An Extension of Ceva's Theorem to n-Simplices

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Abstract. Ceva's theorem, which concerns triangles, is a central result of post-Euclidean plane geometry. The three-dimensional generalization of a triangle is a tetrahedron, and the n-dimensional generalization of these is an n-simplex. We extend Ceva's theorem to n-simplices and in doing so illustrate the considerations and choices that can be made in generalizing from plane geometry to high-dimensional geometries.

1. CEVA'S THEOREM. Ceva's theorem is a central result of post-Euclidean geometry. The theorem was first formulated and proved in the 11th century by Yusuf al-Mu'taman ibn Hud who was a king in Zaragoza. However, it became known in Europe from the proof of the mathematician Giovanni Ceva in the 17th century. Evidence of the centrality of this theorem is that it is the second theorem stated in the book *Geometry Revisited* [**2**] by the famous geometers Coxeter and Grietzer.

The theorem concerns a triangle $p_0p_1p_2$ with vertices p_0, p_1, and p_2, which is depicted in Figure 1. The points p_{01}, p_{12}, and p_{20} are on the sides p_0p_1, p_1p_2, and p_2p_0, respectively. The line segments p_0p_{12}, p_1p_{20}, and p_2p_{01} are called *cevians*. The end point of the cevian that is on the side of the triangle is called the *foot* of the cevian. The foot of a cevian divides the side on which it lies into two line segments. Thus, for example, p_{01} divides the side p_0p_1 into the two line segments p_0p_{01} and $p_{01}p_1$. The ratio of the lengths of these two line segments is $|p_0p_{01}|/|p_{01}p_1|$. Ceva's theorem asserts that the three cevians concur at one point if and only if

$$\frac{|p_0p_{01}|}{|p_{01}p_1|}\frac{|p_1p_{12}|}{|p_{12}p_2|}\frac{|p_2p_{20}|}{|p_{20}p_0|} = 1.$$

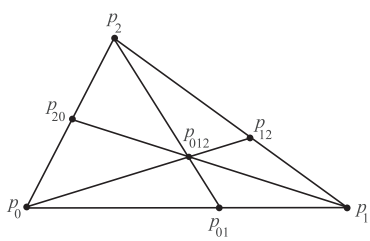

Figure 1. Illustration of Ceva's theorem.

Our purpose is to extend Ceva's theorem, which is part of plane geometry, to high-dimensional geometries. We emphasize the considerations and techniques that are used for such an extension and present the alternative possible extensions of the notions of

doi.org/10.1080/00029890.2021.1896292
MSC: Primary 51M04

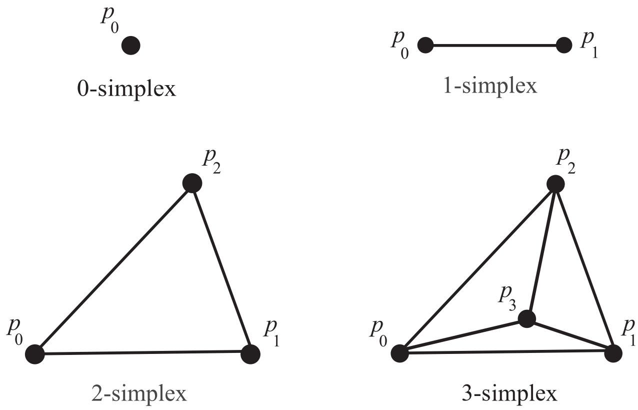

Figure 2. n-simplices for $n = 0, 1, 2, 3$.

plane geometry. Thus, in addition to the particular extension presented here, we aim to provide terminology and tools that can help in thinking of plane geometry as a special case of high-dimensional geometry. Knowledge of n-dimensional real space suffices to follow the arguments of this article.

2. FROM TRIANGLES TO n-SIMPLICES.

The object that Ceva's theorem deals with is a triangle. The first question we should ask when we try to extend the theorem is what are the high-dimensional objects to which the extension will apply. For this we look at the special properties of a triangle that distinguish it from other objects in the plane, say squares or circles. As opposed to a circle, a triangle is a polygon. As opposed to a square, a triangle is a two-dimensional polygon with the smallest number of vertices. The vertices of the triangle are three points that are not collinear. The extension of a polygon to three-dimensional space is a polytope. The minimal three-dimensional polytope must have four vertices that do not lie on a plane. This object is a tetrahedron and it can be easily visualized. There are four subsets of three vertices of the tetrahedron. Each of them is a triangle and they form the sides of the tetrahedron.

