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CONTINUOUS SELECTIONS FOR VECTOR MEASURES*†

DOV SAMET

Bar-Ilan University

A vector measure is a many to one map; it maps many measurable sets onto the same point. A selection for a vector measure is a function which assigns to each point in the range of the vector measure only one measurable set which is mapped onto the point. The existence of a continuous selection for nonatomic vector measures is proved where the distance in the σ-field is the measure (for a given scalar measure) of the symmetric difference.

A stronger version of continuous selection exists for strictly convex ranges.

1. Introduction. Vector measures are used frequently in economics, game theory, statistics and control theory and in many cases a continuous selection for the inverse of the vector measure is meaningful and desirable (see e.g. Tauman [5]). We illustrate this situation by a simple production model. A measurable space (U, Σ, λ) with a nonatomic, σ-additive finite measure λ is representing in this model a continuum of production units or producers. There are n products that can be produced by each producer and the integrable function $a\colon U \to R^n$ gives for each producer t in U his production capacity $a(t)$. We assume further that each producer t in U can produce only vectors in the line segment $[0, a(t)]$. Such an assumption can be made in models of short run production (Hildenbrand [2]). We associate with the function a a nonatomic finite vector measure μ defined by $\mu(S) = \int a(t)\, d\lambda$ for each measurable set S. Let F denote the set of all measurable functions χ satisfying $0 \leqslant \chi \leqslant 1$. A function $\chi \in F$ is called a *production plan*. For each t, $\chi(t)$ represents the proportion of $a(t)$ produced by t and the total production in this case is $\int a(t)\chi(t)\, d\lambda = \int \chi\, d\mu$ which we denote by $\mu(\chi)$. It is well known that $\mu(F)$ is the range $R(\mu)$ of the vector measure μ.

A *production program* φ is a function $\varphi\colon R(\mu) \to F$ such that $\mu(\varphi(x)) = x$ for each $x \in R(\mu)$, i.e., φ assigns to each x a production plan which generates x. Obviously, it is desirable to have a continuous production program which means that small changes in demand (i.e. in x) require only small changes in production plans to match them, where these last changes are naturally measured with respect to the $L_1(\lambda)$ topology on F.

Consider now a further restriction on possible production and assume that each producer t can produce either 0 or $a(t)$. This means that we allow our production plans to get the values 0 and 1 only. Denote by E the set of all 0-1 functions in F. E is naturally identified with Σ and $\mu(\Sigma)$ is of course the range of μ. Production programs are similarly restricted and φ is allowed to have values only from E. A continuous production plan in this case is a continuous selection for μ^{-1}, the inverse of μ. (In particular such a selection is also a production program for the more general case where all production plans in F are allowed.) We prove in Theorem 1 the existence of

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such a selection. To understand the difficulties in finding such a selection we compare the "restricted" problem to the one we started with.

A theorem that can be used to guarantee continuous selections is Michael's Theorem [3]. One of the conditions required in this theorem is the lower semicontinuity of μ^{-1}. It is proved in Samet [4] that μ: $E \to R(\mu)$ is open and similarly one can prove that μ: $F \to R(\mu)$ is open which is equivalent to the lower semicontinuity of μ^{-1}. Another requirement is that $\mu^{-1}(x)$ is convex and closed for each x. This is indeed the case when the whole of F is taken into account and in this case applying Michael's theorem is straightforward. The novelty in Theorem 1 is that a continuous selection can be found with values restricted to E, which means that for each x in $R(\mu)$ φ selects an *extreme point* of $\mu^{-1}(x)$.

Suppose now that only a subset S of U is available for production. The production set of S is $R(\mu, S)$ the range of the restriction of μ to subsets of S. By Theorem 1 there exists a continuous selection $\varphi(S, \cdot)$ defined on the range $R(\mu, S)$. We show in Theorem 2 that when $R(\mu)$ is strictly convex the family of selections $\varphi(S, \cdot)$ (for all measurable S) can be chosen to be simultaneously continuous in S and x.

