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Approximating Common Knowledge with Common Beliefs

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Strict common knowledge seems almost impossible; we can never be sure what others know. It is shown that common knowledge can be approximated by the weaker and more easily obtained condition of common belief. This approximation justifies the standard assumption in game theory that the description of the game is common knowledge. Aumann's result on the impossibility of agreeing to disagree can also be approximated when common knowledge is replaced by common belief. © 1989 Academic Press, Inc.

INTRODUCTION

In most models in game theory it is assumed, either explicitly or implicitly, that common knowledge regarding certain facts is shared by the agents. How crucial is this assumption? Do these theories collapse when common knowledge is absent, or is there any concept weaker than common knowledge that is sufficient to sustain the conclusions of the theories or at least some closely related conclusions? We show that the weaker concept of "common belief" can function successfully as a substitute for common knowledge in the theory of equilibrium of Bayesian games. We also show that Aumann's (1976) no-agreement theorem can be generalized to common belief.

To illustrate the need for a notion weaker than common knowledge we look at the following cases. The most frequent cases of common knowledge are public announcements. Consider, for example, an auction. Once the auctioneer has publicly announced a price, it is assumed by most suppliers of auction models to be common knowledge to the participants of the auction. But is it really? One should always allow for some small probability that a participant was absentminded or deaf at the time of the

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announcement. No matter how small that probability is, the price is not common knowledge. Nobody knows for sure that the others know the price. We cannot even build the first story of this formidable tower of hierarchies of knowledge, "I believe that you believe that I believe . . . ," that is required for common knowledge. One might say, if the probability is very small, why not simply assume it is zero? Such a procedure would be valid only if, by making such an assumption, the conclusions regarding the outcomes of the process do not change dramatically. A stimulating example by Rubinstein (1987) indicates that this might not be the case. In this example, which is a game theoretic formulation of the "coordinating attack" problem (see Halpern, 1986), Nature chooses one of the two two-person games G^1 and G^2 with equal probabilities. Player 1 is informed which game is to be played. He then sends a message to player 2 telling him this information. They then send messages back and forth acknowledging, each in his turn, the previously received message. Each message has a small probability τ of being lost, in which case the whole process stops. When the process stops, neither of the players knows whether it was his message that got lost or his partner's acknowledgment. In this example the players climb, with high probability, the tower of knowledge hierarchy and seem to be closer to common knowledge than the participants of our auction example. Unfortunately, as optimizing agents the players cannot behave as if the games G^1 and G^2 were common knowledge; i.e., they cannot play the "natural" Nash equilibria in G^1 or G^2. Indeed, one can show that even if the players were only ε-optimizers (they do not care about pennies) they still cannot pretend that the games G^1 and G^2 are common knowledge.

How different then are auctions and public announcements in general from the game theoretic version of the coordinated attack problem? Are public announcements stylized but useless fiction? To answer these questions we develop a precise measure of their proximity to common knowledge.

When one examines a truncated hierarchy of knowledge one observes that all is not lost. After several iterations of "I know that you know that I know . . . ," one stops and "he does not know that . . ." from then on. But certainly in this case he may still "believe that . . ." with some certainty. And then he may believe that the other does, and so on. So hierarchies never die; they always extend ad infinitum where beliefs replace knowledge. This opens the door to an iterative definition of a common belief similar to that of common knowledge. The shortcomings of the iterative definitions are that they lack some explanatory and descriptive power. Clearly not even *homo rationalis* checks the validity of infinitely many statements one by one. Still everybody understands public announcements to common knowledge. We seek to capture this intuitive understanding with our definition of an "evidently known" event, as reflected in Aumann's original definition of common knowledge.

We use exactly the same type of events, evidently believed events, to define common beliefs. We find this definition more compelling and eventually most useful in deriving theorems. The difference between these definitions may be of some importance from the viewpoint of the interpretation of the theory of common knowledge and common beliefs. From the formal point of view we show that the two are equivalent.

The model we use is the same as Aumann's, in which knowledge is given by a partition and beliefs are simply posterior probabilities. The definitions of common knowledge and common belief in such a model are strikingly similar. The similarity is that they are both "common." The property of common, which is the same in both models, is expressed by the hierarchies being infinite and by the similarity of evidently known and evidently believed events. The difference between the notions is that belief replaces knowledge.

