Games and Economic Behavior 108 (2018) 585–591

Contents lists available at ScienceDirect

Games and Economic Behavior

www.elsevier.com/locate/geb

# Coalition preferences with individual prospects ☆

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ARTICLE INFO

*Article history:*
Received 8 August 2017
Available online 6 December 2017

*JEL classification:*
D60
D70
D71
D81

*Keywords:*
Preference aggregation
Incomplete preferences
Extended Pareto rule

ABSTRACT

We consider a group of individuals, such that each coalition of them is endowed with a preference relation, which may be incomplete, over a given set of prospects, and such that the extended Pareto rule holds. We assume that each singleton coalition has complete vNM preferences. In this setup, Baucells and Shapley (2008) gave a sufficient condition for a coalition to have complete preferences, in terms of the completeness of preferences of certain pairs of individuals. The new property that we introduce of *individual prospects* requires each individual to have a pair of consequences between which only she is not indifferent. We show that with this property a weaker condition guarantees the completeness of preferences of a coalition: it suffices for a coalition to be a union of a connected family of coalitions with complete preferences.



## 1. Introduction

Consider a group of individuals each possessing a complete vNM preferences over a set of prospects. We further assume that each coalition of individuals is endowed with preferences that may be incomplete, i.e., some prospects may be incomparable. If we think of the preference relation of a coalition as an aggregation of the preferences of its members then the incompleteness of the coalition's preference is the result of the inability to socially compare certain alternatives. Our aim is to provide sufficient conditions for the completeness of the preferences of a coalition in terms of the complete preferences of some of its subcoalitions.

We assume that the coalition preference relations satisfy the Extended Pareto (EP) rule: if two disjoint coalitions agree on the preference relation between two prospects, then the union of these coalitions also has the same preference over the prospects. The EP rule was introduced in Shapley and Shubik (1974, p. 65), and explored by Dhillon (1998), Dhillon and Mertens (1999), Baucells and Sarin (2003) and Baucells and Shapley (2008). It extends and implies the Pareto requirement in Harsanyi (1955), which requires that when all individuals agree on the preference relation between two prospects, then the grand coalition also agree with this preference relation.

☆ This paper builds on Baucells and Shapley (2008), and we are indebted to Lloyd Shapley, who contributed to the start of the project. A prior working version of this paper had the title "Multiperson Utility: The Linearly Independent Case." Samet acknowledges financial support of the Israel Science Foundation, grant 520/16, and the Henry Crown Institute of Business Research in Israel.

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https://doi.org/10.1016/j.geb.2017.10.006

The following claim, which is a simplified version of the main result of Baucells and Shapley (2008), provides a sufficient condition for a coalition $S$ to have complete preferences.[1]

($*$) Consider a connected graph the vertices of which are all the individuals in $S$. If each pair of individuals that form an edge in this graph has complete preferences, then $S$ also has complete preferences.

Here, in addition to the EP rule, we assume the existence of Individual Prospects (IP). That is, for each individual there exists a pair of prospects, such that the individual prefers one to the other, while all other individuals are indifferent between them. If these prospects are lotteries of prizes this assumption holds when for each individual there is a prize that matters only to him. If prizes are monetary, then the assumption would require there to be more prizes than individuals, and have sufficient diversity among individuals' risk preferences. Assuming IP enables us to give the following sufficient condition for the completeness of the preferences of $S$.

($**$) Consider a connected graph, the vertices of which are subcoalitions of $S$, such that the union of these coalitions is $S$ and each pair of coalitions that form an edge in the graph has a non-empty intersection. If each of these coalitions has complete preferences then $S$ also has complete preferences.

The graphs in ($*$) and ($**$) are different. The nodes in the first are individuals, while in our condition, ($**$), they are coalitions. However, the restriction of ($**$) to coalitions of size two is equivalent to ($*$). Thus, our condition ($**$) is more general, and therefore weaker. We manage to reach the same conclusion as Baucells and Shapley (2008) with a weaker condition because of the assumption of IP.

