to the outputs of a multiproduct firm. First we allocate short-run fixed costs through A-S prices which allocate the long-run costs. Long-run cost functions do not generally contain a fixed cost component. We show what part of the A-S prices associated with the long-run cost is allocated to the fixed cost and what part is allocated to the variable cost of the short-run costs. Second, in those cases in which the fixed costs must be allocated directly, we alter the axioms characterizing A-S prices slightly to accommodate cost functions which have a fixed cost component and derive an allocation mechanism characterized by these axioms for cost functions including those with a fixed cost component.

1. Introduction

■ An axiomatic approach to pricing which allocates the full cost of production of a multiproduct firm to all the outputs was proposed by Billera and Heath (1982) and Mirman and Tauman (1982). The axiomatic approach results in a price mechanism which associates a vector of cost sharing prices to each admissible cost function and output vector. These prices, known as Aumann-Shapley (A-S) prices, may be used by regulated monopolies as well as public or quasi-public agencies to allocate the joint costs of production to each of the individual products. To compute A-S prices only the cost structure and the output vector must be known and not the demand functions. These prices avoid the cross subsidization of commodities which would be necessary if the price mechanism were also to depend on demand considerations. Moreover, although the A-S price mechanism does not depend on demand functions, there will exist a vector of quantities and corresponding A-S prices for which markets clear (Mirman and Tauman, 1982).

One major difficulty with this approach is that the set of axioms on which the A-S price mechanism is based is applicable only to cost functions which have no fixed cost component. In fact, if the axioms which define A-S prices are applied to the class of cost functions including those having a fixed cost component, a contradiction is implied. This is not a major obstacle when long-run production functions are considered since in general such functions do not have a fixed cost component. But the implicit optimal short-run technology for producing a particular vector of outputs will, in general, have a fixed cost

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We would like to thank Israel Zang for his helpful comments and the referees and editors for their encouragement in reformulating the results of the article. Finally support from the NSF through grant SES-81-06207 is gratefully acknowledged.

component, which in some circumstances must be allocated among the outputs. Moreover, there are even some long-run cost functions which do have legitimate fixed cost components, e.g., putting the dust of the moon on earth, which must be allocated.

Our first purpose in this article is to find, for a given set of products, a price mechanism which assigns to *each* vector α of outputs, prices which share the *minimal* cost of producing α. For this purpose it is necessary to use, for each vector of outputs, the minimal cost of producing this vector, i.e., to use the long-run cost function. When the long-run cost function does not include a fixed cost component, the A-S price mechanism can be applied to allocate these costs. On the other hand, it is the short-run cost function which is actually used by the firm to calculate the cost of producing α. If the vector α is optimally produced by the firm, this short-run cost function, which, in general, contains a fixed cost component, coincides with the long-run cost function at α (or perhaps a neighborhood of α). Hence the A-S prices which allocate long-run costs also indirectly allocate the short-run costs of α. We shall show what part of the A-S prices associated with the long-run technology at α is allocated to the fixed cost and what part is allocated to the variable cost of the short-run technology at the output vector α.

There are other situations which require a different approach. The first occurs when the long-run cost function has a fixed cost component and thus A-S prices cannot be applied. The other occurs when the producer is not able to use the optimal short-run technology for the actual production level. In this case the short-run cost deviates from the long-run cost, and therefore the allocation of the long-run cost is irrelevant. Thus, the A-S prices which allocate long-run costs are not applicable. Moreover, A-S prices cannot be used to allocate the short-run costs directly since, in general, the short-run cost function contains a fixed cost component. In both of these cases we shall allocate the fixed cost directly by extending the A-S price mechanism to cost functions having a fixed cost component. A change in the axioms characterizing the A-S price mechanism is required to share the fixed cost as well as the variable cost. This change is necessary, since when a fixed cost is present, the consistency and additivity axioms cannot, in general, be satisfied simultaneously. Since the consistency axiom, i.e., two goods having the same impact on costs should have the same price, is the more natural, a change in the additivity axiom must be made to find a reasonable price mechanism on the set of cost functions having a fixed cost component. Fortunately, it is not necessary to do away with the additivity axiom altogether. It is enough to limit the way in which the cost function is allocated between additive components of the variable cost. It turns out that it is not necessary to specify how the fixed cost is split between variable cost components, but rather that there exists a way of splitting the fixed costs. This is analogous to an additivity requirement on the fixed cost. Changing the additivity requirement in this way yields the result that the fixed cost is allocated proportionally to the A-S prices of the variable cost.

2. The axiomatic approach to cost allocation and A-S prices

■ Let F^m be the family of cost functions of m products (i.e., the family of functions defined on E^m_+), and let $F = \bigcup_{m=1}^{\infty} F^m$. We denote by S^m the set $F^m \times E^m_+$ and $S = \bigcup_{m=1}^{\infty} S^m$. The families F^m_+, F_+, S^m_+, and S_+ are defined similarly for nonnegative cost functions. An element of S is a pair (F, α), where F is a cost function of m products, for some m, and α is an m-vector of these products. We shall determine prices for those products when α is produced and the cost function is F. Let T be a subset of S and let $T^m = T \cap S^m$. A *price mechanism for* T is a function P which assigns to each pair (F, α) in T^m, a vector of prices

$$P(F, \alpha) = (P_1(F, \alpha), \dots, P_m(F, \alpha)).$$

Let us now consider those price mechanisms which satisfy the following axioms.