In the same way the intensity of beliefs can be quantified, using probabilities, common belief can be quantified. A common p-belief will be defined for each p in $[0, 1]$. For $p = 1$, common 1-belief is almost the same as common knowledge. For our purposes the differences are irrelevant (common 1-beliefs were studied by Brandenburger and Dekel (1987)). So the concept of common p-belief generalizes the concept of common knowledge. Notions similar to belief and common belief were studied in the framework of logical deductive systems by Gaifman (1986) and Fagin and Halpern (1988).

Does common p-belief approximate common knowledge when p approaches 1? We answer this in the affirmative in Sections 4 and 5.

In Section 4 we consider Aumann's (1976) agreeing to disagree theorem. If agents have the same prior distribution and their posterior probabilities for a certain event are common knowledge then these posteriors must coincide. The theorem can be generalized. If the posteriors are common p-belief then they may differ by at most $2(1 - p)$.

In Section 5 we examine the standard assumption in game theory that for the players to play a Nash equilibrium, the description of the game must be common knowledge. We do this by studying Bayesian games. Nature selects a state. In each state of the world one of finitely many games in normal form is played. Information structures determining what the players know about Nature's choice are given by partitions. If it is common knowledge among players which game is to be played then players can play any Nash equilibrium in each of them. We examine the case where there is only a common p-belief of what game is to be played and where such common beliefs do not necessarily exist in each state but only in a "big" set of measure $1 - \delta$. We show that in this case if players are ε-optimizers at each state ω, they can almost mimic the behavior of players to whom the game played is common knowledge. That is, both their strategies and their payoffs are very close to the strategies and payoffs that would be played if the games played were common knowledge.

1. Preliminaries: Common Knowledge

The results of this section are well known in the literature of common knowledge (see, e.g., Binmore and Brandenburger, 1987; Brandenburger and Dekel, 1987; Geanakoplos, 1988; Tan and Werlang, 1988).

Let I be a finite set of agents and let (Ω, Σ, μ) be a probability space, where Ω is the space of states, Σ is a σ-field of events, and μ is a probability measure on Σ. For each $i \in I$, Π_i is a partition of Ω into measurable sets with positive measure, and therefore a countable partition. For $\omega \in \Omega$ we denote by $\Pi_i(\omega)$ the element of Π_i containing ω. Π_i is interpreted as the information available to agent i; $\Pi_i(\omega)$ is the set of all states which are indistinguishable to i when ω occurs. We denote by $\mathcal{F}_i$ the σ-field generated by Π_i. That is, $\mathcal{F}_i$ consists of all unions of elements of Π_i. We say that *i knows event E at ω*, if $\Pi_i(\omega) \subseteq E$. Let $K_i(E)$ be the event "i knows E." That is,

$$K_i(E) = \{\omega: \Pi_i(\omega) \subseteq E\}.$$

Recall that an event C is *common knowledge at* ω, if there exists an event E such that $\omega \in E \subseteq C$, and $E \in \mathcal{F}_i$ for each $i \in I$ (Aumann, 1976). Alternatively, one can show that C is common knowledge at ω iff for each $n \geq 1$ and agents $i_1, i_2, \ldots, i_n$

$$\omega \in K_{i_1} K_{i_2} \cdots K_{i_n}(C). \tag{1}$$

That is, when ω occurs then for all $n \geq 1$ and for all agents $i_1, i_2, \ldots, i_n$ it is true that

(i_1 knows that [i_2 knows that

[. . . i_{n-1} knows that [i_n knows C]] . . .]).

A different way of expressing the iterative nature of common knowledge is by considering the intersections of the following events:

C^1: Every agent knows C.

C^2: Every agent knows C^1.

C^3: Every agent knows C^2, etc.

PROPOSITION 1. *Let* $E(C) = \bigcap_{n\geq 1} C^n$, *where* $C^n = \bigcap_{i\in I} K_i(C^{n-1})$ *for all* $n \geq 1$, *and* $C^0 = C$. *Then,*

$$C \text{ is common knowledge at } \omega \text{ iff } \omega \in E(C).$$

Proof. Note the following properties of the knowledge operators K_i: For all events $A \subseteq B$, $K_i(A) \subseteq A$, $K_i(A) = A$ iff $A \in \mathcal{F}_i$, and $K_i(A) \subseteq K_i(B)$. Also, for all sequences of events (A^n), $K_i(\bigcap_n (A^n)) = \bigcap_n K_i(A^n)$.