Representation of preferences in terms of linear functions on the prospects plays an important role in our analysis and proofs. Representations of incomplete preferences were developed by Aumann (1962), Shapley and Baucells (1998), and Seidenfeld et al. (1995), and were revisited by Dubra et al. (2004) and Galaabaatar and Karni (2012). Incomplete preferences can be described by cones of utility vectors in the dual space of the space that contains the prospects. For complete preferences the cone is a one dimensional ray. The EP rule can also be described in terms of the cones defining the various coalitional preferences. The IP condition introduced here is equivalent to the requirement for the utility vectors of the individuals to be linearly independent. Linear independence of the utility vectors of each triplet of individuals was assumed in Baucells and Shapley (2008). However, the latter property is expressed in terms of the representation of the preferences and not in terms of the preferences themselves.

## 2. The main result

We consider a set of prospects $\mathcal{M}$, which is a full-dimensional, closed, convex subset of $\mathbb{R}^m$.[2] An *incomplete preference relation* on $\mathcal{M}$ is a binary relation $\succsim \subseteq \mathcal{M} \times \mathcal{M}$ that is reflexive ($\forall p, p \succsim p$), transitive ($\forall p, q, r$, if $p \succsim q$ and $q \succsim r$, then $p \succsim r$), continuous (the set $\{\alpha : p \succsim \alpha q + (1-\alpha)r\}$ is closed), and satisfies the axiom of independence ($\forall p, q, r$ and $\alpha \neq 0$, $p \succsim q$ iff $\alpha p + (1-\alpha)r \succsim \alpha q + (1-\alpha)r$). The relations $\sim$ and $\succ$ are defined as usual. The preference relation is *complete* when for all $p$ and $q$, either $p \succsim q$ or $q \succsim p$. The *trivial preference* is the complete preference that satisfies $p \succsim q$ for all prospects $p$ and $q$, and thus $p \sim q$ for all $p$ and $q$.

Let $N = \{1, \ldots, n\}$ be a set of individuals. Non-empty subsets of $N$ are called *coalitions*. With some abuse of notation we write $i$ for $\{i\}$. We implicitly assume that all individuals agree on $\mathcal{M}$, which is the case when probabilities are objective.[3]

**Definition 1.** A *coalition preference* is an assignment of an incomplete preference relation to each coalition $S$, denoted by $\succsim_S$, such that for each individual $i$, $\succsim_i$ is complete.

We assume that the coalition preference satisfies the following two properties:

**Extended Pareto** (EP)

For all disjoint coalitions $A$ and $B$, and for all $p, q \in \mathcal{M}$, if $p \succsim_A q$ and $p \succsim_B q$, then $p \succsim_{A\cup B} q$, and if $p \succ_A q$ and $p \succsim_B q$, then $p \succ_{A\cup B} q$.

[1] Their theorem is formulated for the grand coalition, but it trivially extends to any coalition $S$. Also, their condition appears to be stronger than ($*$), since it is required to hold for certain small graphs. It is equivalent to the condition presented here, as we show in the discussion that follows Example 1 below. Their theorem also requires a technical "triplet linear independence" assumption, which we discuss below.

[2] For example, $\mathcal{M} = \{p \in \mathbb{R}^m : \sum_{k=1}^m p_k \leq 1, p_k \geq 0\}$ could represent probability mixtures between $m+1$ outcomes. In this case and for $m = 2$, $\mathcal{M}$ is the Marshak triangle.

[3] Having agreement on probabilities puts aside the dilemma between maintaining the Pareto rule but having no group beliefs (Hylland and Zeckhauser, 1979; Mongin, 1995; Nau, 2006), or keeping group beliefs but violating the Pareto rule when individual beliefs differ (Gilboa et al., 2004).

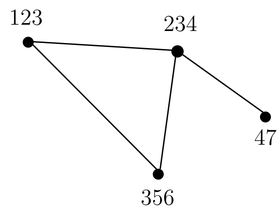


**Fig. 1.** The connected graph associated with $\mathcal{E} = \{123, 234, 356, 47\}$. An edge connects any two nodes (coalitions) that have at least one individual in common.

**Individual Prospect** (IP)

For each individual $i$ there exists a pair of prospects $p, q \in \mathcal{M}$ such that $p \succ_i q$ and $p \sim_j q$ for all $j \neq i$.