2. The main results. Let (U, Σ) be a measurable space and μ: $\Sigma \to R^n$ a nonatomic σ-additive finite vector measure. A function φ: $\mu(\Sigma) \to \Sigma$ is called a *selection* (for $\mu(\Sigma)$) if $\mu(\varphi(x)) = x$ for each $s \in \mu(\Sigma)$. For a given nonatomic σ-additive finite measure λ on (U, Σ) we consider the topology on Σ associated with the pseudo metric d_λ defined by $d_\lambda(S, T) = \lambda(S \Delta T)$ for each $S, T \in \Sigma$, where $S\Delta T$ is the symmetric difference $(S \setminus T) \cup (T \setminus S)$. When φ is continuous with this topology on Σ and the relative topology from R^n on $\mu(\Sigma)$ we say that φ is *continuous with respect to* (*w.r.t.*) λ. Equivalently φ may be considered as a function from $\mu(\Sigma)$ to the set of 0-1 functions in $L_1(\lambda)$ such that $\int \varphi(x)\, d\mu = x$.

THEOREM 1. *Let $\lambda, \mu_1, \ldots, \mu_n$ be nonatomic, σ-additive, finite measures on a measurable space (U, Σ) and let λ be nonnegative. Then there exists a selection for $(\mu_1, \ldots, \mu_n)(\Sigma)$ which is continuous with respect to λ.*

Theorem 2 states a stronger result for strictly convex ranges. We need for this theorem the following notations. For $S \in \Sigma$ we denote by $R(\mu, S)$ the range of the restriction of μ to subsets of S, i.e. $R(\mu, S) = \{\mu(T) | T \subset S\}$. For a scalar measure ν, we write $|\nu|$ for the sum of the positive and negative parts of ν. For a vector measure $\mu = (\mu_1, \ldots, \mu_n)$, $|\mu| = \Sigma_{i=1}^n |\mu_i|$. The *complete range* of $\mu = (\mu_1, \ldots, \mu_n)$ is the subset $R^c(\mu)$ of $\Sigma \times R^n$ defined by

$$R^c(\mu) = \{(S, x) | S \in \Sigma, x \in R(\mu, S)\}.$$

A function φ: $R^c(\mu) \to \Sigma$ is called a *selection* (for $R^c(\mu)$) if for each $S \in \Sigma$, $\varphi(S, \cdot)$ is a selection for $R(\mu, S)$ i.e. if for each $(S, x) \in R^c(\mu)$, $\varphi(S, x) \subset S$ and $\mu\varphi(S, x) = x$. The topology of $R^c(\mu)$ is induced by the d_λ-topology on Σ and the usual topology on R^n. We say that φ *is continuous w.r.t.* λ if it is continuous with the above mentioned topology on $R^c(\mu)$ and the d_λ-topology on Σ.

The range $R(\mu)$ is *strictly convex* if every nontrivial convex combination of points in the range falls in the relative interior of $R(\mu)$. We say in such a case that μ *is strictly convex*.

THEOREM 2. *Let $\mu = (\mu_1, \ldots, \mu_n)$ be a nonatomic, σ-additive, finite and strictly convex vector measure and let λ be a nonnegative, nonatomic, σ-additive and finite measure such that $|\mu|$ is absolutely continuous with respect to λ ($|\mu| \ll \lambda$). Then there exists a selection for the complete range of μ, $R^c(\mu)$ which is continuous with respect to λ.*

3. Proofs. We start by proving Theorems 1 and 2 for $\lambda = |\mu|$. When no measure is mentioned we refer by 'almost everywhere' ('a.e.') to the measure $|\mu|$. We use in the proof some well-known properties of the extreme points of vector measure ranges which we describe now. For further details see Bolker [1]. We denote the Radon-Nikodym derivative of $\mu = (\mu_1, \ldots, \mu_n)$ w.r.t. $|\mu|$ by $f = (f_1, \ldots, f_n)$ (i.e., $f_i = d\mu_i/d|\mu|$). For each $S \in \Sigma$, $\mu(S) = \int_S f\,d|\mu|$. For $d \neq 0$ in R^n and $T \in \Sigma$ the face of $R(\mu, T)$ in the direction d is the set

$$F(T, d) = \left\{ x | \langle d, x \rangle = \max_{y \in R(\mu, T)} \langle d, y \rangle \right\}$$

where $\langle d, x \rangle$ is the scalar product of d and x. A measurable set $S \subset T$ for which $\mu(S) \in F(T, d)$ must satisfy

$$\langle d, \mu(S) \rangle = \int_S \langle d, f \rangle \, d|\mu| = \max_{S' \subset T} \int_{S'} \langle d, f \rangle \, d|\mu|,$$

which implies that

$$\{ t \in T | \langle d, f(t) \rangle > 0 \} \subset S \subset \{ t \in T | \langle d, f(t) \rangle \geqslant 0 \}$$

a.e. w.r.t. $|\mu|$. It follows then that:

LEMMA 1. *The face of $R(\mu, T)$ in the direction d is a singleton (i.e. it consists of an extreme point) if and only if $|\mu| (\{ t \in T | \langle d, f(t) \rangle = 0 \}) = 0$. Moreover, if for $S \subset T$, $\mu(S)$ is an extreme point of $R(\mu, T)$ then $S = \{ t \in T | \langle d, f(t) \rangle \geqslant 0 \}$ a.e. w.r.t. $|\mu|$.*

By Lyapunov's Theorem the range $R(\mu, T)$ is convex. We note that the range is strictly convex iff each face of it, except $R(\mu, T)$ itself, is an extreme point. It easily follows from Lemma 1 that if $R(\mu, U)$ is strictly convex and $|\mu|(S) > 0$ then $R(\mu, S)$ is also strictly convex and of the same dimension as $R(\mu, U)$. We denote by $\partial R(\mu, T)$ the relative boundary of $R(\mu, T)$ which is the union of all its nontrivial faces.

We give now some simple properties of $R^c(\mu)$.

LEMMA 2. *$R^c(\mu)$ is closed. Moreover, if μ is strictly convex, $(T_k, x_k) \in R^c(\mu)$ for each $k \geqslant 0$, $(T_k, x_k) \to (T_0, x_0)$ and $x_k \in \partial R(\mu, T_k)$ for $k \geqslant 1$ then $\{x_0\} = \partial R(\mu, T_0)$ and there are directions d_k for each $k \geqslant 0$, such that $\{x_k\} = F(T_k, d_k)$ for each $k \geqslant 0$, and a subsequence of d_k converges to d_0.*

PROOF. To prove closedness of $R^c(\mu)$ let $(T_k, x_k) \to (T_0, x_0)$ and suppose for each $k \geqslant 1$, $x_k = \mu(S_k)$ for $S_k \subset T_k$. Define $S_k' = S_k \cap T_0$ and $x_k' = \mu(S_k)$. Then $x_k' - x_k \to 0$. But $x'_k \in R(\mu, T_0)$ for each $k \geqslant 1$ and therefore $x_0 \in R(\mu, T_0)$. Suppose now in addition that μ is strictly convex and for $k \geqslant 1$, $x_k \in \partial R(\mu, T_k)$. We can choose directions d_k for $k \geqslant 1$ with $\|d_k\| = 1$ such that $\{x_k\} = F(T_k, d_k)$. When $R(\mu)$ is not of full dimension we select a point $z \neq 0$ in R^n which is normal to $R(\mu)$ and choose the directions d_k such that $\langle d_k, z \rangle = 0$ for each $k \geqslant 1$. Since there exists a converging subsequence of d_k we may assume without loss of generality that for some $d \neq 0$, $d_k \to d$. Let $y \in R(\mu, T_0)$ and $y = \mu(S)$ for $S \subset T_0$. For each $k \geqslant 1$ define $S_k = S \cap T_k$, and $y_k = \mu(S_k)$. Then $y_k \in R(\mu, T_k)$ and $y_k \to y$. Therefore $\langle d_k, x_k \rangle \geqslant \langle d_k, y_k \rangle$ and in the limit $\langle d, x \rangle \geqslant \langle d, y \rangle$ which shows that $x \in F(T_0, d)$. The point x is not in the relative interior of $F(T_0, d)$ because otherwise $\langle d, R(\mu, T_0) \rangle = 0$ and since also $\langle d, z \rangle = 0$, $d \neq 0$ is contradicted. Therefore $\{x_0\} = F(T_0, d)$. Q.E.D.

We denote by $\partial R^c(\mu)$ the set $\{(T, x) | x \in \partial R(\mu, T)\}$ and restrict ourselves in the next lemma to this set only, and to strictly convex ranges.

LEMMA 3. *Let μ be a strictly convex vector measure. Then there exists a selection ψ: $\partial R^c(\mu) \to \Sigma$ which is continuous with respect to $|\mu|$.*

PROOF. For each $(T, x) \in \partial R^c(\mu)$ choose $S_{T,x} \subset T$ such that $\mu(S_{T,x}) = x$ and define $\psi(T, x) = S_{T,x}$. Let $\{(T_k, x_k)\}_{k=1}^{\infty}$ be a sequence in $\partial R^c(\mu)$ such that $(T_k, x_k) \to (T_0, x_0)$. By Lemma 1 (and by choosing a subsequence if necessary) there are directions d_k for $k \geqslant 0$ such that $\{x_k\} = F(T_k, d_k)$ and $d_k \to d_0$. For each $k \geqslant 0$ let $S_k = \{t \in T_k | \langle d_k, f(t)\rangle \geqslant 0\}$. By Lemma 1 for each $k \geqslant 0$, $\psi(T_k, x_k) = S_k$ a.e. w.r.t. $|\mu|$, and therefore it suffices to show that $|\mu|(S_k \Delta S_0) \to 0$.

