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Agreeing to Disagree in Infinite Information Structures

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1 Introduction

Several authors have recently studied the game theory aspects of generalized information structures, that is, information structures that are not partitions. Such structures are needed when we wish to impose some restrictions on the concept of knowledge or bound rationality (see [3] and [4]). Some theories that were developed for partitions do not hold for generalized information structures or at least require changes and adjustments (e.g., correlated equilibria in [2] and the 'no-trade' theorem in [5]). Other results continue to hold for some families of generalized information structures. Thus, for example, it has been shown in [4] that the impossibility of 'agreeing to disagree', which was proved in [1] for partitions, also holds for more general information structures.

We want to draw attention to some non-trivial differences between finite and infinite structures where generalized information structures are concerned. For partitions, there is almost no interesting distinction between finite structures and infinite ones. Theorems for the finite case can be repeated and proved almost verbatim for the infinite, countable case. We show here that this is no longer true for generalized structures. More specifically, we show that the impossibility of 'agreeing to disagree', which holds invariably for finite and infinite partitions, holds generally for reflexive-transitive structures only in the finite case. For infinite structures one has to impose some restrictions on the infiniteness of the structure. Thus in [4], in order to prove the 'agreeing to disagree' theorem an assumption is made that agents' knowledge is finitely generated. This means that for each state and each agent there is a finite number of facts from which everything he knows is implied. Here we show, by constructing a counter-example, that with no restriction on the infinite structure the 'agreeing to disagree' theorem fails.

Infinite information structures are by no means a mathematical luxury. They are indispensable in the theory of information and knowledge in many cases. For example, in developing models in which the description of a state of the world includes the state of knowledge of various agents (as in [1], [4] and many others) we can hardly avoid using infinite information structures. This is also the case for models of the 'coordinated attack' type in which communication is iterated with no bound on the number of iterations (see [6]). Finally, we should note that models with infinite information structures are commonplace in the literature of economic

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theory and game theory. Such, for example, are models in which agents are informed by observing a random variable with infinitely many values, or models in which agents are of infinitely many types.

2 *RT* Information Structures

Let Ω be a state space with a σ-field Σ. We define a *reflexive transitive (RT) information structure* to be a function $P: \Omega \to \Sigma$ which satisfies for each ω in Ω:

$$\omega \in P(\omega), \tag{1}$$

for each

$$\omega' \in P(\omega), \quad P(\omega') \subseteq P(\omega). \tag{2}$$

$P(\omega)$ is interpreted as the set of all states which are considered *possible* to the agent when state ω is realized. The relation 'ω' is possible at ω' is reflexive and transitive by (1) and (2). These properties can be shown to be equivalent to the claims that whatever is known by the agent is true, and that the agent always knows that he knows a fact, when this is the case (see [3], [4]).

If we require that the possibility relation is also symmetric then P represents a partition, that is, the range of P is a partition of Ω. We call P *finite* when its range is finite. We say that P is *finitely nested* if there is no infinite sequence in the range of P that is strictly decreasing with respect to inclusion.

Consider now two agents 1 and 2, with information structures P_1 and P_2 respectively. We say that event E is *common knowledge* at ω if there exists some event C such that $\omega \in C \subseteq E$ and for each $\omega' \in C$, $P_i(\omega') \subseteq C$ for $i = 1, 2$. When the range of P is a partition, this definition is equivalent to the one given in [1]. See also [3] and [4] for equivalent definitions of common knowledge in terms of iterated knowledge operators.

3 Agreeing to Disagree in Infinite *RT* Information Structures

Suppose the agents have a common prior probability distribution μ over Σ. For a given event X and number r let $E_i(X, r)$ for $i = 1, 2$, be the event that the posterior probability that agent i assigns to X is r, i.e., $E_i(X, r) = \{\omega \mid \mu(X \mid P_i(\omega)) = r\}$.

We can now state a general theorem, for *RT* structures, about the impossibility of agreeing to disagree.

Proposition 1. If P_1 and P_2 are finitely nested RT information structures with a countable range, and for some event X and numbers r and s, $E_1(X, r) \cap E_2(X, s)$ is common knowledge at some state, then $r = s$.

A similar theorem is proved in [4] for a model in which the information structures P_i are derived from propositional knowledge operators. If we assume in this model that knowledge acquired by an agent, even if it is infinite, is a logical consequence of finitely many propositions, then the countability and the finite nestedness of the derived information structures is guaranteed. The proof of Proposition 1 is similar to that given in [4]. Note that finite information structures are trivially finitely nested and of countable range, and therefore the proposition holds for them.

Proposition 2. The claim of Proposition 1 does not hold in general for infinite RT information structures even when the ranges are countable.

Proof: We construct an example with two agents in which both information structures are countable though one of them is not finitely nested, and where a disagreement of the agents about the prior of a given event X is common knowledge. Let Ω be the unit interval $[0, 1]$ and μ the Lebesgue measure on it. We define two RT-information structures P_1 and P_2 and an event X as follows.