An n-simplex is defined in a plane of dimension n in some real space, similarly to a triangle in a plane and a tetrahedron in a three-dimensional space. An n-simplex has $n + 1$ vertices, $p_0, \dots, p_n$, which do not lie in an $(n - 1)$-dimensional plane. This is equivalent to saying that the n vectors $p_0 - p_1, p_0 - p_2, \dots, p_0 - p_n$ are linearly independent. Thus, a 3-simplex is a tetrahedron, a 2-simplex is a triangle, a 1-simplex is a line segment, and a 0-simplex is a point (see Figure 2). Any subset of $k + 1$ vertices of the n-simplex, for $0 \le k \le n$, forms a k-simplex. We call such a k-simplex a *k-face* of the n-simplex. Thus, the n-face of an n-simplex S is S. An $(n - 1)$-face is called a *facet*. We say that a facet is *opposite* the vertex that is not included in the facet. An n-simplex has $n + 1$ facets. A 2-simplex in S is called an *edge* of S. Finally, the 0-faces of S are its vertices. The edges of a 2-simplex, i.e., a triangle, are also its facets. In a 3-simplex, i.e., a tetrahedron, the facets are four triangles and they are different from the edges.

There are several extensions of Ceva's theorem to n-simplices: see [**1**, **3**–**5**].

3. CEVIANS.

After fixing n-simplices as the objects to which we want to extend Ceva's theorem, we need to know how we define cevians for n-simplices. A cevian in a 2-simplex connects a vertex to an edge and is 1-dimensional. We can define a cevian in the n-simplex in exactly the same way. But after reflecting on this possibility, we do

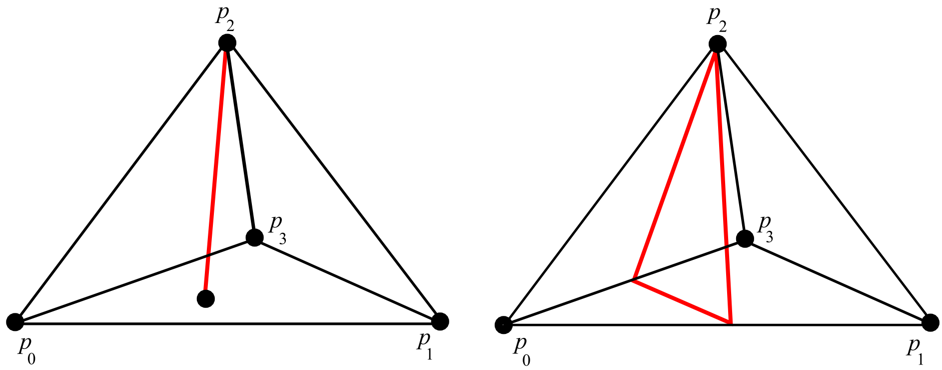

Figure 3. Two possible extensions of cevians. Left: In [3] and in this article. Right: In [1, 4, 5].

not see anything of interest that we can say for such cevians. Remembering that the edges of the 2-simplex are also its facets, we can define a cevian to be a line segment that connects a vertex with a point on the facet opposite the vertex. This is how [3] defines a cevian in an n-simplex. Yet another alternative is to note that the dimension of a cevian in a 2-simplex is one less than the dimension of the 2-simplex. Thus, we can define a cevian in an n-simplex to be an $(n-1)$-dimensional plane that connects a vertex to the facet opposite the vertex. This is the way a cevian is defined in [4] and [5] for $n = 4$ and in [1] for general n.

In this article, we adopt the approach of [3] and define a cevian in an n-simplex $S = p_0 \cdots p_n$ to be a line segment that connects a vertex p_i of S and a point, called the *foot* of the cevian, in the interior of the facet opposite p_i. The two ways of defining cevians are depicted for tetrahedra in Figure 3.

4. FEET. Ceva's theorem concerns three points, one in each of the interiors of the sides of a triangle S. With an eye to the extension to an n-simplex, we refer to these points as 1-feet because they lie on a 1-simplex. We say that these points are *induced* by a point p_S in the interior of the triangle when they are the feet of the cevians that pass through p_S. Ceva's theorem gives a necessary and sufficient condition for such three points to be induced by a point p_S.

In the obvious extension to a tetrahedron S, we consider four points, one in each of the interiors of the facets of S, and look for a necessary and sufficient condition for these points to be induced by a point p_S in the interior of S. Figure 4 depicts three points out of the four. They are the feet of three cevians, depicted in red, which concur at p_S. That is, these points are induced by p_S. In [3], $n + 1$ points are considered, one in each of the interiors of the facets of an n-simplex. A necessary and sufficient condition is given for such $n + 1$ points to be induced by a point inside the simplex. However, the condition there is not formulated in terms of a product of ratios.