Now

$$S_0 \setminus S_k \subset \{t \in T_k \cup T_0 | \langle d_0, f(t)\rangle \geqslant 0\} \setminus \{t \in T_k \cap T_0 | \langle d_k, f(t)\rangle \geqslant 0\}$$
$$\subset \{t | \langle d_0, f(t)\rangle \geqslant 0, \langle d_k, f(t)\rangle < 0\}$$
$$\cup \{t \in T_k \Delta T_0 | \langle d_0, f(t)\rangle \geqslant 0\}.$$

Denote $A_k = \{t | \langle d_0, f(t)\rangle \geqslant 0$ and $\langle d_k, f(t)\rangle < 0\}$. Clearly $A_k = \{t | 0 \leqslant \langle d_0, f(t)\rangle < \langle d_0 - d_k, f(t)\rangle\}$. Notice that $\Sigma_{i=1}^n |f_i(t)| = 1$ a.e. and thus a.e. in A_k, $0 \leqslant \langle d_0 - d_k, f(t)\rangle \leqslant \|d_0 - d_k\| = c_k$ where $\|\cdot\|$ is the sup norm on R^n. Thus $A_k \subset B_k = \{t | 0 \leqslant \langle d_0, f(t)\rangle \leqslant c_k\}$ a.e.. But since $c_k \to 0$, B_k converges a.e. to $\{t | \langle d_0, f(t)\rangle = 0\}$ which by Lemma 1 is of $|\mu|$ measure 0. Also by our assumption $|\mu|(T_k \Delta T_0) \to 0$ and hence $|\mu|(S_0 \setminus S_k) \to 0$. Similarly, one can show that $|\mu|(S_k \setminus S_0) \to 0$ which completes the proof. Q.E.D.

We show now in Lemmas 4-6 that the complete range of a strictly convex vector measure can be continuously imbeded in the boundary of the complete range of a strictly convex vector measure of higher dimension. The continuous selection guaranteed for the boundary of the latter by Lemma 3 is then used to construct a continuous selection for the first.

Notice first that if $R(\mu, U)$ is strictly convex and of full dimension then by Lemma 1 $|\mu|(\{t | \langle a, f(t)\rangle = 0\}) = 0$ for each $a \neq 0$ in R^n. In other words the functions $f_1, \ldots, f_n$ are linearly independent over any set of positive $|\mu|$ measure. We call such functions *completely independent* (w.r.t. $|\mu|$).

LEMMA 4. *Let $f_1, \ldots, f_n$ be measurable functions which are completely independent w.r.t. to a nonatomic nonnegative measure λ. Then there exists a bounded measurable function f_{n+1} such that $f_1, \ldots, f_{n+1}$ are completely independent w.r.t. λ.*

PROOF. Since λ is nonatomic we can choose a measurable function h such that for each real c, $\lambda(\{t | h(t) = c\}) = 0$. (Note that for $n \geqslant 2$ the nonatomicity of λ follows from the complete independence of $f_1, \ldots, f_n$.) To see that such h exists we construct a sequence $\{\pi_n\}$ of measurable partitions of U and a sequence χ_n of characteristic functions. For $n = 1$, $\pi_1 = \{U\}$ and $\chi_1 \equiv 1$. The partition π_n refines π_{n-1} by dividing each set in π_{n-1} into two subsets of equal measure. χ_n is defined to be 1 over one of these subsets and 0 over the other. The function $h = \Sigma 2^{-n}\chi_n$ satisfies the requirement. Consider the functions $g_i = h^i$ $(i = 1, \ldots, n+1)$. These functions are completely independent. Indeed for $a \neq 0$ the set $\{t | \Sigma_{i=1}^{n+1} a_i h^i(t) = 0\}$ consists of all points at which h equals one of the finitely many roots of the polynomial $\Sigma_{i=1}^{n+1} a_i x^i$. Thus the measure of this set is 0. Clearly for each S whith $\lambda(S) > 0$ there exists some i $(1 \leqslant i \leqslant n+1)$ such that $f_1, \ldots, f_n, g_i$ are linearly independent over S. We claim that for each S with $\lambda(S) > 0$ there exist a subset $T = T(S)$ and a function g_T in $\{g_1, \ldots, g_{n+1}\}$ such that $\lambda(T) > 0$ and $f_1, \ldots, f_n, g_T$ are linearly independent over any subset of T with positive measure. Suppose on the contrary that for some S with