Suppose C is common knowledge at ω. Then there exists $\omega \in E \subseteq C$ such that $E \in \mathcal{F}_i$ for all $i \in I$. Let $i \in I$. As $E \subseteq C$, $E = K_i(E) \subseteq K_i(C)$. Therefore,

$$E \subseteq \bigcap_{i\in I} K_i(C) = C^1.$$

Continue to prove by induction on n that $E \subseteq C^n$ for all $n \geq 1$. That is, $E \subseteq E(C)$. Therefore $\omega \in E(C)$.

Conversely, it suffices to prove that $E(C) \subseteq C$ and that $E(C) \in \mathcal{F}_i$ for all $i \in I$. Obviously, $E(C) \subseteq C^1 \subseteq K_i(C) \subseteq C$. To show that $E(C) \in \mathcal{F}_i$ observe that for all $n \geq 1$

$$E(C) \subseteq C^{n+1} \subseteq K_i(C^n).$$

Hence,

$$E(C) \subseteq \bigcap_{n\geq 1} K_i(C^n) = K_i\left(\bigcap_{n\geq 1} C^n\right) = K_i(E(C)) \subseteq E(C). \quad \blacksquare$$

We say that the event E is *evident knowledge* if for each $i \in I$

$$E \subseteq K_i(E). \tag{2}$$

An event is evident knowledge if whenever it occurs all agents know it. The typical evident knowledge events are public announcements. When the event E is "the teacher says in class that at least one student has a red spot on his head," then this event, just by its occurrence, implies that every agent (student) knows it. Note that (2) is equivalent to $K_i(E) = E$, but it is the one-sided implication of (2) that captures the nature of evident knowledge. The other implication $K_i(E) \subseteq E$ is a property of knowledge not related to evident knowledge. It is easy to see now that the following proposition holds.

PROPOSITION. *C is common knowledge at* ω *iff there exists an evident knowledge event E such that* $\omega \in E$ *and for all* $i \in I$

$$E \subseteq K_i(C). \tag{3}$$

The interpretation of common knowledge as stated in (3) is the following: The event C is common knowledge in a certain state of affairs ω if a certain event (E) occurs in this state, which is evidently known, and which implies that everyone knows C. As an illustration, let C be the event "the price of the picture is $1000," and let E be the event "the auctioneer announces that the price of the picture is $1000." ω is one of the states of the world in which E occurs. Obviously, when E occurs everyone knows that it has occurred ($E \subseteq K_i(E)$). Moreover, once E occurs everyone knows C ($E \subseteq K_i(C)$). Thus, C is common knowledge at each state of the world ω in which E occurs ($\omega \in E$). Note that C itself is not evident knowledge; the fact that the price is $1000 is not necessarily known to all agents, whenever this is indeed the price. The two alternative definitions of common knowledge, the iterative definition given in Proposition 1 (or in (1)) and that given in (3), represent two different aspects of this notion. The iterative definition is considered by many as the natural definition of common knowledge. On the other hand, it is hard to believe that people identify common knowledge by checking the infinitely many conditions imposed by (1) one by one, or even any finite number of them beyond $n \geq 3$ (see, e.g., Clark and Marshall, 1981). The definition given in (3) (which is basically Aumann's definition), with its economic means, is the one that is used to recognize and understand common knowledge.

2. BELIEFS AND COMMON BELIEFS

As before, (Ω, Σ, μ) is a probability space, I is a finite set of agents, and the information available to agent i is given by a partition Π_i of Ω to measurable sets with positive probabilities. We start by replacing the phrase "i knows E at ω" by the phrase "i believes E with probability at least p at ω," where $0 \leq p \leq 1$, abbreviated to "i p-believes E at ω." Formally, *i p-believes E at* ω if $\mu(E|\Pi_i(\omega)) \geq p$, i.e., if the posterior of E given that ω has occurred is at least p (for agent i). We denote by $B_i^p(E)$ the event "i p-believes E." That is,

$$B_i^p(E) = \{\omega: \mu(E|\Pi_i(\omega)) \geq p\}.$$

We now record the following properties of B_i^p for later use. The proof of these properties is straightforward, and therefore will be omitted.