Here we consider a set of points larger than the set of $n + 1$ points studied in [3]. In our theory, the set that extends the set of three points in Ceva's theorem includes one point in each of the interiors of the k-faces of an n-simplex, for all $1 \le k \le n - 1$. We call such a set a *multipede* and define what it means for a point in the interior of the simplex to *induce* a multipede. The set of three points on the sides of a triangle in Ceva's theorem is a special case of a multipede for $n = 2$.

We generalize Ceva's theorem to n-simplices by giving a necessary and sufficient condition for a multipede to be induced by a point in the interior of the simplex. This condition is given in terms of a product of certain ratios defined by the points of the multipede. Ceva's theorem turns out to be a special case of our main theorem for $n = 2$.

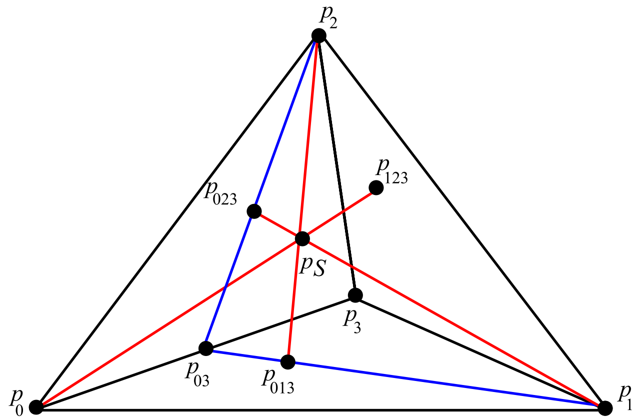

Figure 4. Feet induced by p_S on various k-faces of a tetrahedron.

Before we formally define the inducement of a multipede by a point in the interior of a simplex, we illustrate it in Figure 4 for a tetrahedron. The point p_S in the interior of the tetrahedron induces four 2-feet on the 2-faces of the tetrahedron. Three of these points are depicted. For example, the point p_{013} is the 2-foot of the cevian $p_2 p_S$, depicted in red, on the 2-face $p_0 p_1 p_3$. The point p_{013}, being in the interior of the 2-face $p_0 p_1 p_3$, induces the point p_{03} on the 1-face $p_0 p_3$, which is the foot of the cevian $p_1 p_{013}$, depicted in blue. The point p_{03} is also in the multipede induced by p_S.

Next, we formally define multipedes and their inducement. We denote by $\mathcal{F}$ the family of faces of S of dimension k with $1 \leq k \leq n-1$.

Definition. A *multipede* is a set of points $\{p_F\}_{F \in \mathcal{F}}$ where for each F, the point p_F is in the interior of F. A point p_S in the interior of S *induces* the multipede if the feet of the cevians in S that pass through p_S are the points p_G where G is a facet of S, and for each face $F \in \mathcal{F}$ of dimension $k > 1$ the feet of the cevians in F that pass through p_F are the points p_G where G is a facet of F.

For the case of a triangle ($n = 2$), any point in the interior of the triangle induces one and only one multipede. The next proposition generalizes this claim for n-simplices. Before we state and prove this proposition we explain why it is not as straightforward for $n \geq 3$ as it is for $n = 2$.

It seems at first glance that exactly as in the case for $n = 2$ we can construct the unique multipede induced by p_S. The construction is illustrated in Figure 4. The 2-feet induced by p_S on the facets of S are determined by p_S. The foot on each facet of S, for example p_{013} on $p_0 p_1 p_3$, determines the 1-feet of the multipede in this facet. For larger n, this construction can go on until the 1-faces are reached. While this construction shows the *uniqueness* of the multipede induced by a point p_S, that is, that there cannot be more than one multipede induced by p_S, it fails to show that there *exists* such a multipede. The reason is that a k-face of an n-simplex can be reached by different cevians. In Figure 4, for example, the 1-face $p_0 p_3$ can be reached by the cevian $p_0 p_{013}$ but also by the cevian $p_2 p_{023}$. Since a multipede should contain only one point on the 1-face $p_0 p_3$, the two cevians should end at the same point in order to define a multipede. In Figure 4, they do end at the same point p_{03}, but this does not follow straightforwardly from the construction.

We prove the existence of a multipede induced by a point p_S by describing explicitly the points of the multipede in terms of the barycentric coordinates of p_S. First we recall the notion of barycentric coordinates. Let $S = p_0 \cdots p_d$ be a d-simplex. Then each point p in S can be written as a combination $p = \sum_{i=0}^{d} \beta_i p_i$, where $\beta_i \geq 0$ for $i = 1, \ldots, d$ and $\sum_{i=0}^{d} \beta_i = 1$. The coefficients $\beta_0, \ldots, \beta_d$ are uniquely determined by p and they are called the *barycentric coordinates* of p. The converse also holds: if $\beta_i \geq 0$ for $i = 1, \ldots, d$ and $\sum_{i=0}^{d} \beta_i = 1$, then $p = \sum_{i=0}^{d} \beta_i p_i$ is in S. A point p is in the interior of S if and only if its barycentric coordinates are strictly positive.