$\lambda(S) > 0$ there is no such T. Then we can construct inductively sets $S = S_0 \supset S_1 \supset \cdots \supset S_{n+1}$ and indices $i_0, \dots, i_{n+1}$ from $\{1, \dots, n+1\}$ such that for each $j = 0, \dots, n+1$, $f_1, \dots, f_n, g_{i_j}$ are linearly independent over S_j but are linearly dependent over S_{j+1} ($j = 0, \dots, n$). Since there are $n+2$ indices there exist i_j and i_k, $k > j$ such that $g_{i_j} = g_{i_k}$. But g_{i_j} is linearly dependent over S_{j+1} and therefore also over S_k which contradict the choice of g_{i_k}.

Consider now the family $F = \{T(S) | \lambda(S) > 0\}$ and the collection $\mathscr{C} = \{B | B \subset F$, the sets in B are disjoint$\}$. Elements of $\mathscr{C}$ are ordered by set inclusion and by Zorn's lemma there exists a maximal element B_0 in $\mathscr{C}$. Since the sets in B_0 are disjoint and are all of positive measure, B_0 has at most countably many sets and by the definition of F, $\Sigma_{T \in B_0} \lambda(T) = \lambda(U)$, otherwise $\lambda(U \setminus \bigcup_{T \in B_0} T) > 0$ and thus $B_0 \cup \{T(U \setminus \bigcup_{T' \in B_0} T')\}$ is in $\mathscr{C}$ contradicting the maximality of B_0. Now set $f_{n+1} = \Sigma \chi_T g_T$ where χ_T is the indicator function of T. Q.E.D.

LEMMA 5. *Let $\mu = (\mu_1, \dots, \mu_n)$ be a vector measure with an n-dimensional strictly convex range. Then there exists a vector measure $\bar{\mu} = (\mu, \mu_{n+1})$ such that $R(\bar{\mu}, U)$ is also strictly convex of dimension $n + 1$.*

PROOF. Let $f = d\mu/d|\mu|$. By Lemma 4 there exists f_{n+1} such that $f_1, \dots, f_{n+1}$ are completely independent. Let μ_{n+1} be defined by $d\mu_{n+1} = f_{n+1}\, d|\mu|$, and let $\bar{\mu} = (\mu, \mu_{n+1})$. Now

$$d\bar{\mu}/d|\bar{\mu}| = (d\bar{\mu}/d|\mu|)(d|\mu|/d|\bar{\mu}|) = (d\mu/d|\mu|, f_{n+1})/(1 + |f_{n+1}|)$$
$$= (f, f_{n+1})/(1 + |f_{n+1}|).$$

Thus for each $d = (d_1, \dots, d_{n+1}) \neq 0$ in R^{n+1}

$$\{t | \langle d, d\bar{\mu}/d|\bar{\mu}| \rangle = 0\} = \left\{ t \,\middle|\, \sum_{i=1}^{n+1} d_i f_i(t) = 0 \right\}$$

and by the construction of f_{n+1} the latter set is of $|\mu|$-measure 0 and, therefore also of $|\bar{\mu}|$-measure 0. This shows, by Lemma 1, that $R(\bar{\mu}, U)$ is strictly convex and of dimension $n + 1$. Q.E.D.

LEMMA 6. *Let $R^c(\mu)$ be the complete range of a strictly convex vector measure μ. Then there exists a selection for R^c which is continuous w.r.t. $|\mu|$.*

PROOF. Assume first that $R(\mu, U)$ is of full dimension and consider the vector measure $\bar{\mu}$ whose existence is guaranteed by Lemma 5. Note that the topologies defined by $d_{|\mu|}$ and $d_{|\bar{\mu}|}$ on Σ are the same. Let m: $R^c(\mu) \to R$ be the function defined by:

$$m(T, x) = \min\{z | (x, z) \in R(\mu, T)\}.$$

We show that m is continuous. Let $(T_k, x_k) \to (T_0, x_0)$ be a converging sequence in $R^c(\mu)$. Denote for $k \geqslant 1$, $z_k = m(T_k, x_k)$ and assume that $z_k \to z_0$. We show that $z_0 = m(T_0, x_0)$. Clearly $(x_k, z_k) \in \partial R(\mu, T_k)$ for each $k \geqslant 1$ and thus, by Lemma 2, $(x_0, z_0) \in \partial R(\mu, T_0)$.