PROPOSITION 2. *For each* $0 \leq p \leq 1$, $i \in I$, *and* $E, F \in \Sigma$,

$$B_i^p(E) \in \mathcal{F}_i. \tag{4}$$

$$\text{If } E \in \mathcal{F}_i \text{ then } B_i^p(E) = E. \tag{5}$$

$$B_i^p(B_i^p(E)) = B_i^p(E). \tag{6}$$

$$\textit{If } E \subseteq F \textit{ then } B_i^p(E) \subseteq B_i^p(F). \tag{7}$$

If (E^n) *is a decreasing sequence of events then* (8)

$$B_i^p\left(\bigcap_n E^n\right) = \bigcap_n B_i^p(E^n).$$

$$\mu(E \mid B_i^p(E)) \geq p. \tag{9}$$

Note that (4)–(8) of Proposition 2 hold if one substitutes K_i for B_i^p. K_i also has the property $K_i(E) \subseteq E$ for each E, which is the hallmark of knowledge; whenever one knows E then E (is true). Property (9) generalizes this axiom of knowledge. It states that given that one believes E with a probability of at least p (this is the event $B_i^p(E)$) then the probability of E is indeed at least p.

The case $p = 1$ is very similar to knowledge. We say that $E = F$ a.s. (respectively $E \subseteq F$ a.s.) if $\mu(E \Delta F) = 0$ (respectively $\mu(F \setminus E) = 0$). With this notation, (9) implies that

$$B_i^1(E) \subseteq E \qquad \text{a.s.} \tag{10}$$

While in finite models there is no point in distinguishing between knowledge and 1-belief, one may wish to preserve the distinction in continuous models. For example, we 1-believe that a number picked at random from the interval [0, 1] will be irrational, but we do not know it.

E is an *evident p-belief* if for each $i \in I$

$$E \subseteq B_i^p(E). \tag{11}$$

That is, whenever E occurs everyone assigns a probability of at least p to its occurrence (cf. (1)).

We now define common p-belief, generalizing the definition of common knowledge in (3).

Definition 1. An event C is common p-belief at ω if there exists an evident p-belief event E such that $\omega \in E$, and for all $i \in I$,

$$E \subseteq B_i^p(C). \tag{12}$$

To illustrate the notions of evident p-belief and common p-belief let us reconsider auctions. Let C be the event "the price of the picture is \$1000." Let E be the event "the auctioneer announces that the price of the picture is \$1000." If everyone must hear the announcements of the auctioneer, then E is evident knowledge. That is, whenever E occurs everyone knows that it has occurred. However, if there is some positive (possibly small) probability ε that not all the audience are hearing, then E is not evident knowledge. Moreover, E may not be evident p-belief for high p. This is the case, for example, if E occurs and one of the agents who assigns a low probability to C does not hear the announcement. However, if F is the event "the audience are all hearing, and the auctioneer announces that the price of the picture is \$1000," then F is evident $(1 - \varepsilon)$-belief. Therefore, C is common $(1 - \varepsilon)$-belief at each state of the world in which F occurs.

Common p-beliefs also have the following iterative interpretation:

Definition 2. For every event C and every $0 \leq p \leq 1$ let

$$E^p(C) = \bigcap_{n \geq 1} C^n, \tag{13}$$

where $C^0 = C$, and, for $n \geq 1$, $C^n = \bigcap_{i \in I} B_i^p(C^{n-1})$.

The following proposition is a counterpart of Proposition 1.

Proposition 3. *For every event C and for every* $0 \leq p \leq 1$,

(I) $E^p(C)$ *is evident p-belief, and* $E^p(C) \subseteq B_i^p(C)$ *for all* $i \in I$.

(II) C *is common p-belief at* ω *iff* $\omega \in E^p(C)$.

Proof. (I) First we show that $(C^n)_{n=1}^{\infty}$ is a decreasing sequence. Indeed, for each i and for each $n \geq 1$, $C^n \subseteq B_i^p(C^{n-1})$. Therefore by (7) and (6)

$$B_i^p(C^n) \subseteq B_i^p(B_i^p(C^{n-1})) = B_i^p(C^{n-1}).$$

Hence for all $n \geq 1$,

$$C^{n+1} = \bigcap_{i \in I} B_i^p(C^n) \subseteq \bigcap_{i \in I} B_i^p(C^{n-1}) = C^n.$$

Let $i \in I$. For all $n \geq 1$,

$$E^p(C) \subseteq C^{n+1} \subseteq B_i^p(C^n).$$

Therefore,

$$E^p(C) \subseteq \bigcap_{n \geq 1} B_i^p(C^n),$$

which implies (by (8)) that

$$E^p(C) \subseteq B_i^p\left(\bigcap_n C^n\right) = B_i^p(E^p(C)).$$

Thus, we have proved (11).