Axiom 1. Cost sharing. The prices determined by P for the pair (F, α) cover the cost of production $F(\alpha)$, i.e., for each $(F, \alpha) \in T$,

$$\alpha P(F, \alpha) = F(\alpha).$$

Axiom 2. Rescaling. If the scales of measurement of the commodities are changed, then the prices determined by P are changed accordingly. To state the axiom formally, let $\lambda * \alpha = (\lambda_1\alpha_1, \ldots, \lambda_m\alpha_m)$, for $\lambda, \alpha \in E^m$. Assume that $\lambda \in E^m_{++}$, (G, α), $(F, \lambda * \alpha) \in T^m$, and

$$G(x) = F(\lambda * x),$$

i.e., the cost functions G and F differ only in the scales of the commodities (the λ_i being the rescaling factors). Then,

$$P(G, \alpha) = \lambda * P(F, \lambda * \alpha).$$

The next axiom requires that each unit of the "same good" have the same price. The question is: what is the criterion for being the "same good"? Since the price mechanism yields prices which depend on the cost function and not on demand functions, being the "same good" will mean playing the same role in the cost function. As an illustration, suppose that red and blue cars are produced. The cost function is a two-variable function $F(x_1, x_2)$, where x_1 and x_2 are the quantities of red and blue cars, respectively. But, in fact, the cost of producing a red car is the same as the cost of producing a blue car. This can be formulated as follows: There is a one-variable function G for which $G(x)$ is the cost of producing the total of x cars (red ones, blue ones, or both) and

$$F(x_1, x_2) = G(x_1 + x_2).$$

In this case the axiom asserts that the price of a blue car is the same as the price of a red car, i.e.,

$$P_1(F, (\alpha_1, \alpha_2)) = P_2(F, (\alpha_1, \alpha_2)).$$

Axiom 3. Consistency. Let

$$F(x_1, \ldots, x_m) = G(\sum_{j=1}^{m} x_j)$$

for each $x \in E^m_+$. If (F, α) and $(G, \sum_{j=1}^{m} \alpha_j)$ are in T^m and T^1, respectively, then for each i, $1 \le i \le m$,

$$P_i(F, \alpha) = P(G, \sum_{j=1}^{m} \alpha_j).$$

Let $\alpha \in E^m_+$, and denote by C_α the set of all points in E^m_+ which do not exceed α. Namely,

$$C_\alpha = \{x \in E^m_+ | x_j \leqq \alpha_j, \quad j = 1, \ldots, m\}.$$

The next axiom asserts that if F increases at least as rapidly as G on C_α, then the prices determined for (F, α) are at least as high as those determined for (G, α).

Axiom 4. Positivity. Let (F, α) and (G, α) be two elements in T such that $F(0) \geq G(0)$. If $F - G$ is nondecreasing on C_α, then

$$P(F, \alpha) \geq P(G, \alpha).$$

The last axiom requires that whenever a cost function can be broken into two components F and G (e.g., management and production), then calculating the prices determined by the cost function at a point α can be accomplished by adding the prices determined by F and G, respectively, at this point.

Axiom 5. Additivity. If (F, α), (G, α), and $(F + G, \alpha)$ are elements of T,

$$P(F + G, \alpha) = P(F, \alpha) + P(G, \alpha).$$

The existence and uniqueness of a price mechanism which obeys these five axioms can be proved for continuously differentiable cost functions with no fixed components (Mirman and Tauman, 1982; Samet and Tauman, 1982). This price mechanism is called the Aumann-Shapley (A-S) price mechanism. To state this result rigorously, let us denote by T_0 the subset of S which contains all pairs of the form (F, α), where F is continuously differentiable with $F(0) = 0$ and $\alpha \in E^m_{++}$.

Theorem 1. There exists one and only one price mechanism $\hat{P}(\cdot, \cdot)$ for T_0 which obeys axioms 1–5. This is the Aumann-Shapley price mechanism given by

$$\hat{P}_i(F, \alpha) = \int_0^1 \frac{\partial F}{\partial x_i}(t\alpha)dt, \tag{1}$$

for each $(F, \alpha) \in T_0$.

3. Allocation of fixed costs using the long-run cost function

■ The axiomatic approach to the determination of a price mechanism is applied only to the family T_0 of continuous cost function G satisfying $G(0) = 0$. The restriction of $G(0) = 0$ is plausible when G is the long-run cost function. The long-run cost function may be thought of as the envelope of "short-run" cost functions as is illustrated in Figure 1 in the one-dimensional case. The efficient portion of a short-run technology is that which coincides with the long-run technology. All other points are not efficiently produced by this short-run technology.