Consider first the case that $x_0 \in \partial R(\mu, T_0)$. In this case the whole interval between (x_0, z_0) and $(x_0, m(T_0, x_0))$ is contained in $\partial R(\mu, T_0)$ and by the strict convexity of $R(\mu, T_0)$ it follows that $z_0 = m(T_0, x_0)$. Secondly, suppose that $x_0 \notin \partial R(\mu, T_0)$. It follows by Lemma 2 (and by choosing a subsequence if necessary) that $x_k \notin \partial R(\mu, T_k)$ and that there are directions d_k in R^{n+1} for $k \geqslant 0$ such that $\{(x_k, z_k)\} = F(T_k, d_k)$ and $d_k \to d_0$. Now, since x_k is an interior point of $R(\mu, T_k)$ and $R(\mu, T_k)$ is of full dimension it follows that $\{z|(x_k, z) \in R(\mu, \hat{T}_k)\}$ is a nontrivial interval and therefore the $n + 1$ coordinate of d_k satisfies $d_k^{n+1} \leqslant 0$ (otherwise $\langle d_k, (x_k, z)\rangle$ is not maximized at $z = z_k$). Thus, we conclude that $d_0^{n+1} \leqslant 0$.

Now x_0 is an interior point of $R(\mu, T_0)$ and (x_0, z_0) is in $\partial R(\mu, T_0)$ and thus z_0 is either $m(T_0, x_0)$ or $\max\{z|(x_0, z) \in R(\mu, T_0)\}$. But if z_0 is the latter point then by the same argument as above $d_0^{n+1} \geqslant 0$ and moreover $d_0^{n+1} > 0$ because when $d_0^{n+1} = 0$, $x_0 \in \partial R(\mu, T_0)$.

We proved that $z_0 = m(T_0, x_0)$ and therefore m is continuous on $R^c(\mu)$.

The functions $(S, x) \to (S, (x, m(S, x)))$ is thus a continuous imbedding of $R^c(\mu)$ in $\partial R^c(\mu)$. By Lemma 5 there exists a selection ψ: $\partial R^c(\mu) \to \Sigma$ which is continuous w.r.t. $|\bar{\mu}|$ (and therefore also w.r.t. $|\mu|$). We define now $\varphi(S, x) = \psi(S, (x, m(S, x)))$ which is the required selection.

If $\mu = (\mu_1, \ldots, \mu_n)$ is of dimension $k < n$ then without loss of generality we can assume that for $\mu' = (\mu_1, \ldots, \mu_k)$, $R(\mu', U)$ is of full dimension and is strictly convex. Since $\mu_{k+1}, \ldots, \mu_n$ are linearly dependent on $\mu_1, \ldots, \mu_k$ the topologies induced by $d_{|\mu|}$ and $d_{|\mu'|}$ are equivalent. The map $(S, x) \to (S, \pi(x))$ where π projects x on its first k coordinates is a continuous μ-preserving map of $R^c(\mu)$ onto $R^c(\mu')$. For $R^c(\mu')$ we have a continuous selection ψ and therefore $\varphi(S, x) = \psi(S, \pi(x))$ is a continuous selection for $R^c(\mu)$. Q.E.D.

To prove the theorem for general ranges we use the following lemma from Samet [4].

LEMMA 7. *There is a countable decomposition* $R(\mu, U) = \sum_{i \in I} R(\mu, S_i)$ *such that* $\{S_i\}_{i \in I}$ *is a partition of* U *and for each* $i \in I$, $R(\mu, S_i)$ *is strictly convex.*

In general we have for a point in $R(\mu, U)$ many representations as a sum of points from the ranges $R(\mu, S_i)$. In the next lemma we show that we can continuously select a unique representation for each x.

LEMMA 8. *Let* $\{C_i\}_{i \in I}$ *be a finite or denumerable family of compact strictly convex sets in* R^n *and let* $M = \sum_i C_i$ *be compact too. Then there exists for each* i *a continuous function* f_i: $M \to C_i$ *such that for each* $m \in M$, $m = \sum_i f_i(m)$.