Clearly, $E^p(C) \subseteq C^1 \subseteq B_i^p(C)$.

(II) If $\omega \in E^p(C)$, then by (I) C is common p-belief at ω. As for the converse, suppose C is common p-belief at ω. Let $\omega \in E$ and E satisfies (11) and (12). We show by induction on n that $E \subseteq C^n$ for all $n \geq 1$. This will imply that $\omega \in E^p(C)$. By (12) $E \subseteq C^1$. If $E \subseteq C^n$, then by (11) and (7) $E \subseteq B_i^p(E) \subseteq B_i^p(C^n)$. As this is true for each $i \in I$,

$$E \subseteq \bigcap_{i \in I} B_i^p(C^n) = C^{n+1}. \quad \blacksquare$$

By (II), $E^p(C)$ is the event "C is common p-belief," and by (I) we find that this event itself is common p-belief at each of its states.

At this point it is useful to illustrate the new definitions with an example.

EXAMPLE 1. There are two agents 1 and 2 ($I = \{1, 2\}$). The agents either hear (H) or do not hear (D) a given announcement. The probability that an agent hears is $1 - \varepsilon$. Thus (assuming independence between the players),

$$\Omega = \{HH, HD, DH, DD\},$$

$\mu(HH) = (1 - \varepsilon)^2$, $\mu(HD) = \mu(DH) = \varepsilon(1 - \varepsilon)$, and $\mu(DD) = \varepsilon^2$.

Each agent knows his type only. That is,

$$\Pi_1 = \{\{HH, HD\}, \{DH, DD\}\}$$

and

$$\Pi_2 = \{\{HH, DH\}, \{HD, DD\}\}.$$

Let $A = \{HH\}$. That is, A is the event "everyone hears." A is not common knowledge at any $\omega \in \Omega$ (as there does not exist a ω in which even one of the agents knows that A occurs). However, A is common $(1 - \varepsilon)$-belief at HH. To verify this, note that $B_i^p(A) = A$ for $i = 1, 2$ and $p = 1 - \varepsilon$, and therefore $E^p(A) = A$.

Although the notion of common p-belief is defined for every $0 \leq p \leq 1$ it

is not of much interest unless $p > 0.5$. When we know that an event occurs with probability 0.1 we are not likely to say that we believe in this event. Indeed, if $p \leq 0.5$ we may p-believe (and even common p-believe) in two contradictory events. For example, in the current example let $\varepsilon = 0.5 = p$, and let $E = \{HD, DH\}$ and $F = \{HH, DD\}$. Then, $E^p(E) = E^p(F) = \Omega$. That is, at every $\omega \in \Omega$ it is common p-belief that both agents have the same type, and it is also common p-belief that both agents do not have the same type.

We end this section with simple observations about the relations between common knowledge and common belief. If C is common knowledge at ω, then C is common p-belief at ω for every $0 \leq p \leq 1$. If C is common 1-belief at ω, then there exists an event D which is common knowledge at ω such that $C = D$ a.s. However, note that for a given event C, the event "C is common knowledge" does not necessarily coincide a.s. with the event "C is common 1-belief."

3. A DIGRESSION: THE AXIOMATIC APPROACH

In this section we view the previous results from a different perspective. In Sections 1 and 2 the operators K_i and B_i^p were derived from a given partition. Now we assume that these operators are given as primitives and that they are referred to as knowledge and belief because of some axioms they satisfy. Analysis of abstract knowledge operators on events is used by Bacharach (1985) and by Brown and Geanakoplos (1987). A different axiomatic approach is suggested by Samet (1987) and by Shin (1987).

Let (Ω, Σ) be a measurable space. An operator $B: \Sigma \to \Sigma$ is called a *belief operator* if the following conditions are satisfied for all $E, F, E^1, E^2, \ldots, E^n, \ldots$ in Σ,

$$B(B(E)) \subseteq B(E), \tag{14}$$

$$B(E) \subseteq B(F) \quad \text{whenever } E \subseteq F, \tag{15}$$

$$B\left(\bigcap_n E^n\right) = \bigcap_n B(E^n), \tag{16}$$

whenever (E^n) is a decreasing sequence.

$B(E)$ is interpreted as the event that one believes E.

Several remarks are in order: (14) states that if one believes that he believes E, then he believes E. That is, one's beliefs concerning his beliefs are always correct. Equation (15) states that if one believes in the event E

and E implies F, then he believes F. For finite models (16) is equivalent to (15). Therefore (16) is just a continuity axiom.