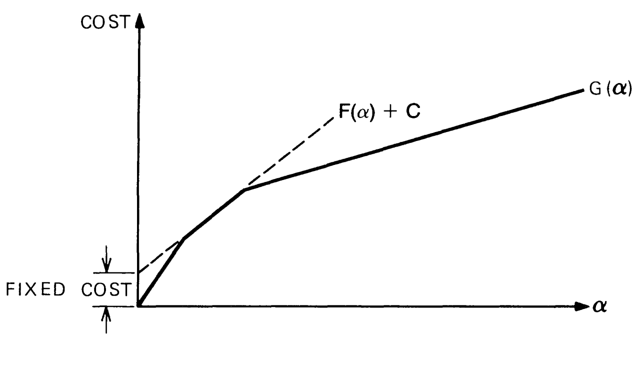
In general, the short-run cost function will have a fixed cost component. However, using the A-S prices for the long-run cost function G to allocate the minimal cost $G(\alpha)$ of producing the vector α, an allocation of the fixed cost associated with the efficient (short-run) technology of producing α can actually be determined. To do this consider the production of a specific output vector α, using the optimal (short-run) technology, T for producing α. Notice that with this technology other levels of outputs could be produced, however not necessarily efficiently. Let us denote by C the fixed cost of this technology and by $F(x)$ the variable cost of producing x with technology T. Clearly at the point α,

$$G(\alpha) = F(\alpha) + C.$$

The price mechanism determines prices for α using the (long-run) cost function G. Since these prices cover the cost $G(\alpha)$, they cover both the variable cost $F(\alpha)$ and the fixed cost C of producing α with T. One may ask what part of these prices, determined by the pair (G, α), covers the variable cost $F(\alpha)$ and what part covers the fixed cost C.

FIGURE 1

LONG–RUN COSTS AS AN ENVELOPE OF SHORT–RUN COSTS



Using G to determine the cost sharing prices $p_1, \ldots, p_m$ when α is produced, we find from (1) that

$$p_i = \int_0^1 \frac{\partial G}{\partial x_i}(t\alpha)dt.$$

Since $G(\alpha) = F(\alpha) + C$ at α, these prices cover both the variable cost plus the fixed cost of the short-run cost function (i.e., $\sum_{i=1}^{m} p_i\alpha_i = G(\alpha) = F(\alpha) + C$). It is possible to calculate another set of prices $\hat{p}$ due only to the variable cost function F, i.e.,

$$\hat{p}_i = \int_0^1 \frac{\partial F}{\partial x_i}(t\alpha)dt, \qquad i = 1, \ldots, m. \tag{2}$$

These prices cover the variable cost $F(\alpha)$ of producing α (i.e., $\sum_{i=1}^{m} \hat{p}_i\alpha_i = F(\alpha)$). Obviously,

$$\sum_{i=1}^{m} (p_i - \hat{p}_i)\alpha_i = G(\alpha) - F(\alpha) = C,$$

which means that $p_i - \hat{p}_i$ is that part of the price p_i which may be thought of as covering the fixed cost C.

We interpret this in a specific case in which the long-run technology is piecewise linear and all short-run technologies are linear. Let T_1 be the most efficient technology for producing all output vectors $t\alpha$, $t_0 \leq t \leq t_1$, and T_2 the most efficient technology for producing output vectors $t\alpha$, $t_1 \leq t \leq t_2, \ldots,$ and finally T_k the most efficient technology for producing $t\alpha$, $t_{k-1} \leq t \leq t_k$, where $t_0 = 0$ and $t_k = 1$. Assume that the technology T_j implies a linear cost function H_j of the form

$$H_j(x_1, \ldots, x_m) = F_j(x_1, \ldots, x_m) + C_j = \sum_{i=1}^{m} a_i^j x_i + C_j,$$

where $C_1 = 0$. Let G be the long-run cost function of producing these goods. The A-S price of the ith good determined by (G, α) is given by

$$p_i = \sum_{j=0}^{k-1} (t_{j+1} - t_j)\frac{\partial F_{j+1}}{\partial x_i}. \tag{3}$$

On the other hand, the part $\hat{p}_i$ of the price p_i used to cover the variable cost is the marginal cost of the ith good under T_k, the best technology to produce α, i.e.,

$$\hat{p}_i = \frac{\partial F_k}{\partial x_i} = a_i^k.$$

The difference

$$p_i - \hat{p}_i = \sum_{j=0}^{k-1} (t_{j+1} - t_j)\frac{\partial F_{j+1}}{\partial x_i} - \frac{\partial F_k}{\partial x_i}$$

is the part of p_i which covers the fixed cost C_k of the technology T_k. Note that the part $p_i - \hat{p}_i$ of the price p_i which is used to cover the fixed cost C_k of producing α under T_k depends on the difference between the variable cost function F_k of the technology T_k and the long-run cost function G, for output levels which cannot be efficiently produced by the short-run technology T_k. This is a property of general technologies as well.[1]