The first property is called extended Pareto because it implies Harsanyi's Pareto requirement that when all individuals agree on the preference between two given prospects, then this will be the preference of the grand coalition. The second property, which distinguishes this work from Baucells and Shapley (2008), requires that each individual has some prospects that only she cares about.

All our results consider a coalition preference satisfying EP and IP. The following proposition is the key to our main result.

**Proposition 1.** *If $\succsim_A$ and $\succsim_B$ are complete and $A \cap B \neq \emptyset$, then $\succsim_{A\cup B}$ is complete.*

We extend Proposition 1 to a set of coalitions $\mathcal{E} = \{T_1, ..., T_k\}$ by associating $\mathcal{E}$ with a graph $G_{\mathcal{E}}$ with $k$ nodes, one for each coalition in $\mathcal{E}$, where the edges are the pairs $(T_i, T_j)$ with $i \neq j$ for which $T_i \cap T_j \neq \emptyset$. A set of coalitions $\mathcal{E}$ is *connected* if its associated graph $G_{\mathcal{E}}$ is connected.

**Theorem 1.** *Let $\mathcal{E}$ be a connected set where for each $T \in \mathcal{E}$, $\succsim_T$ is complete. Then for $S = \cup_{T\in\mathcal{E}} T$, the preference relation $\succsim_S$ is complete.*

**Example 1.** Let $n = 7$. We denote a coalition by the list of its members. Thus, for example, the coalition $\{1, 2, 3\}$ is denoted by 123. Let $\mathcal{E} = \{123, 234, 356, 47\}$. The edges of the associated graph are (123, 234), (123, 356), (234, 356), and (234, 37), yielding the connected graph shown in Fig. 1. If all the coalitions in $\mathcal{E}$ have complete preference relations, then 1234567 also has a complete preference relation.

Of course, Theorem 1 also applies to any connected set $\mathcal{E}' \subseteq \mathcal{E}$. In Example 1, the subgraph associated with $\mathcal{E}' = \{123, 356\}$ is connected, and therefore 12356 also possesses a complete preference relation.

The main result of Baucells and Shapley (2008) concerns coalitions of size two with complete preference relations, and graphs with nodes that are individuals. Our main result, in Theorem 1, concerns coalitions of any size that have complete preference relations and the graphs with nodes that are coalitions. In order to compare these two results we consider the following three conditions.

1. There exists a connected set of coalitions $\mathcal{E}$ such that $\cup_{T\in\mathcal{E}} T = S$, and for each $T \in \mathcal{E}$, $T$ has two members and $\succsim_T$ is complete.
2. There exists a connected graph $G_S$, the nodes of which are the individuals in $S$ and the edges are pairs of individuals, such that for each edge $\{i, j\}$ of the graph, $\succsim_{\{i,j\}}$ is complete.
3. There exists a graph $G_S$ as in condition 2, with $n - 1$ edges.

These three conditions are equivalent. To see this, assume that 1 holds. Define a graph $G_S$ such that $\{i, j\}$ is an edge when $\{i, j\} = T$ for some $T \in \mathcal{E}$. It is easy to see that this graph is connected and therefore 2 holds. When 2 holds, define $\mathcal{E}$ to be the set of all pairs $\{i, j\}$ that are edges in $E$. Again, it is straightforward to see that $\mathcal{E}$ is connected and therefore 1 holds. Finally, if 2 holds, then a spanning tree for $G_S$ satisfies 2 and it has $n - 1$ edges.

Both the main theorem of Baucells and Shapley (2008) and our Theorem 1 give conditions that guarantee the completeness of $\succsim_S$. The stipulation in Baucells and Shapley (2008) is condition 3, while the stipulation in Theorem 1 is more general than condition 1 which is equivalent to 3. The weaker stipulation of Theorem 1 suffices to reach the same conclusion as Baucells and Shapley (2008) because of the additional assumption of IP.

## 3. Utility representation

Preference relations can be described in terms of linear functions on $\mathcal{M}$. It is this description of preferences that we use to prove our claims.