Let I be a finite set of agents. For each $i \in I$ let B_i be a belief operator interpreted as the belief operator of i. Let $E \in \Sigma$. E is *evident belief* if $E \subseteq B_i(E)$ for all $i \in I$. Let $C \in \Sigma$ and let $\omega \in \Omega$. C is *common belief* at ω if there exists an evident belief event E such that $\omega \in E$ and $E \subseteq B_i(C)$ for all $i \in I$.

Proposition 4. *Let* $E(C) = \bigcap_{n\geq 1} C^n$, *where* $C^0 = C$ *and for all* $n \geq 1$, $C^n = \bigcap_{i\in I} B_i(C^{n-1})$. *Then,*

$$C \text{ is common belief at } \omega \text{ iff } \omega \in E(C).$$

Proof. The proof is very similar to the analogous proof of Proposition 3 and therefore will be omitted. ■

The p-beliefs are less general than abstract belief operators, but they allow us to quantify beliefs. In the following sections we show that in many applications common p-belief is a good approximation to common knowledge when p is close to 1.

4. Agreeing to Disagree: The Case of Beliefs

Aumann (1976) has shown that two agents with common priors cannot agree to disagree. That is, if the posteriors of a certain event are common knowledge, then these posteriors must coincide. We will show that if the posteriors are common p-belief for large enough p, then these posteriors cannot differ significantly.

Fix an event X and define functions f_i for all agents i by

$$f_i(\omega) = \mu(X|\Pi_i(\omega)).$$

Let r_i, $i \in I$, be numbers in the interval [0, 1], and consider the event $C = \bigcap_{i\in I} \{u \in \Omega: f_i(u) = r_i\}$. We say that the posteriors of X are common p-belief at ω if C is common p-belief at ω.

Theorem A. *If the posteriors of the event X are common p-belief at some $\omega \in \Omega$, then any two posteriors can differ by at most $2(1 - p)$.*

Proof. As C is common p-belief at ω, there exists an event E such that $\omega \in E$, $E \subseteq B_i^p(E)$, and $E \subseteq B_i^p(C)$ for all $i \in I$. Since f_i is Π_i-measurable, f_i has a constant value on each $S \in \Pi_i$. But f_i is constantly r_i on C and therefore $f_i(u) = r_i$ for every $u \in S$ whenever $S \in \Pi_i$ and $S \cap C \neq \emptyset$. Since for every $u \in B_i^p(C)$, $\Pi_i(u) \cap C \neq \emptyset$ (since $\mu(C|\Pi_i(u)) \geq p$), it follows that $f_i = r_i$ on $B_i^p(C)$. Therefore $f_i = r_i$ on E. Applying the previous arguments once more yields $f_i = r_i$ on $B_i^p(E)$.

As $B_i^p(E)$ is a union of sets S from Π_i for which $\mu(X|S) = r_i$, we have

$$\mu(X|B_i^p(E)) = r_i.$$

Let

$$x = \mu(X|E) = \frac{\mu(X \cap E)}{\mu(E)}.$$

Then,

$$x = \frac{\mu(X \cap B_i^p(E))}{\mu(E)} - \frac{\mu(X \cap [B_i^p(E)\setminus E])}{\mu(E)}$$
$$= \mu(X|B_i^p(E)) \frac{\mu(B_i^p(E))}{\mu(E)} - \frac{\mu(X \cap [B_i^p(E)\setminus E])}{\mu(E)}.$$

Hence,

$$x = r_i \frac{\mu(B_i^p(E))}{\mu(E)} - \frac{\mu(X \cap [B_i^p(E)\setminus E])}{\mu(E)}.$$