[1] To show this fact let t_{k-1} be the minimal t for which the technology T_k is the most efficient for producing the output level $t\alpha$. Then

$$p_i - \hat{p}_i = \left[\int_0^1 \frac{\partial G}{\partial x_i}(t\alpha)dt - \int_0^1 \frac{\partial F_k}{\partial x_i}(t\alpha)dt\right] = \int_0^{t_{k-1}} \frac{\partial G}{\partial x_i}(t\alpha)dt - \int_0^{t_{k-1}} \frac{\partial F_k}{\partial x_i}(t\alpha)dt$$

$$= t_{k-1}\left[\int_0^1 \frac{\partial G}{\partial x_i}(tt_{k-1}\alpha) - \frac{\partial F_k}{\partial x_i}(tt_{k-1}\alpha)\right]dt.$$

Example. Consider the production of two goods. Assume that there are only two available technologies T_1 and T_2 for producing these goods. Let H_1 and H_2 be the cost functions associated with T_1 and T_2, respectively, and assume that

$$H_1(x_1, x_2) = 3x_1 + 2x_2 = F_1(x_1, x_2)$$

and

$$H_2(x_1, x_2) = x_1 + x_2 + 9 = F_2(x_1, x_2) + 9.$$

Since $H_1 \leq H_2$ whenever $2x_1 + x_2 \leq 9$, the minimum cost $G(x_1, x_2)$ of producing the output level (x_1, x_2) is given by

$$G(x_1, x_2) = \begin{cases} 3x_1 + 2x_2, & 2x_1 + x_2 \leq 9 \\ x_1 + x_2 + 9, & \text{otherwise.} \end{cases}$$

Consider now the production of six units of each commodity, namely $\alpha = (6, 6)$. The total production cost is then

$$G(6, 6) = 6 + 6 + 9 = 21.$$

Clearly as Figure 2 illustrates,

$$t_0 = 0, \qquad t_1 = \frac{1}{2}, \qquad \text{and} \qquad t_2 = 1.$$

Thus by (3) the A-S cost sharing prices for (G, α) are

$$p_1 = (t_1 - t_0)\frac{\partial F_1}{\partial x_1} + (t_2 - t_1)\frac{\partial F_2}{\partial x_1} = \frac{1}{2}\cdot 3 + \frac{1}{2}\cdot 1 = 2$$

$$p_2 = (t_1 - t_0)\frac{\partial F_1}{\partial x_2} + (t_2 - t_1)\frac{\partial F_2}{\partial x_2} = \frac{1}{2}\cdot 2 + \frac{1}{2}\cdot 1 = 1.5.$$

Also

$$\hat{p}_1 = \frac{\partial F_2}{\partial x_1} = 1$$

and

$$\hat{p}_2 = \frac{\partial F_2}{\partial x_2} = 1.$$

Hence the parts of p_1 and p_2 that are used to cover the fixed cost of nine units are

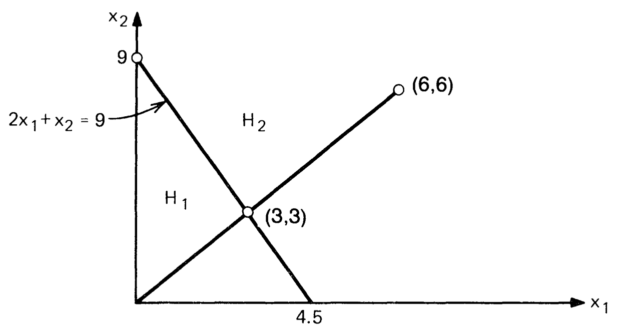
$$p_1 - \hat{p}_1 = 2 - 1 = 1$$

$$p_2 - \hat{p}_2 = 1.5 - 1 = .5.$$

Thus, while the price vector (2, 1.5) covers the total cost of producing the output

FIGURE 2

AN EXAMPLE TO COMPUTE AS PRICES FOR A PIECEWISE LINEAR FUNCTION



level (6, 6), half of the price of the first good and one-third of the second good may be attributed to covering the fixed cost, i.e., $\left[1(6) + \frac{1}{2}(6) = 9\right]$.

Finally, it turns out by (3) that the A-S prices have the following interpretation. Charge each unit of the first $t_1\alpha_i$ units of the ith good the marginal cost of producing this unit, namely $\frac{\partial F_1}{\partial x_i}$, then charge each unit of the next $(t_2 - t_1)\alpha_i$ units its marginal cost, namely $\frac{\partial F_2}{\partial x_i}$, etc. Thus, the α_i units of the ith good are charged, altogether,

$$t_1\alpha_i \frac{\partial F_1}{\partial x_i} + (t_2 - t_1)\alpha_i \frac{\partial F_2}{\partial x_i} + \cdots + (1 - t_{k-1})\alpha_i \frac{\partial F_k}{\partial x_i},$$

which, by (3), equals $\alpha_i p_i$. Therefore, the charge per unit is p_i, the A-S price of the ith good.