Any vector $u \in \mathbb{R}^m$ can be viewed as a linear function $u : \mathcal{M} \to \mathcal{R}$ by defining $u(p) = \langle u, p \rangle$, where the latter is the scalar product of $u$ and $p$. The vector $u$, in its role as a linear function on $\mathcal{M}$, is called a *utility*. A utility $u$ defines a binary relation $\succsim^u$ on $\mathcal{M}$:

**Definition 2.** For utility $u$, the binary relation $\succsim^u$ is defined by $p \succsim^u q$ whenever $u(p) \geq u(q)$.

A non-empty, closed, and convex cone $U \subseteq R^m$ is called a *utility cone*. A utility cone $U$ defines a binary relation $\succsim^U$ on $\mathcal{M}$:

**Definition 3.** For a utility cone $U$, the binary relation $\succsim^U$ is defined by $p \succsim^U q$ whenever for all $u \in U$, $u(p) \geq u(q)$ or equivalently, $p \succsim^u q$.

We can rephrase Definition 2 in terms of utility cones rather than utilities. We say that the utility cone $U$ is a *ray*, if $U = \{tu \mid t \geq 0\}$ for some utility $u$. We say in this case that $U$ is *generated* by $u$.

**Observation 1.** *If the utility cone $U$ is a ray generated by $u$, then $\succsim^U = \succsim^u$.*

The following theorem, which expresses completeness and incompleteness of preference relations in terms of utility, is stated in Baucells and Shapley (2008), and is comparable to Aumann (1962), Shapley and Baucells (1998), Dubra et al. (2004), and Galaabaatar and Karni (2012).

**Proposition.**

- *A preference relation $\succsim$ is incomplete if and only if $\succsim = \succsim^U$ for some utility cone $U$.*
- *A preference relation $\succsim$ is complete if and only if $\succsim = \succsim^u$ for some utility $u$, or equivalently, $\succsim = \succsim^U$ for a utility cone $U$ generated by a utility.*

Note, that the trivial preference relation, which is complete, is defined by $u = 0$ or equivalently by the cone generated by 0, namely $\{0\}$.

In view of this theorem, we can describe coalition preference in terms of utilities.

**Corollary 1.** *An assignment of a binary relation $\succsim_S$ on $\mathcal{M}$ to each coalition $S$ is a coalition preference if for each $S$, $\succsim_S = \succsim^{U_S}$ for some utility cone $U_S$, and for each singleton $\{i\}$, $\succsim_i = \succsim^{U_i}$ for some utility cone $U_i$ generated by a utility, or equivalently, $\succsim_i = \succsim^{u_i}$ for some utility $u_i$.*

The following characterization of the EP rule in terms of utility cones is a somewhat simplified version of the one given in Baucells and Shapley (2008). We recall that a vector $u$ is in the relative interior of $U$, $Ri(U)$, if for each $v \in U$ there exists $\alpha > 1$ such that $\alpha u + (1 - \alpha)v \in U$.

**Proposition.** *The* EP *rule holds if and only if for any two disjoint coalitions $A$ and $B$,*

$$U_{A \cup B} \subseteq U_A + U_B, \text{ and} \tag{1}$$

$$U_{A \cup B} \cap Ri(U_A + U_B) \neq \emptyset. \tag{2}$$

IP can also be described in terms of individuals' utilities.

**Proposition 2.** IP *holds if and only if the utilities of the individuals are linearly independent.*

In light of Proposition 2, in order for IP to hold, the dimension of the space of utilities and the set of prospects, $m$, should be at least as large as the number of individuals, $n$.

Since the trivial preference is defined by the 0 utility, we conclude from Proposition 2:

**Corollary 2.** *The individuals' preferences $\succsim_i$ are not trivial.*

The following proposition, which follows immediately from Corollary 2 and Lemma 3 in Section 5, extends Corollary 2.

**Proposition 3.** *For each coalition $S$, $\succsim_S$ is not trivial.*

By Proposition 2, IP can hold only if the number of individuals $n$ is not lower than the dimension of $\mathcal{M}$, which we have assumed to be the case. In Baucells and Shapley (2008), where IP was not assumed, linear independence of the utilities of

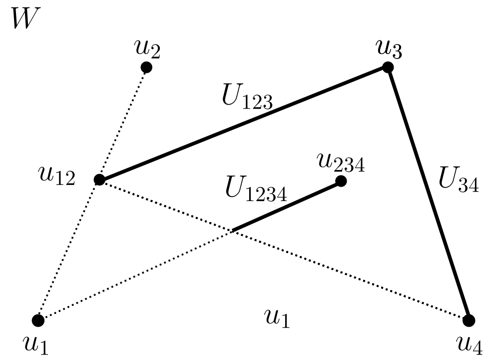


**Fig. 2.** $\succsim_{12}$ and $\succsim_{234}$ are complete, but $\succsim_{1234}$ is incomplete.

each three individuals was assumed. However, this property was not described in terms of preferences. Here independence holds for the set of all individuals' utilities, and this property is described by IP solely in terms of the preferences.