Proposition 1. *Each point p_S in the interior of an n-simplex S induces one and only one multipede.*

Proof. We first prove the existence of a multipede induced by a point p_S in the interior of the n-simplex $S = p_0 \cdots p_n$. Let $\beta_0, \ldots, \beta_n$ be the barycentric coordinates of p_S. For each face $F \in \mathcal{F}$, define

$$p_F = \frac{\sum_{p_i \in F} \beta_i p_i}{\sum_{p_i \in F} \beta_i}. \tag{1}$$

Observe first that since p_S is in the interior of S, $\beta_i > 0$ for each i, and therefore the multipede is well-defined. We show that $\{p_F\}_{F \in \mathcal{F}}$ is a multipede induced by p_S. Consider a face F of dimension $d > 1$ and facet G of F opposite the vertex $p_i \in F$. Then $G \in \mathcal{F}$. Let $\alpha = \sum_{p_j \in G} \beta_j / \sum_{p_j \in F} \beta_j$. Thus $p_F = \alpha p_G + (1 - \alpha) p_i$. Hence p_i, p_F, and p_G are collinear, and therefore the cevian $p_i p_G$ passes thorough p_F.

To see that only one multipede can be induced by p_S, note that the $(n-1)$-feet of the multipede are, by definition, the feet of the cevians that go through p_S, which are uniquely determined. For each facet F of S, the $(n-2)$-feet in F are, by definition, the feet of the cevians that go through p_F, which are uniquely determined, and so on. ∎

We return to Ceva's theorem and prove the "only if" part of the theorem that makes use of the barycentric coordinates of p_{012}. This will lead us to the last two steps in extending the theorem to n-simplices.

Ceva's theorem (The "only if" part). *If a multipede $\{p_{01}, p_{12}, p_{20}\}$ in the triangle $p_0 p_1 p_2$ is induced by a point p_{012} in the interior of the triangle, then*

$$\frac{|p_0 p_{01}|}{|p_{01} p_1|} \frac{|p_1 p_{12}|}{|p_{12} p_2|} \frac{|p_2 p_{20}|}{|p_{20} p_0|} = 1.$$

Proof. The proof is carried out in three steps.

Step 1: Let $\beta_0, \beta_1, \beta_2$ be the barycentric coordinates of p_{012}. By Proposition 1, $p_{01} = (\beta_0/(\beta_0 + \beta_1)) p_0 + (\beta_1/(\beta_0 + \beta_1)) p_1$. Similar equations hold for p_{12} and p_{20}.

Step 2: By Step 1, p_{01} divides the interval (1-simplex) $p_0 p_1$ into two subintervals (1-simplices) $p_0 p_{01}$ and $p_{01} p_1$ with a ratio of lengths

$$\frac{|p_0 p_{01}|}{|p_{01} p_1|} = \frac{\beta_1}{\beta_0}. \tag{2}$$

Similar equations hold for the other two sides of the triangle.

Step 3: By Step 2, multiplying the ratio of each side along the cycle of the sides results in

$$\frac{|p_0p_{01}|}{|p_{01}p_1|}\frac{|p_1p_{12}|}{|p_{12}p_2|}\frac{|p_2p_{20}|}{|p_{20}p_0|} = \frac{\beta_1}{\beta_0}\frac{\beta_2}{\beta_1}\frac{\beta_0}{\beta_2} = 1.$$

■

5. LOBES. We now extend Step 2 in the proof to n-simplices. The 1-foot p_{01} divides the 1-simplex p_0p_1 it lies on into two 1-simplices which we will call 1-*lobes*. In the mulitpede of an n-simplex, with $n > 2$, we have k-feet for $k > 1$. Such feet also partition the k-simplex they lie on into $k + 1$ k-lobes. Figure 5 illustrates a partition of a 1-simplex into two 1-lobes by a 1-foot, and a partition of a 2-simplex into three 2-lobes by a 2-foot.

Lobes are defined in general as follows.

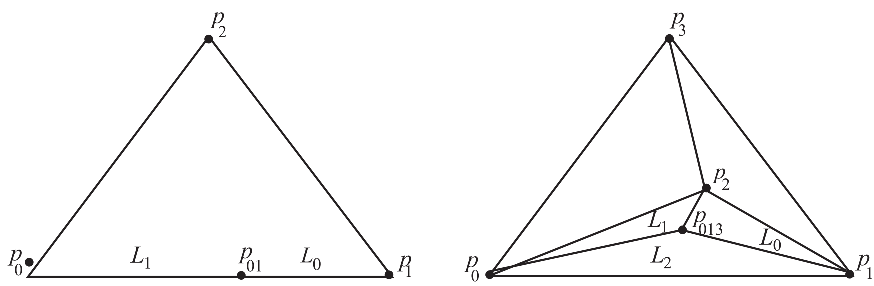

Figure 5. Partition into 1-lobes by a 1-foot on the left and into 3-lobes by a 2-foot on the right.