PROOF. Consider the Cartesian product $C = \Pi_i C_i$, equipped with the product topology. Define s: $C \to M$ by $s(c) = \sum c(i)$. We use Michael's selection theorem [3] for s. For this purpose we have to show that $s^{-1}(m)$ is a convex and closed set in a Banach space and s^{-1} is lower semicontinuous. Indeed consider the space $(R^n)^I$ with the norm $\|(x(i))_{i \in I}\| = \operatorname{Sup}_i \|x(i)\|$ where $\|x(i)\|$ is the Euclidean norm in R^n. This is a Banach space and it is easy to see that the product topology of C is equivalent to the topology induced on C by the norm on $(R^n)^I$. Clearly, for each $m \in M$, $s^{-1}(m)$ is convex and closed in C. We show now that s is open which is equivalent to the lower semicontinuity of s^{-1}. We prove that s is open by induction on l the dimension of M. For $l = 0$, C_i is a singleton for each i and the claim is trivial. Suppose it is proved for all dimensions less than l and let M be of dimension l. Let $c \in C$ be a point for which $m = s(c)$ is in the relative interior of M. If N is a neighborhood of c, then for sufficiently small $\delta > 0$, $(1 - \delta)c + \delta C$ is in N.

Now $s(N) \supset s((1 - \delta)c + \delta C) = (1 - \delta)m + \delta M$ which is a neighborhood of m. Suppose now that $m = s(c)$ is a point in a face of M of dimension less than l in the direction d. For a set K we denote by $F(K, d)$ the face of K in the direction d.

Since for each i, C_i is strictly convex it follows that for each i either $C_i = F(C_i, d)$ or $F(C_i, d)$ consists of a single point in C_i.

Define

$$I_1 = \{i|F(C_i, d) \neq C_i\}, \quad I_2 = \{i|F(C_i, d) = C_i\}, \quad C^1 = \prod_{i \in I_1} C_i, \quad C^2 = \prod_{i \in I_2} C_i,$$

and for each $k \in C$ denote by k^1 and k^2 its projection on C^1 and C^2, respectively. We use s to denote the natural restriction of s to C^1 and C^2 and we denote $s(C^1) = M^1$, $s(C^2) = M^2$, $s(c^1) = m^1$ and $s(c^2) = m^2$. Clearly $m^1 + m^2 = m$. We note that $F(C^1, d) = s^{-1}(F(M^1, d))$. Since $F(C^1, d)$ is a singleton and $m^1 \in F(M^1, d)$ it follows that $F(C^1, d) = \{c^1\}$ and $F(M^1, d) = \{m^1\}$.

Proving that s maps a neighborhood of c onto a neighborhood of $s(c) = m$ is equivalent to showing that for any sequence $m_j \to m$ in M there exists a sequence c_j in C such that $s(c_j) = m_j$ and $c_j \to c$. Let m_j be such a sequence. For each j choose $m_j^1 \in M^1$ and $m_j^2 \in M^2$ such that $m_j^1 + m_j^2 = m_j$. We observe that any limit point of m_j^1 must be in $F(M^1, d)$ and hence $m_j^1 \to m^1$. From this and $m_j \to m$ we conclude that $m_j^2 \to m^2$.

We build now the sequence c_j. First we choose a sequence c_j^1 in C^1 such that $s(c_j^1) = m_j^1$. Clearly $c_j^1 \to c^1$. The dimension of C^2 is less than l and thus by applying the induction hypothesis to m^2, c^2 and the sequence m_j^2 we find a sequence c_j^2 in C^2 such that $s(c_j^2) = m_j^2$ and $c_j^2 \to c^2$. Define now $c_j \in C$ by $c_j(i) = c_j^1(i)$ for $i \in I_1$ and $c_j(i) = c_j^2(i)$ for $i \in I_2$. Thus, $s(c_j) = s(c_j^1) + s(c_j^2) = m_j^1 + m_j^2 = m_j$ and $c_j \to c$.

Since s is open and $s^{-1}(m)$ is closed and convex for each $m \in M$ we can apply Michael's theorem and find a function f: $M \to C$ such that $s(f(m)) = m$ i.e. $\Sigma_i f(m)(i) = m$. The functions $f_i(\cdot) = f(\cdot)(i)$ satisfy the requirements. Q.E.D.