4. An extension of the axiomatic approach

■ We turn now to the question of determining cost-sharing prices for cost functions which include a fixed cost component. Such is the case, for example, when the actual demand α deviates from the expected demand and the short-run technology which is optimal for the expected demand but not for α is used to produce α. In this case the long-run cost function is irrelevant for the allocation of the cost of producing α. Consequently, the fixed cost must be allocated directly from the short-run cost function rather than through the long-run cost function as was done in Section 3. Also, in some cases even the long-run cost function that gives the minimal cost of producing any output level has a fixed cost component (for example, consider the product which is the moon's dust on earth). Unfortunately, the five axioms as stated in Section 2 are inconsistent with respect to such a wide class of cost functions. We propose in the sequel a natural change in the additivity axiom which enables us to derive the existence and uniqueness of a price mechanism on this family of cost functions. Naturally, for cost functions which have no fixed cost component, the prices determined by this price mechanism coincide with A-S prices. First, we shall point out why the additivity axiom is the natural candidate for a change.

Consider the set T_1 which contains all pairs of S of the form $(F + C, \alpha)$, where $(F, \alpha) \in T_0$, $F \in F_+$ (i.e., F is a nonnegative function), $F(\alpha) \neq 0$, $\alpha \neq 0$, and C is a nonnegative number.

To verify that the above five axioms are inconsistent on T_1 let us consider the additivity axiom. By breaking a cost function into additive components we can, using the additivity axiom, compute the prices for the original function by computing the prices for each of the components. This procedure is immediately applicable to cost functions without a fixed cost because, in this case, the result is independent of the decomposition. When there is a fixed cost, different decompositions of the cost function will in general yield different results. In fact, over T_1 the consistency and additivity axioms are incompatible as is shown by the following example.

Consider, for example, the single cost function $x_1 + x_2 + C$ and assume $\alpha = (1, 1)$. By the consistency axiom the prices of the two commodities should be the same. But now let us decompose this cost function into x_1 and $x_2 + C$. Applying additivity and cost sharing, we find that the price of the first commodity covers only the variable cost of producing one unit which is 1. The price of the second commodity $1 + C$ covers all of the fixed cost. Of course, this contradicts the consistency axiom. Thus, we have to give up one of the two: additivity or consistency. It seems that consistency is the appropriate axiom to keep since, as the above example shows, there is no reason to differentiate between the commodities x_1 and x_2.

The additivity axiom has some practical importance because it enables us to reduce the computation of prices for complex cost functions into simple components. This reduction is important both from the practical point of view and from the theoretical tractability of the prices. Hence it would be beneficial to retain additivity in some form.

An alternative approach is to keep additivity on the variable cost, but allocate the fixed cost independently of the variable cost. In particular, the following two price mechanisms which allocate the fixed cost independently of its variable part fail to have the desired properties. Both price mechanisms allocate the variable part through A-S prices and allocate the fixed part as follows.

i. $$Q(C, \alpha) = \left(\frac{C}{\sum \alpha_i}, \ldots, \frac{C}{\sum \alpha_i}\right), \qquad \alpha = (\alpha_1, \ldots, \alpha_m)$$

ii. $$Q(C, \alpha) = \left(\frac{C}{m\alpha_1}, \ldots, \frac{C}{m\alpha_m}\right).$$

For the first price mechanism the fixed cost is allocated equally among *all units* of the various commodities. The second one divides the fixed cost equally among the various commodities, e.g., the ith commodity is charged in total $\frac{C}{m}$ and thus its price is $\frac{C}{m\alpha_i}$.

The first price mechanism depends strongly on the definition of a unit of the commodity; changing the scale will not yield an appropriate change in price, i.e., it violates the rescaling axiom.

The second price mechanism violates the consistency axiom; by splitting a commodity into two irrelevant classifications one can manipulate the prices. For example, consider again the production of red cars and blue cars in amounts α_1 and α_2, where $\alpha_1 \neq \alpha_2$. By the consistency axiom the price of red cars and blue cars must be the same. But the second mechanism treats these two types of cars unequally by imposing a higher proportion of the fixed costs on the cars which are produced in the smaller amount.

In fact, it can easily be shown that *each* price mechanism which allocates the fixed cost independently of its variable cost violates either the rescaling axiom or the consistency axiom.

There is no need, however, to discard additivity altogether, since it turns out that only a small change in the additivity axiom is necessary to make it compatible with the other axioms. The idea is to limit the way in which the fixed cost is allocated between additive components of the variable cost. In the example $x_1 + x_2 + C$, if C were split equally between x_1 and x_2, then additivity would be consistent with the other axioms. However, the question remains of how to split C in general. It turns out that it is not necessary to specify, *a priori, how* we split the fixed cost between different variable cost components. It is sufficient to require only that there *exists* a way to do it! This is spelled out in Axioms 5* and 6*.