Since $\mu(B_i^p(E)) \geq \mu(E) \geq p\mu(B_i^p(E))$ we get

$$xp \leq r_i \leq x + 1 - p$$

or

$$-x(1 - p) \leq r_i - x \leq 1 - p.$$

Hence,

$$|r_i - x| \leq 1 - p.$$

Since the last inequality holds for all $i \in I$,

$$|r_i - r_j| \leq 2(1 - p)$$

for all $i, j \in I$. ■

5. Common p-Beliefs in Bayesian Games

Let $\mathcal{J} = \{1, 2, \ldots, m\}$. Let $\mathcal{G}^1, \mathcal{G}^2, \ldots, \mathcal{G}^m$ be finite games in strategic (normal) form. For every $j \in \mathcal{J}$ the players set is $I = \{1, 2, \ldots, n\}$, the pure and mixed strategies sets of player i are A_i and Δ_i, respectively, and the payoff function of player i is $H_i^j: \Delta \to R$, where $\Delta = \otimes_{i \in I} \Delta_i$ and R is the set of real numbers. Let (Ω, Σ, μ) be a probability space. For a random variable f we will denote the integral $\int_\Omega f(u)d\mu(u)$ either by $E(f)$ or by $E_\mu(f)$. Similarly, for $S \subseteq \Omega$ with $\mu(S) > 0$ we will denote the expression $(1/\mu(S)) \int_S f(u)d\mu(u)$ by $E(f|S)$ or by $E_\mu(f|S)$. Let $J: \Omega \to \{1, 2, \ldots, m\}$ be a random variable that determines the game to be played. That is, if $J(\omega) = j$, then $\mathcal{G}^j$ is played when ω occurs. A partition Π of Ω into measurable subsets with positive probabilities is called an *information structure*.

Given information structures Π_i, $i \in I$, we define a game Γ as follows: The players set is I. For each $i \in I$ the strategies set of player i is $\bar{\Delta}_i$, where $\bar{\Delta}_i$ is the set of all Π_i-measureable functions $\sigma_i: \Omega \to \Delta_i$. For each $\sigma_i \in \bar{\Delta}_i$ and each $S \in \Pi_i$ we denote $\sigma_i(S) = \sigma_i(\omega)$, where ω is an arbitrary state in S. Let $\bar{\Delta} = \otimes_{i \in I} \bar{\Delta}_i$ be the set of strategy profiles. The payoff of player i in the game Γ is the function $F_i: \bar{\Delta} \to R$ defined by

$$F_i(\sigma) = E(H_i^J(\sigma)),$$

where for $\omega \in \Omega$, $\sigma(\omega) = (\sigma_1(\omega), \sigma_2(\omega), \ldots, \sigma_n(\omega))$. That is,

$$F_i(\sigma) = \int_\Omega H_i^{J(\omega)}(\sigma_1(\omega), \sigma_2(\omega), \ldots, \sigma_n(\omega))d\mu(\omega). \tag{17}$$

Let $\sigma_i \in \bar{\Delta}_i$, let $\sigma_{-i} \in \bar{\Delta}_{-i} = \otimes_{k \neq i} \bar{\Delta}_k$, and let $\omega \in \Omega$. We say that *σ_i is a best response strategy (b.r.s) against σ_{-i} at ω* if

$$E_\mu(H_i^J(\sigma_i(\omega), \sigma_{-i})|\Pi_i(\omega))$$

maximizes

$$E_\mu(H_i^J(s_i, \sigma_{-i})|\Pi_i(\omega))$$

over all $s_i \in \Delta_i$.

Obviously, σ_i is a b.r.s against σ_{-i} iff it is a b.r.s at each $\omega \in \Omega$. That is, σ is an *ex ante* equilibrium point iff σ is an *ex post* equilibrium point.

We now consider ε-equilibrium points. We say that σ is an *ex ante ε-equilibrium point* if for all $i \in I$

$$E_\mu(H_i^J(\sigma_i(\omega), \sigma_{-i})) \geq E_\mu(H_i^J(s_i, \sigma_{-i})) - \varepsilon$$

for all $s_i \in \Delta_i$.

We say that σ is an *ex post ε-equilibrium point* if for all $i \in I$ and for all $\omega \in \Omega$

$$E_\mu(H_i^J(\sigma_i(\omega), \sigma_{-i})|\Pi_i(\omega)) \geq E_\mu(H_i^J(s_i, \sigma_{-i})|\Pi_i(\omega)) - \varepsilon$$

for all $s_i \in \Delta_i$.

Clearly every *ex post* ε-equilibrium point is an *ex ante* ε-equilibrium point, but not vice versa.

Let G^j be the set of all states ω in which the game played is $\mathcal{G}^j$. That is

$$G^j = \{\omega \in \Omega: J(\omega) = j\}.$$

Let $G = \{G^1, G^2, \ldots, G^m\}$ and let $p > 0.5$. We say that *the game $\mathcal{G}^j$ is common knowledge (common p-belief) at ω*, if the event G^j is common knowledge (common p-belief) at ω. Note that if the game $\mathcal{G}^j$ is common knowledge at ω, then $\mathcal{G}^j$ is played at ω (i.e., $J(\omega) = j$), while if $\mathcal{G}^j$ is only common p-belief at ω, then the game played at ω may be different from $\mathcal{G}^j$.