Definition. Let $F \in \mathcal{F}$ be a k-face with a vertex p_i, and p a point in the interior of F. Let L_i be the k-simplex whose vertices are p and all the vertices of F other than p_i. We call L_i the k-*lobe* of F at p, and say that L_i is opposite vertex p_i.

As 1-lobes are intervals, they have lengths, and the ratio of these lengths plays a role in Ceva's theorem. 2-lobes are two-dimensional and they have area. Thus, the natural extension of the ratio of lengths of 1-lobes to 2-lobes is the ratio of their areas. The extension of length and area to three-dimensional objects is volume, and we use this word for n-dimensional objects.

We denote the volume of a k-simplex S by $\mathrm{Vol}(S)$. Note that if S is in a space of dimension larger than k, then its volume in this space is 0. But S can be embedded isometrically (that is, with distance preserved) in $\mathbb{R}^k$. The volume of S we refer to is its volume as a k-dimensional object in $\mathbb{R}^k$. The volume of S can be computed by a determinant as follows. We recall that the determinant is a real-valued function, which we denote by det, defined on arrays of k vectors in $\mathbb{R}^k$. The function det is multilinear in the following sense. For vectors $v_1, \ldots, v_k$, scalar α and vector u, $\det(v_1, \ldots, v_i + u, \ldots, v_k) = \det(v_1, \ldots, v_i, \ldots, v_k) + \det(v_1, \ldots, u, \ldots, v_k)$, and $\det(v_1, \ldots, \alpha v_i, \ldots, v_k) = \alpha\det(v_1, \ldots, v_i, \ldots, v_k)$. When two of the vectors v_i are the same, then the determinant is 0. These are the properties of the determinant we use in the sequel. Now it turns out that if $S = p_0 \cdots p_k$, then $\mathrm{Vol}(S) = |\det(p_1 - p_0, p_2 - p_0, \ldots, p_k - p_0)|/k!$.

Step 2 in the proof of Ceva's theorem culminates in equation (2) which states that the ratio of the lengths of two 1-lobes is the ratio of the barycentric coordinates of the vertices of the 1-simplex containing these lobes. The extension of this property to

k-lobes in equation (3) is well known and its short proof is included here for completeness.

Proposition 2. *Let L_i and L_j be k-lobes of F at p opposite a vertex p_i and p_j, respectively. Let β_i and β_j be the coordinates of p_i and p_j, respectively, in the barycentric representation of p in F. Then*

$$\frac{\mathrm{Vol}(L_i)}{\mathrm{Vol}(L_j)} = \frac{\beta_i}{\beta_j}. \tag{3}$$

Proof. Assume, without loss of generality, that F is the simplex $p_0 \cdots p_k$ in $\mathbb{R}^k$. Thus $\mathrm{Vol}(F) = |\det(p_1 - p_0, p_2 - p_0, \dots p_k - p_0)|/k!$. Suppose, without loss of generality, that $i = 1$. The vertices of L_1 are p and the vertices $p_0 p_2, \dots, p_k$, and therefore

$$\begin{aligned}
\mathrm{Vol}(L_1) &= \Big|\det(\sum_{l=0}^{k} \beta_l p_l - p_0, p_2 - p_0, \dots, p_k - p_0)\Big|/k! \\
&= \Big|\det(\sum_{l=0}^{k} \beta_l (p_l - p_0), p_2 - p_0, \dots, p_k - p_0)\Big|/k! \\
&= \Big|\sum_{l=0}^{k} \beta_l \det(p_l - p_0, p_2 - p_0, \dots, p_k - p_0)\Big|/k! \\
&= \Big|\beta_1 \det(p_1 - p_0, p_2 - p_0, \dots, p_k - p_0)\Big|/k! \\
&= \beta_1 \mathrm{Vol}(F).
\end{aligned}$$

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6. FANS. We are now ready to extend the third and final step in the proof of Ceva's theorem. We know by Proposition 2 that the ratio of the volumes of k-lobes behaves similarly to the ratio of the lengths of 1-lobes. In Ceva's theorem ratios of lengths of 1-feet are multiplied along a cycle of sides of the triangle. The order in which the ratios are taken matters, or else the product will not be 1. We extend the idea of taking ratios in some order along cycles of faces of an n-simplex. Here we call them *fans*.