*Axiom 5**. Let F be a function in F_+^m and let C be a nonnegative number. Then, for each G in F_+^m such that $G \leq F$ there is a nonnegative number C_G such that if $F = \sum_{i=1}^{n} G_i$ and $G_i \in F_+^m$, then $C = \sum_{i=1}^{n} C_{G_i}$ and

$$P(F + C, \alpha) = \sum_{i=1}^{n} P(G_i + C_{G_i}, \alpha).$$

The second axiom asserts that the part C_{G_i} of the fixed cost C that is associated with the component G_i should be at least as large as C_{G_j} whenever the part $G_i(\alpha)$ in the total variable cost $F(\alpha)$ is at least as large as $G_j(\alpha)$.

*Axiom 6**. Let F, $G_1, \ldots, G_n$, and C be as stated in Axiom 5*. Then $G_i(\alpha) \geq G_j(\alpha)$ implies $C_{G_i} \geq C_{G_j}$.

Theorem 2. There exists a unique price mechanism $Q(\cdot, \cdot)$ on T_1 obeying Axioms 1, 2, 3, 4, 5*, and 6*. This mechanism is defined by the formula

$$Q(F + C, \alpha) = \hat{P}(F, \alpha) + \frac{C}{F(\alpha)} \hat{P}(F, \alpha) = \left(1 + \frac{C}{F(\alpha)}\right)\hat{P}(F, \alpha),$$

where $\hat{P}(F, \alpha)$ is the A-S price vector associated with the variable part F for the quantity vector α. That is,

$$\hat{P}_i(F, \alpha) = \int_0^1 \frac{\partial F}{\partial x_i}(t\alpha)dt, \qquad i = 1, \ldots, m.$$

In other words the fixed cost is allocated proportionally to the A-S prices of the variable cost $F(\alpha)$, and thus $Q(F + C, \alpha)$ is a scalar multiple of the A-S price vector $\hat{P}(F, \alpha)$.

Example. Consider an additive cost function

$$F(x_1, \ldots, x_m) = \sum_{i=1}^{m} a_i x_i.$$

Let $\alpha \in E^m_{++}$ and let $C > 0$. Then

$$p_i = Q_i(F + C, \alpha) = a_i + a_i \frac{C}{F(\alpha)}.$$

Let $\hat{p}_i$ be the part of the price p_i covering the variable cost. In this case $\hat{p}_i = a_i$. The part $p_i - \hat{p}_i$ that covers the fixed cost C is thus

$$p_i - \hat{p}_i = a_i \frac{C}{F(\alpha)}.$$

Notice that for any two goods i and j, $\frac{\hat{p}_i}{\hat{p}_j}$ and $\frac{(p_i - \hat{p}_i)}{(p_j - \hat{p}_j)}$ are both equal to the ratio of the marginal costs of these two goods. Namely,

$$\frac{\hat{p}_i}{\hat{p}_j} = \frac{p_i - \hat{p}_i}{p_j - \hat{p}_j} = \frac{a_i}{a_j}.$$

In particular, if $a_i = a_j$, the variable parts of p_i and p_j as well as the fixed cost parts are equal.

The proof of Theorem 2 is given in the Appendix.

Appendix

Proof of Theorem 2

■ First, it is easy to verify that the price mechanism $Q(\cdot, \cdot)$ defined on T_1 by

$$Q(F + C, \alpha) = \left(1 + \frac{C}{F(\alpha)}\right)\hat{P}(F, \alpha)$$

obeys the six axioms 1, 2, 3, 4, 5*, and 6*. Notice that the number C_G appearing in Axiom 5* is defined, for the price mechanism $Q(\cdot, \cdot)$, by

$$C_G = C\frac{G(\alpha)}{F(\alpha)}.$$

Thus, it remains only to prove the uniqueness part. To that end we first prove the following two lemmas.

Lemma 3. Let F and G be in F_+^m and let $C \geq 0$. If $G \leq F$ and $F(\alpha) \neq 0$, then the number C_G whose existence is guaranteed by Axiom 5* is given by

$$C_G = C\frac{G(\alpha)}{F(\alpha)}.$$

Proof. First let us show that C_G is an additive operator on the class of all functions $G \in F_+^m$ with $G \leq F$. Indeed, let G_1 and G_2 be any two functions in F_+^m with $G_1 + G_2 \leq F$. Since the function G_3 defined by

$$G_3 = F - (G_1 + G_2)$$

also belongs to this class, we have by Axiom 5*

$$P(F + C, \alpha) = \sum_{i=1}^{3} P(G_i + C_{G_i}, \alpha),$$

where

$$C = \sum_{i=1}^{3} C_{G_i}. \tag{A1}$$

On the other hand, again by Axiom 5*,

$$P(F + C, \alpha) = P(G_1 + G_2 + C_{G_1+G_2}, \alpha) + P(G_3 + C_{G_3}, \alpha),$$

where

$$C = C_{G_1+G_2} + C_{G_3}. \tag{A2}$$

Equalities (A1) and (A2) imply

$$C_{G_1+G_2} = C_{G_1} + C_{G_2}. \tag{A3}$$

Thus C_G is an additive operator. Axiom 6* clearly implies that C_G is a nondecreasing function of $G(\alpha)$. Thus, the operator C_G can be written as