Finally, using the description of coalition preferences in terms of utility, we can strengthen Theorem 1 by relating the complete preference of the union of the elements of $\mathcal{E}$ with the complete preference associated with the union of the elements of any connected subset of $\mathcal{E}$. Recall that a coalition preference assumes complete preferences for individuals, and hence a set of individual utilities, $u_i$, $i \in S$.

**Theorem 2.** *Let $\mathcal{E}$ be a connected set where for each $T \in \mathcal{E}$, $\succsim_T$ is complete. Then there are weights $\lambda_i > 0$, for all $i \in S = \cup_{T \in \mathcal{E}} T$, such that $u_S = \sum_{i \in S} \lambda_i u_i$ generates the cone $U_S$. Moreover, if $\mathcal{E}' \subset \mathcal{E}$ is a connected set and $S' = \cup_{T \in \mathcal{E}'} T$, then $u_{S'} = \sum_{i \in S'} \lambda_i u_i$ generates the cone $U_{S'}$.*

## 4. A counterexample

In the following example there are four individuals. The utilities of any three of them are independent, which is the assumption in Baucells and Shapley (2008). However, the utilities of all four individuals are dependent. In the example there is a connected set of coalitions each of which has complete preferences. The union of these coalitions is all four individuals. The conclusion of Theorem 1, that the grand coalition has a complete preference, fails to hold. This example demonstrates that the property of IP, which is equivalent to the independence of individual's utilities, cannot be omitted in Theorem 1. Note, that the connected set in the example necessarily has a coalition with more than two players, because if all of them were pairs, the grand coalition would have complete preferences by Baucells and Shapley (2008).

**Example 2.** Consider a coalition preference with four individuals and a set of prospects in $\mathbb{R}^3$. Thus, the utility cones are also in $\mathbb{R}^3$. Obviously, the four utilities of the individuals cannot be linearly independent. However, any three utilities are independent, which is the requirement in Baucells and Shapley (2008). The utility cones of the coalitions are described by their intersection with a plane $W$ that does not contain the origin, and is depicted in Fig. 2. We denote by $u_S$ the cones of coalitions $S$ with complete preferences, namely all the singletons, as well as the coalitions 12 and 234, and denote by $U_S$ the cones of coalitions $S$ that have incomplete preferences, namely coalitions 123, 34, and 1234. The cones of the coalitions 13, 14, 23, 24, 134, and 124 are not displayed, and we set for each $S$ of this family, $U_S = \sum_{i \in S} U_i$. Note, that the EP rule holds because $U_{1234}$ is equal to the intersection of $u_1 + U_{234}$, $u_{12} + U_{34}$, and $U_{123} + u_4$. For all other decompositions of 1234, to coalitions $S$ and $T$ which are not depicted, the sum $U_S + U_T$ will be the cone generated by $u_i$, $i \in S$, of which $U_{1234}$ is a subset. Now, $\succsim_{12}$ and $\succsim_{234}$ are complete because their cone is defined by a single utility, but $\succsim_{1234}$ is incomplete because $U_{1234}$ is not generated by a utility, that is, it is not a point in $W$. Thus, without linear independence the conclusion of Theorem 1 fails to hold.

## 5. Proofs

Proposition 1 is formulated in terms of the coalition preferences. However, to prove it we use the utility representation of these preferences, and in particular the linear independence of individual utilities stated in Proposition 2, which we prove first.