Definition. Fix a multipede $\{p_F\}_{F \in \mathcal{F}}$ of an n-simplex S. A *uniform m-fan* of k-lobes, for $m \geq 3$ and $k \geq 1$, is a set $\{(F_z, L_z, M_z) \mid z \in \mathbb{Z}/m\mathbb{Z}\}$, indexed by the integers modulo m, such that for each z,

- F_z is a face of S, and L_z and M_z are lobes in F_z at p_{F_z};
- $M_z \cap L_{z+1}$ is a $(k-1)$-face.

We show later that the faces F_z in a uniform m-fan of k-lobes must satisfy, for each z, that $F_z \neq F_{z+1}$. Moreover, $F_z \cap F_{z+1}$ is a facet of each of these two k-faces. Thus, F_{z+1} is obtained from F_z by replacing one vertex in F_z by another vertex that is not in F_z.

We illustrate the notion of a uniform fan via several examples in the tetrahedron. An example of a uniform 3-fan of 1-lobes is

$$(p_0 p_1, p_0 p_{01}, p_{01} p_1), (p_1 p_2, p_1 p_{12}, p_{12} p_2), (p_2 p_0, p_2 p_{20}, p_{20} p_0).$$

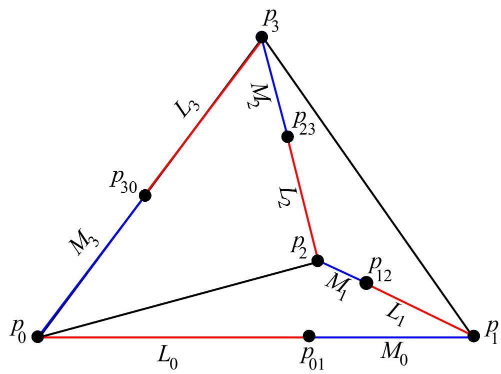

Figure 6. A uniform 4-fan of 1-lobes.

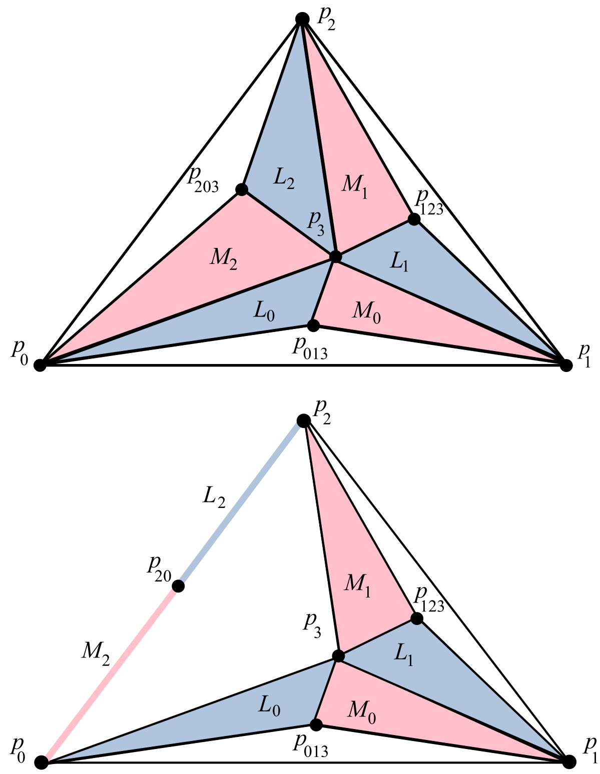

Figure 7. Top: A uniform 3-fan of 2-lobes. Bottom: A mixed 3-fan obtained from the uniform fan at the top.

This fan is on the facet $p_0p_1p_2$ of the tetrahedron, which is a triangle, and therefore it is subject to the conclusion of Ceva's theorem. Figure 6 depicts a uniform 4-fan of 1-lobes. A uniform 3-fan of 2-lobes is depicted in Figure 7 at the top. The fan in Figure 8 is a uniform 4-fan of 2-lobes. To visualize this fan more easily, the surface of the tetrahedron, namely its four facets, is rolled flat in a plane by cutting the surface along the edges p_3p_0, p_0p_1, and p_1p_2.

The uniformity of a fan refers to the fact that all the faces in the fan have the same dimension. We next define a mixed fan in which faces may have different dimensions, but must be obtained from a uniform fan as we define next.