$$C_G = f(G(\alpha)),$$

where f is a nondecreasing function on the interval $[0, F(\alpha)]$. By (A3) it follows that f is also an additive function on $[0, F(\alpha)]$. Therefore, it follows that f is of the form

$$f(x) = ax, \qquad 0 \leq x \leq F(\alpha). \tag{A4}$$

Using the fact that $C_F = C$ (which is implied by Axiom 5* when taking $n = 1$) (A4) implies that

$$C = C_F = f(F(\alpha)) = aF(\alpha).$$

Since $F(\alpha) \neq 0$, $a = \dfrac{C}{F(\alpha)}$, and thus (A4) can be written as

$$f(x) = \frac{C}{F(\alpha)}x.$$

This implies that

$$C_G = f(G(\alpha)) = \frac{C}{F(\alpha)}G(\alpha),$$

as was claimed.

Definition 1. Let $\beta \in E^m_{++}$ and let $A_\beta = \{x \in E^m_+ | x \leq \beta\}$. The norm $\| \quad \|_{C^1}$ on the set of all continuously differentiable functions F on A_β is defined by

$$\|F\|_{C^1} = \sup |F| + \sum_{i=1}^{m} \sup \left| \frac{\partial F}{\partial x_i} \right| ,$$

where the sup is taken over A_β.

Definition 2. Let $P(\cdot, \cdot)$ be a price mechanism on T_1. We say that $P(\cdot, \cdot)$ is continuous at (F, α) if whenever $(F_n, \alpha) \in T_1$, $(F, \alpha) \in T_1$, and $\|F_n - F\|_{C^1} \to 0$ on A_α, then

$$\lim_{n \to \infty} P(F_n, \alpha) = P(F, \alpha).$$

$P(\cdot, \cdot)$ is continuous on T_1 if it is continuous at every point of T_1.

Lemma 4. Let $P(\cdot, \cdot)$ be a price mechanism on T_1 which obeys the cost-sharing and positivity axioms (Axioms 1 and 4). Then $P(\cdot, \cdot)$ is continuous on T_1.

Proof. The positivity axiom implies that $P(F, \alpha) = P(G, \alpha)$ if $F(x) = G(x)$ for each $x \in A_\alpha$. Since any continuously differentiable function F on A_α can be extended to a continuously differentiable function F on E^m_+ (Whitney, 1934), the price mechanism $P(\cdot, \cdot)$ can be considered as a positive functional on pairs (F, α) where the domain of F is restricted to the box A_α.

Let (F_n, α) and (F, α) be in T_1 and assume that $\|F_n - F\|_{C^1} \to 0$ on A_α as $n \to \infty$. Thus, by the definition of the C^1 norm, there are two sequences $(\epsilon_n)_{n=1}^{\infty}$ and $(C_n)_{n=1}^{\infty}$ of positive numbers such that $\epsilon_n \to 0$, $C_n \to 0$, as $n \to \infty$, and

$$\frac{\partial G_n}{\partial x_j} \geq \frac{\partial F}{\partial x_j} \tag{A5}$$

$$G_n(0) \geq F(0) \tag{A6}$$

on A_α, where

$$G_n(x) = F_n(x) + \epsilon_n \sum_{j=1}^{m} x_j + C_n. \tag{A7}$$

By (A5), $G_n - F$ is nondecreasing. This, together with (A6) and the positivity axiom, implies that

$$P(G_n, \alpha) \geq P(F, \alpha). \tag{A8}$$

Let

$$\beta_j = \limsup_n P_j(G_n, \alpha)$$

and $\beta = (\beta_1, \ldots, \beta_m)$. Clearly by (A8), $\beta \geq P(F, \alpha)$. In fact, we claim that $\beta = P(F, \alpha)$. Indeed, if there exists a j, $1 \leq j \leq m$, such that $\beta_j > P_j(F, \alpha)$, then by the cost-sharing axiom and the fact that $\alpha \in E^m_{++}$,

$$\alpha\beta > \alpha P(F, \alpha) = F(\alpha). \tag{A9}$$

On the other hand, (A7), (A8), and the cost-sharing axiom imply that

$$\alpha\beta = \limsup_n \alpha P(G_n, \alpha) = \limsup_n G_n(\alpha) = F(\alpha).$$