**Proof of Proposition 2.** Let $L_S$ be the linear space spanned by the vectors $u_i$ for $i \in S$.

Assume that IP holds but linear independence does not. Then, for some $i$, $u_i \in L_{N \setminus \{i\}}$. Suppose that $p \sim_j q$ for all $j \neq i$. Then, for all $j \neq i$, $u_j(p) = u_j(q)$ and hence $\langle u_j, p - q \rangle = 0$. Thus, $p - q \in L^{\perp}_{N \setminus \{i\}}$. However, $L_N = L_{N \setminus \{i\}}$, and thus $p - q \in L^{\perp}_N$. Therefore, $u_i(p) = u_i(q)$, and $p \sim_i q$, contradicting IP.

Suppose that linear independence holds. Then, for each $i$, the projection of $u_i$ on $L^{\perp}_{N\setminus\{i\}}$ is different from zero. By the full dimensionality of $\mathcal{M}$, we can choose $p$ in the interior of $\mathcal{M}$ and $q \neq p$ in the ball around $p$ such that $p-q$ is collinear with this projection. Thus, $\langle u_i, p-q\rangle > 0$, and hence $p \succ_i q$. However, since $p-q \in L^{\perp}_{N\setminus\{i\}}$, it follows that for all $j \neq i$, $\langle u_j, p-q\rangle = 0$ and hence $p \sim_j q$. Thus, IP holds. $\square$

**Lemma 1.** *For each $S$, $U_S \subseteq \sum_{i\in S} U_i$ and thus for each $u \in U_S$ there are $\lambda_i \geq 0$ for $i \in S$ such that $u = \sum_{i\in S} \lambda_i u_i$.*

**Proof.** Prove by induction on the size of the coalition $S$, using the EP rule, Equation (1). $\square$

**Lemma 2.** *For each coalition $S$ and $T \subset S$, if $\sum_{i\in S} \lambda_i u_i \in U_S$, then $\sum_{i\in T} \lambda_i u_i \in U_T$.*

**Proof.** Suppose $\sum_{i\in S} \lambda_i u_i \in U_S$. By (1), $U_S \subseteq U_T + U_{S\setminus T}$. Therefore, $\sum_{i\in S} \lambda_i u_i = v + w$, where $v \in T$ and $w \in S \setminus T$. By Lemma 1, $v = \sum_{i\in T} \alpha_i u_i$ with $\alpha_i \geq 0$ for each $i \in T$, and $w = \sum_{i\in S\setminus T} \alpha_i u_i$ with $\alpha_i \geq 0$ for each $i \in S \setminus T$. By linear independence, $\lambda_i = \alpha_i$ for each $i \in S$, implying that $\sum_{i\in T} \lambda_i u_i = v \in U_T$. $\square$

**Lemma 3.** *For each coalition $S$, there are $\lambda_i > 0$ for all $i \in S$, such that $\sum_{i\in S} \lambda_i u_i \in U_S$.*

**Proof.** The proof is by induction on the size of the coalition. The claim holds trivially for $|S| = 1$. For $|S| \geq 2$, assume it holds for coalitions smaller than $S$. Let $\emptyset \subset T \subset S$. Then, $S$ is the disjoint union of the smaller coalitions $T$ and $S \setminus T$. By Equation (2), we can choose $u \in U_S \cap Ri(U_T + U_{S\setminus T})$. By Lemma 1, write $u = \sum_{i\in S} \alpha_i u_i$, with $\alpha_i \geq 0$ for each $i \in S$. By the induction hypothesis, choose $x = \sum_{i\in T} \lambda_i u_i \in U_T$, with $\lambda_i > 0$ for each $i \in T$; and $y = \sum_{i\in S\setminus T} \lambda_i u_i \in U_{S\setminus T}$, with $\lambda_i > 0$ for each $i \in S \setminus T$. Use $x + y \in U_T + U_{S\setminus T}$ and the definition of relative interior to conclude that $\alpha u + (1-\alpha)(x+y) \in U_T + U_{S\setminus T}$ for some $\alpha > 1$. Thus, $\alpha u + (1-\alpha)(x+y) = v + w$ for $v \in U_T$ and $w \in U_{S\setminus T}$. Again, by Lemma 1, $v = \sum_{i\in S} \beta_i u_i$ and $w = \sum_{i\in S\setminus T} \beta_i u_i$ with $\beta_i \geq 0$ for each $i \in S$. Thus, $\alpha u + (1-\alpha)(x+y) = \sum_{i\in S} \beta_i u_i$. By linear independence, for each $i \in S$, $\alpha\alpha_i + (1-\alpha)\lambda_i = \beta_i \geq 0$. Therefore, $\alpha_i = (\beta_i + (\alpha-1)\lambda_i)/\alpha > 0$, $i \in S$, and the result follows. $\square$