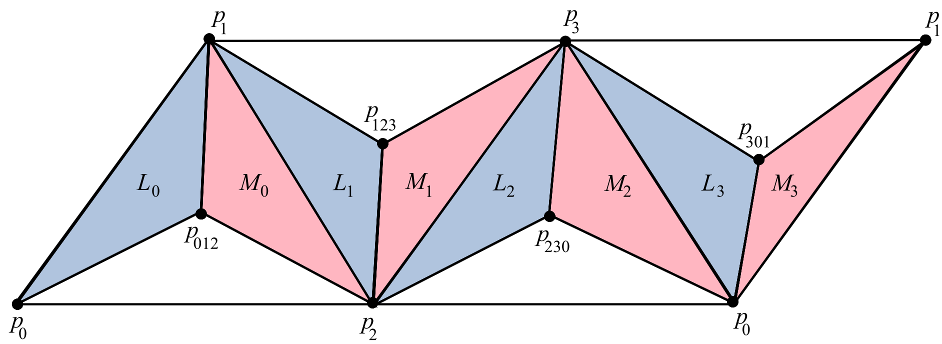

Figure 8. A uniform 4-fan of 2-lobes.

Definition. A *mixed m-fan*, for $m \geq 3$, is a set $\{(F_z, L_z, M_z) \mid z \in \mathbb{Z}/m\mathbb{Z}\}$ such that

- for each z, F_z is a face of S, and L_z and M_z are lobes in F_z at p_{F_z};
- there exists a uniform m-fan of k-lobes, $\{(F'_z, L'_z, M'_z) \mid z \in \mathbb{Z}/m\mathbb{Z}\}$, such that for each z, L_z and L'_z are opposite the same vertex, and M_z, and M'_z are opposite the same vertex.

The fan at the bottom in Figure 7 is mixed. It is obtained from the uniform fan at the top by replacing the two-dimensional face $p_0p_3p_2$ by the one-dimensional face p_0p_2, and the pair of 2-lobes in the original face by a pair of 1-lobes in the new face. Both pairs are opposite the vertices p_0 and p_2.

The generalized Ceva's theorem. *A mulitpede $\{p_F\}_{F\in\mathcal{F}}$ in an n-simplex S is induced by a point in the interior of S if and only if, for each uniform or mixed fan* $\{(F_z, L_z, M_z) \mid z \in \mathbb{Z}/m\mathbb{Z}\}$,

$$\prod_{z\in\mathbb{Z}/m\mathbb{Z}} \frac{\mathrm{Vol}(L_z)}{\mathrm{Vol}(M_z)} = 1. \tag{4}$$

Proof. We first prove the "only if" part. Suppose P is a multipede induced by p_S in the interior of S, and let $\beta_0, \ldots, \beta_n$ be the barycentric coordinates of p_S. By Proposition 2, the left-hand side of equation (4) for a mixed fan is the same as the left-hand side for the uniform fan from which it was obtained. Thus it is enough to prove that (4) holds for uniform fans. Let Γ be such a fan. Let p_{i_z} and p_{j_z} be the vertices in F_z opposite L_z and M_z, respectively. By Propositions 1 and 2,

$$\frac{\mathrm{Vol}(L_z)}{\mathrm{Vol}(M_z)} = \frac{\beta_{i_z}}{\beta_{j_z}}.$$

The number of times β_i appears in the numerator of equation (4) is the number of z's for which $i = i_z$ and the number of times it appears in the denominator is the number of z's for which $i = j_z$. We show that for each i these two numbers are the same, which completes the proof of the "only if" part.

For each z, $F_z \neq F_{z+1}$. Otherwise, M_z and L_{z+1} are lobes in F_z, and hence they have in common the point p_{F_z} which is not on a $(k-1)$-face of S, contrary to the requirement of the definition. Let $G_z = M_z \cap L_{z+1}$. Then G_z is a $(k-1)$-face. As $F_z \cap F_{z+1} \supset M_z \cap L_{z+1} = G_z$, it follows that G_z is a facet of F_z and of F_{z+1}.

Thus, the vertices of F_z consist of p_{j_z} and the vertices of G_z. The vertices of F_{z+1} consist of $p_{i_{z+1}}$ and the vertices of G_z. That is, F_{z+1} is obtained from F_z by replacing the vertex p_{j_z} by the vertex $p_{i_{z+1}}$ in F_z. We need to show that each p_i replaces a vertex the same number of times it is replaced by a vertex.

Fix i and let $f : \mathbb{Z}/m\mathbb{Z} \to \{0, 1\}$ be defined by $f(z) = 1$ when $p_i \in F_z$ and $f(z) = 0$ when $p_i \notin F_z$. The number of times p_i is replaced by another vertex is the number of z's for which $f(z) - f(z+1) = 1$. The number of times p_i replaces another vertex is the number of z's for which $f(z) - f(z+1) = -1$. The number of times p_i is neither replaced nor replaces is the number of z's for which $f(z) - f(z+1) = 0$. But, because of the cyclic structure of the group of integers modulo n,

$$\sum_{z \in \mathbb{Z}/m\mathbb{Z}} (f(z) - f(z+1)) = 0,$$

and hence p_i replaces a vertex the same number of times it is replaced by a vertex.