LEMMA 9. *For each vector measure μ there exists a selection for $R(\mu)$ which is continuous w.r.t. $|\mu|$.*

PROOF. Let $R(\mu, U) = \Sigma R(\mu, S_i)$ be the decomposition of Lemma 7. By Lemma 8 there are functions f_i: $R(\mu, U) \to R(\mu, S_i)$ such that $\Sigma f_i(x) = x$ for each $x \in R(\mu)$. Denote by $R^C(\mu, S_i)$ the complete range of μ restricted to subsets of S_i. Since $R(\mu, S_i)$ is strictly convex there are, by Lemma 6, selections ψ_i: $R^c(\mu, S_i) \to \{T | T \subset S_i\}$ which are continuous w.r.t. $|\mu|$ (restricted to $\{T | T \subset S_i\}$). We define $\varphi_i(\cdot) = \psi_i(\cdot, S_i)$. For each $x \in R(\mu)$ define $\varphi(x) = \cup \varphi_i(f_i(x))$. Clearly

$$\mu(\varphi(x)) = \Sigma \mu(\varphi_i(f_i(x)) = \Sigma f_i(x) = x$$

and each function $\varphi_i(f_i(\cdot))$ is continuous on $R(\mu)$. Viewed as a series of functions with values in $L_1(|\mu|)$, $\varphi(x)$ is uniformly converging and therefore φ is continuous. Q.E.D.

PROOF OF THEOREM 1. Decompose U into two disjoints sets U_0 and U_1 such that $|\mu|(U_0) = 0$ and $\lambda \ll |\mu|$ on U_2. By Lemma 9 there exists a selection φ for the range of μ restricted to measurable subsets of U_1 which is continuous w.r.t. $|\mu|$ and therefore also w.r.t. λ. Since $R(\mu, U) = R(\mu, U_1))$ the proof is complete. Q.E.D.

PROOF OF THEOREM 2. Let $U = U_0 \cup U_1$ be the decomposition of the previous proof. Since $|\mu| \ll \lambda$ and, on U_1, $\lambda \ll |\mu|$, λ is equivalent to $|\mu|$ on U_1. Let ψ be the selection of Lemma 6 for $R^C(\mu, U_1)$. Define $\varphi(S, x) = \psi(S \cap U_1, x)'$. It can be easily verified that φ is continuous w.r.t. λ. Q.E.D.

4. Possible extensions. The existence of a selection for the complete range is proved in Theorem 2 only for strictly convex ranges. We do not know whether such a selection exists in general. This open problem is related to the following possible extension. A general range μ can be decomposed by Lemma 7 into a sum $\Sigma R(\mu, S_i)$ of strictly convex ranges. By Lemma 8 we can present uniquely each $x \in R(\mu)$ as a sum Σx_i with $x_i \in R(\mu, S_i)$ for each i such that x_i varies continuously with x. Consider now pairs (x, S) in the complete range $R^C(\mu)$. Can we assign continuously to each pair (x, S) a unique decomposition Σx_i such that $x_i \in R(\mu, S_i \cap S)$? It is easy to see that an answer to this question in the affirmative will enable us to extend Theorem 2 to general ranges.

The range of a vector measure can be presented as the integral of set valued function (see §3). Thus an extension of the notion of a selection for integrals of set valued functions in general can be readily formulated. Let $A(t)$ be a set valued function on U with $A(t) \subset R^n$ for each t and let λ be a measure on (U, Σ). Denote by F the set of all λ integrable functions h which satisfy $h(t) \in A(t)$ a.e. w.r.t. λ. The set $\{\int h\,d\lambda | h \in F\}$ is denoted by $\int A\,d\lambda$. A selection for $\int A$ is a map φ: $\int A \to F$ which assigns to each $x \in \int A$ a "source" $\varphi(x)$ in F (i.e. $\int \varphi(x)(t)\,d\lambda(t) = x$). When $A(t) = \{0, f(t)\}$ for a given measurable f: $U \to R^n$ then $\int A\,d\lambda$ is the range of the measure μ defined by $d\mu = f\,d\lambda$. In this case Theorem 1 guarantees the existence of a selection which is continuous with the relative topology of $L_1(\lambda)$ on F. However a function $A(t)$ can be constructed for which no continuous selection exists. A natural question is then: under what conditions on $A(t)$ might a continuous selection exist?

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DEPARTMENT OF ECONOMICS; BAR-ILAN UNIVERSITY, RAMAT GAN, ISRAEL

CURRENT ADDRESS: KELLOGG GRADUATE SCHOOL OF MANAGEMENT, NORTHWESTERN UNIVERSITY, EVANSTON, ILLINOIS 60201