**Proof of Proposition 1.** Assume that $A \cap B \neq \emptyset$, and that the cones $U_A$ and $U_B$ are rays. By Lemma 3, there is a vector $\sum_{i\in A} \alpha_i u_i$ in $U_A$ such that $\alpha_i > 0$, for all $i \in A$, and a vector $\sum_{i\in B} \beta_i u_i$ in $U_B$ with $\beta_i > 0$ for all $i \in B$. We show that $U_{A\cup B}$ is also a ray. Let $u \in U_{A\cup B}$. By Lemma 1, $u = \sum_{i\in A\cup B} \lambda_i u_i$. By Lemma 2, $\sum_{i\in A} \lambda_i u_i \in U_A$ and $\sum_{i\in B} \lambda_i u_i \in U_B$. Since $U_A$ and $U_B$ are rays, there are $s, t \geq 0$ such that

$$\sum_{i\in A} \lambda_i u_i = s \sum_{i\in A} \alpha_i u_i \quad \text{and} \quad \sum_{i\in B} \lambda_i u_i = t \sum_{i\in B} \beta_i u_i.$$

By linear independence, $\lambda_i = s\alpha_i$ for each $i \in A$, and $\lambda_i = t\beta_i$ for each $i \in B$. Let $i^* \in A \cap B$. Then, $\lambda_{i^*} = s\alpha_{i^*} = t\beta_{i^*}$. Since $\beta_{i^*} > 0$, $t = s\alpha_{i^*}/\beta_{i^*}$. Thus,

$$\begin{aligned}\sum_{i\in A\cup B} \lambda_i u_i &= \sum_{i\in A} \lambda_i u_i + \sum_{i\in B\setminus A} \lambda_i u_i \\ &= s\sum_{i\in A} \alpha_i u_i + t \sum_{i\in B\setminus A} \beta_i u_i \\ &= s\sum_{i\in A} \alpha_i u_i + s \sum_{i\in B\setminus A} \frac{\alpha_{i^*}}{\beta_{i^*}} \beta_i u_i\end{aligned}$$

Denote for $i \in B \setminus A$, $\alpha_i = \alpha_{i^*}\beta_i/\beta_{i^*}$. Then, $u = \sum_{i\in A\cup B} \lambda_i u_i = s\sum_{i\in A\cup B} \alpha_i u_i$. Therefore, each vector $u$ in $U_{A\cup B}$ is collinear with $\sum_{i\in A\cup B} \alpha_i u_i$, which shows that $U_{A\cup B}$ is a ray. $\square$

**Proof of Theorem 1.** We prove by induction on the size of $\mathcal{E}$. The claim holds trivially for $k = 1$. Suppose the claim holds for $k$ and consider a connected set $\mathcal{E} = \{T_1, \ldots, T_{k+1}\}$. We can assume without loss of generality that $T_1 \cap T_2 \neq \emptyset$. Consider the set $\mathcal{E}' = \{T_1 \cup T_2, \ldots, T_{k+1}\}$. By Proposition 1, $T_1 \cup T_2$ has complete preferences. The graph associated with $\mathcal{E}'$ is connected and hence by the induction hypothesis for $T = \cup_{i=1}^{k+1} T_i$, $\succsim_T$ is complete. $\square$

**Proof of Theorem 2.** By Lemma 3 there are $\lambda_i > 0$ for all $i \in S$ such that $u_S = \sum_{i\in S} \lambda_i u_i$ is in $U_S$. Since, by Theorem 1, $U_S$ is generated by a utility vector, it is generated by $u_S$. As $S' \subseteq S$, it follows by Lemma 2 that $u_{S'} = \sum_{i\in S'} \lambda_i u_i$ is in $U_{S'}$ and again, by Theorem 1, it follows that $u_{S'}$ generates $U_{S'}$. $\square$

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