To prove the "if" part we assume that equation (4) holds for all fans and we construct a point p_S in the interior of S that induces the multipede. Denote by p_{ij} the point in the multipede that is on $p_i p_j$, and consider the ratio $r_{ij} = |p_j p_{ij}|/|p_i p_{ij}|$. Obviously, $r_{ji} = 1/r_{ij}$.

Starting with some $c > 0$, we define a point p_S by its barycentric coordinates $\beta_0, \ldots, \beta_n$ as follows:

$$\beta_0 = c,$$
$$\beta_{k+1} = \beta_k r_{(k+1)k} \text{ for } k \geq 0.$$

Thus, for $k > 0$, $\beta_k = c r_{k(k-1)} \cdots r_{10}$. The constant c is chosen to guarantee that $\sum_i \beta_i = 1$.

We show that for $i \neq j$, $\beta_j/\beta_i = r_{ji}$. Since $r_{ij} = 1/r_{ji}$ it is enough to consider only $j > i$. If $j = i+1$, then $\beta_j/\beta_i = \beta_{i+1}/\beta_i = r_{(i+1)i} = r_{ji}$. If $j > i+1$, then $\beta_j/\beta_i = r_{j(j-1)} \cdots r_{(i+1)i}$. By (4), $r_{j(j-1)} \cdots r_{(i+1)i} r_{ij} = 1$. Thus $\beta_j/\beta_i = 1/r_{ij} = r_{ji}$. Therefore, by Proposition 1, p_{ij} is a point in the multipede induced by p_S. This shows that all the points of the multipede that are on the 1-faces of S are in the multipede induced by p_S.

Next, consider a point p_F of the multipede on a k-face F for $1 < k < n$. We show that p_F is induced by p_S. Let γ be the vector of barycentric coordinates of p_F. By Proposition 2, for any pair of vertices p_i and p_j in F, $\mathrm{Vol}(L_i)/\mathrm{Vol}(L_j) = \gamma_i/\gamma_j$, where L_i and L_j are the k-lobes of F at p_F opposite the vertices p_i and p_j, respectively. We will further show that for any such pair $\mathrm{Vol}(L_i)/\mathrm{Vol}(L_j) = \beta_i/\beta_j$. This implies that γ is proportional to β. That is, for some constant c and for each i such that $p_i \in F$, we have $\gamma_i = c\beta_i$. Since γ is normalized, it follows that $c = 1/\sum_{p_i \in F} \beta_i$. Thus, by (1), p_F is induced by p_S.

To complete the proof we need to show that $\mathrm{Vol}(L_i)/\mathrm{Vol}(L_j) = \beta_i/\beta_j$ for all p_i and p_j in F. For this we first create a uniform 3-fan of 1-lobes. We then create a mixed fan by changing one of the elements of the uniform fan to (F, L_j, L_i). We deduce the required equality by applying equation (4) to both fans.

Let p_k be a vertex not in F, and consider the uniform 3-fan of 1-lobes

$$\big((p_i p_j, p_i p_{ij}, p_j p_{ij}), (p_j p_k, p_j p_{jk}, p_k p_{jk})(p_k p_i, p_k p_{ki}, p_i p_{ki})\big).$$

By (4), $r_{ij} r_{jk} r_{ki} = 1$. Next consider the mixed 3-fan obtained from the previous one by replacing $(p_i p_j, p_i p_{ij}, p_j p_{ij})$ with (F, L_j, L_i). By (4),

$$\mathrm{Vol}(L_i)/\mathrm{Vol}(L_j) r_{jk} r_{ki} = 1.$$

This implies that $\mathrm{Vol}(L_i)/\mathrm{Vol}(L_j) = r_{ij} = \beta_i/\beta_j$, which completes the proof. ■

Remark. The condition in equation (4) in the main theorem is for *all* m-fans of k-lobes. We can have m-fans with arbitrarily large m. For the "only if" part it makes the claim strong. But for the "if" part we can make the claim stronger by requiring that equation (4) holds only for some m's. Indeed it follows from the proof that it is enough to require equation (4) to hold only for *simple* uniform m-fans of k-lobes and the mixed m-fans that are derived from them. By "simple" we mean that all m faces F_z in the fan are distinct. Thus, m is bounded by the number of k-faces of the n-simplex.

ACKNOWLEDGMENTS. The author wishes to thank Ehud Lehrer for helpful discussions and comments. The author acknowledges financial support from Israel Science Foundation, Grant #722/18.

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