The information structure P_2 is trivial: $P_2(\omega) = \Omega$ for all ω. The event X will be constructed such that $\mu(X) = \frac{1}{2} - d$, for some $d > 0$, and therefore for each ω:

$$\mu(X | P_2(\omega)) = \frac{1}{2} - d.$$

The information structure P_1 will be chosen such that for each $\omega \in \Omega$:

$$\mu(X | P_1(\omega)) = \frac{1}{2}. \tag{3}$$

Before we construct X and P_1 formally we give a sketch of the construction. Suppose we define $P_1(\omega) = [0, \frac{2}{3})$ for each $\omega \in [0, \frac{1}{3})$ and $P_1(\omega) = (\frac{1}{3}, 1]$ for each $\omega \in (\frac{2}{3}, 1]$. We choose X such that

$$\mu\left(X \,\middle|\, \left[0, \frac{1}{3}\right)\right) = \mu\left(X \,\middle|\, \left(\frac{2}{3}, 1\right]\right) = \frac{1}{2} - 3d,$$

while

$$\mu\left(X \,\middle|\, \left(\frac{1}{3}, \frac{2}{3}\right)\right) = \frac{1}{2} + 3d.$$

Clearly $\mu(X) = \frac{1}{2} - d$ and (3) is satisfied for each $\omega \in [0, \frac{1}{3}) \cup (\frac{2}{3}, 1]$. To take care of ω's in $(\frac{1}{3}, \frac{2}{3})$ we have to define P_1 and X over this interval. This problem is analogous to the one we started with. We could devide the interval into three parts, decrease the amount of mass in the middle one by d and add d in each of the other two intervals. But we can not continue this construction by induction since d is fixed and the interval length approaches zero. This can be solved if in each step we construct more and more overlapping components of P_1. In this case the amount of mass that we have to remove and add also diminishes. The formal construction is as follows.

Let $(n_i)_{i\geq 1}$ be an increasing sequence of positive integers. Denote

$$s_i = \prod_{k=1}^{i} \frac{1}{n_k+1}$$

and

$$t_i = \prod_{k=1}^{i} \frac{1}{n_k-1}.$$

Further, let $t_0=1$. The sequence $(s_i/t_i)_{i\geq 1}$ is decreasing, but by allowing $(n_i)_{i\geq 1}$ to increase fast enough we can guarantee that $\lim s_i/t_i = L > 0$. (Indeed L can be chosen to be as close as we wish to 1). Let $d=L/2$.

We now construct sets B^i, A^i_j for $i\geq 1$ and $j=1,\ldots,n_i$ by induction on i, as follows. $B^1=\Omega$. When B^i is constructed, the sets $A^i_1,\ldots,A^i_{n_i}$, B^{i+1} are chosen to be a partition of B^i into n_i+1 sets of equal measure. Clearly for all $i\geq 1$ and $j=1,\ldots,n_i$:

$$\mu(A^i_j)=\mu(B^{i+1})=s_i. \tag{4}$$

Moreover, we choose the decreasing sequence of sets B^i such that $\bigcap_{i\geq 1} B^i=\emptyset$ and therefore $\{A^i_j \mid i\geq k, j=1,\ldots,n_i\}$ is a partition of B^k. For each $i\geq 1$ and $j=1,\ldots,n_i$ let

$$C^i_j = A^i_j \cup B^{i+1}.$$

We define the information structure P_1 as follows: For $\omega\in A^i_j$,

$$P_1(\omega)=C^i_j.$$

Since $\{A^i_j \mid i\geq 1,\ j=1,\ldots,n_i\}$ is a partition of Ω, the information structure P_1 is defined for all ω and it is easy to see that P_1 is an RT information structure.

To demonstrate that 1 and 2 can agree to disagree with these information structures we construct an event X such that at each ω the posteriors of 1 and 2 are different and yet are common knowledge. For $i\geq 1$ and $j=1,\ldots,n_i$ let X^i_j be a subset of A^i_j with

$$\mu(X^i_j) = \frac{\mu(A^i_j)}{2} + (-1)^i t_i d.$$

To prove that such sets exist we have to show that $t_i d \leq \mu(A^i_j)/2$. This follows from (4) and the fact that s_i/t_i is decreasing. Define

$$X = \bigcup_{\substack{i \geq 1 \\ j=1,\ldots,n_i}} X^i_j.$$

To compute $\mu(X|P_1(\omega))$ we note first that:

$$\mu(X \cap B^{i+1}) = \sum_{k=i+1}^{\infty} n_k(-1)^k t_k d + \frac{\mu(B^{i+1})}{2}. \tag{5}$$

To evaluate the series on the right hand side of (5) we observe that for each $k \geq 1$

$$n_k t_k = t_{k-1} + t_k.$$

Thus the series is reduced to $(-1)^{i+1} t_i d$. Now for $\omega \in A^i_j$:

$$\begin{aligned}
\mu(X \cap P_1(\omega)) &= \mu(X \cap C^i_j) \\
&= \mu(X \cap A^i_j) + \mu(X \cap B^{i+1}) \\
&= \frac{\mu(A^i_j)}{2} + (-1)^i t_i d + \frac{\mu(B^{i+1})}{2} + (-1)^{i+1} t_i d \qquad (6) \\
&= \frac{\mu(P_1(\omega))}{2}.
\end{aligned}$$

Hence for each $\omega \in \Omega$:

$$\mu(X|P_1(\omega)) = \frac{1}{2}.$$

For $i=0$, in (5), we have $\mu(X) = \frac{1}{2} - d$, and therefore for each ω:

$$\mu(X|P_2(\omega)) = \frac{1}{2} - d.$$

We have shown that $E_1(X, \frac{1}{2}) = E_2(X, \frac{1}{2} - d) = \Omega$ and therefore $E_1(X, \frac{1}{2}) \cap E_2(X, \frac{1}{2} - d)$ is common knowledge but $\frac{1}{2} \neq \frac{1}{2} - d$, and thus it is possible to agree to disagree in Ω. □

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