This contradicts (A9). Thus, we conclude that

$$\beta = P(F, \alpha). \tag{A10}$$

This implies the existence of the limit and that

$$\lim_n P(G_n, \alpha) = P(F, \alpha). \tag{A11}$$

By (A5), (A6), and the positivity axiom

$$P(G_n, \alpha) \geq P(F_n, \alpha) \qquad \text{for each } n. \tag{A12}$$

Hence by (A11) and (A12)

$$\liminf_n P(F_n, \alpha) \leq P(F, \alpha).$$

It can be shown, along the same lines as in the derivation of equality (A10), by using the cost sharing axiom that

$$\liminf_n P(F_n, \alpha) = P(F, \alpha). \tag{A13}$$

On the other hand, by (A11) and (A12)

$$\limsup_n P(F_n, \alpha) \leq P(F, \alpha),$$

which together with (A13) implies the continuity of P, i.e.,

$$\lim_n P(F_n, \alpha) = P(F, \alpha) \qquad \text{as } n \to \infty.$$

We are now ready to prove the uniqueness part of the theorem. Let $Q^1(\cdot, \cdot)$ and $Q^2(\cdot, \cdot)$ be two price mechanisms on T_1 which obey the six axioms. Let us first show that they coincide on polynomials. Indeed, let F be a polynomial on E^m_+ with $F(0) = 0$ and let C be a nonnegative number. The polynomial F is a linear combination (with coefficients $+1$ or -1) of polynomials L of the form

$$L(x_1, \ldots, x_m) = (n_1x_1 + \cdots + n_mx_m)^l, \tag{A14}$$

where the n_j's are nonnegative numbers and l is a positive integer (e.g., see Aumann and Shapley (1974, p. 41). Thus, F can be written as

$$F = \sum_i F_i - \sum_j G_j, \tag{A15}$$

where both F_i and G_j are of the form (A14). Let $\alpha \in E^m_{++}$ be such that $F(\alpha) \neq 0$ and let $d = \dfrac{\sum F_i(\alpha)}{F(\alpha)} C$. Then by Lemma 3 and Axiom 5*

$$Q^k(F + \sum_j G_j + d, \alpha) = Q^k(F + C, \alpha) + \sum_j Q^k\left(G_j + C\frac{G_j(\alpha)}{F(\alpha)}, \alpha\right) \tag{A16}$$

for $k = 1, 2$. On the other hand, by (A15)

$$Q^k(F + \sum_j G_j + d, \alpha) = Q^k(\sum_i F_i + d, \alpha), \qquad k = 1, 2.$$

Again, using Axiom 5* and Lemma 3, we obtain

$$Q^k(F + \sum_j G_j + d, \alpha) = \sum_i Q^k\left(F_i + C\frac{F_i(\alpha)}{F(\alpha)}, \alpha\right), \qquad k = 1, 2.$$

This together with (A16) implies that

$$Q^k(F + C, \alpha) = \sum_i Q^k\left(F_i + C\frac{F_i(\alpha)}{F(\alpha)}, \alpha\right) - \sum_j Q^k\left(G_j + C\frac{G_j(\alpha)}{F(\alpha)}, \alpha\right), \qquad k = 1, 2.$$

Thus, to prove that Q^1 and Q^2 coincide on $(F + C, \alpha)$, it is sufficient to prove that the two coincide on $(L + C, \alpha)$, where L has the form (A14). Moreover, using the continuity of $Q^k(\cdot, \cdot)$ on T_1 ($k = 1, 2$) as stated in Lemma 4, one can show by using the same arguments as in Samet and Tauman (1982) that it is sufficient to deal with the case in which the n_j's appearing in (A14) are all positive. Let H be the function on E_+^m defined by

$$H(x_1, \ldots, x_m) = (\sum_j x_j)^l + C.$$

Since

$$L(x_1, \ldots, x_m) = H(n_1x_1, \ldots, n_mx_m),$$

the rescaling axiom implies for each j, $1 \leq j \leq m$, and $k = 1, 2$ that

$$Q_j^k(L + C, \alpha) = n_j Q_j^k(H, n * \alpha), \tag{A17}$$

where $n * \alpha = (n_1\alpha_1, \ldots, n_m\alpha_m)$. Applying the consistency axiom

$$Q_j^k(H, n * \alpha) = Q_{j'}^k(H, n * \alpha)$$

for $1 \leq j, j' \leq m$, and $k = 1, 2$. Let

$$u^k = Q_j^k(H, n * \alpha), \qquad k = 1, 2 \text{ and } j = 1, \ldots, m. \tag{A18}$$

The cost-sharing axiom implies that

$$u^k = \frac{H(n * \alpha)}{\sum_j n_j \alpha_j}, \qquad k = 1, 2,$$

in particular, $u^1 = u^2$ and thus by (A17) and (A18)

$$Q_j^1(L + C, \alpha) = Q_j^2(L + C, \alpha), \qquad j = 1, \ldots, m.$$

Consequently, $Q^1(\cdot, \cdot)$ and $Q^2(\cdot, \cdot)$ coincide on $(\hat{F}, \alpha)$ in T_1, where $\hat{F}$ is a polynomial. Let (F, α) be an arbitrary element in T_1 where $\alpha \in E_{++}^m$. The polynomials in m variables are dense in the set of all continuously differentiable functions on A_α with respect to the C^1 norm (for a proof see Courant and Hilbert (1953, p. 68)). Thus, by the continuity of $Q^1(\cdot, \cdot)$ and $Q^2(\cdot, \cdot)$ (Lemma 4) and by the fact that $Q^1(\cdot, \cdot)$ and $Q^2(\cdot, \cdot)$ coincide on polynomials, it follows that the two coincide on T_1 as was claimed.

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