Set $p^j = \mu(G^j)$. For each $j \in \mathcal{J}$ let $s^j \in \Delta$ be an equilibrium point in the game $\mathcal{G}^j$ with the associated payoff

$$H^j = (H_1^j(s^j), H_2^j(s^j), \ldots, H_n^j(s^j)). \tag{18}$$

Define a function (not necessarily a strategy profile!) $\sigma^*: \Omega \to \Delta$ by

$$\sigma^*(\omega) = (\sigma_1(\omega), \sigma_2(\omega), \ldots, \sigma_n(\omega)) = (s_1^{J(\omega)}, s_2^{J(\omega)}, \ldots, s_n^{J(\omega)}). \tag{19}$$

Obviously,

$$E(H^J(\sigma^*)) = \sum_j p^j H^j.$$

If the game played is common knowledge at each $\omega \in \Omega$, then each of the partitions Π_i, $i \in I$, refines G, and therefore σ_i^* is a strategy for player i (i.e., σ_i^* is Π_i-measurable). Obviously σ^* turns out to be an equilibrium point with the associated payoffs $F(\sigma^*) = \sum_j p^j H^j$.

We now examine the common p-belief case.

THEOREM B. *Let $p > 0.5$. Set*

$$1 - \delta = \mu\{\omega \in \Omega: \text{For some } j, G^j \text{ is common } p\text{-belief at } \omega\},$$

and let

$$M = \max_{i,j,s} |H_i^j(s)|. \tag{20}$$

Then there exists $\Omega' \subseteq \Omega$ with $\mu(\Omega') \geq (2p - 1)(1 - \delta)$ such that the following holds: For any selection $(s^j)_{j \in \mathcal{J}}$ of equilibrium points in the games $(\mathcal{G}^j)_{j \in \mathcal{J}}$ there exists a strategy profile $\sigma \in \bar{\Delta}$ satisfying:

B-1: *$\sigma = s^J$ on Ω'.*

B-2: *$|F_i(\sigma) - \sum_j p^j H_i^j(s^j)| \leq 2M[1 - (2p - 1)(1 - \delta)]$ for all $i \in I$.*

B-3: *σ is an ex post ε-equilibrium point whenever $\varepsilon > 2M(1 - (2p - 1))$.*

Proof. For each $j \in \mathcal{J}$ denote $E^j = E^p(G^j)$. That is, by Proposition 3

$$E^j = \{\omega \in \Omega: G^j \text{ is common } p\text{-belief at } \omega\}.$$

Let $i \in I$. If $\omega \in B_i^p(E^j)$, then $\mu(E^j|\Pi_i(\omega)) \geq p > 0.5$. Therefore, for $l \neq j$, $\mu(E^l|\Pi_i(\omega)) < 0.5$, which implies $\omega \notin B_i^p(E^l)$. Thus the sets $B_i^p(E^j)$, $j \in \mathcal{J}$, are mutually disjoint. Set $\Omega_i = \bigcup_j B_i^p(E^j)$. Define $\sigma_i(\omega) = s_i^j$ whenever $\omega \in B_i^p(E^j)$. Later we will define $\sigma_i(\omega)$ for ω in the complementary set Ω_i^c of Ω_i, but whatever our definition may be, we are now able to prove B-1 and B-2 with $\Omega' = \bigcup_j(E^j \cap G^j)$. Indeed, since for all $i \in I$, $E^j \subseteq B_i^p(E^j)$, then for each $\omega \in E^j \cap G^j$, $\sigma(\omega) = s^{J(\omega)}$. As for all $i \in I$, $E^j \subseteq B_i^p(G^j)$ and $E^j \subseteq B_i^p(E^j)$, it is easily verified that $E^j \subseteq B_i^{2p-1}(E^j \cap G^j)$ and therefore $E^j \cap G^j \subseteq B_i^{2p-1}(E^j \cap G^j)$. Hence by (9),

$$\mu(E^j \cap G^j | B_i^{2p-1}(E^j \cap G^j)) \geq 2p - 1.$$

Therefore,

$$\mu(E^j \cap G^j) \geq (2p - 1)\mu(B_i^{2p-1}(E^j \cap G^j)) \geq (2p - 1)\mu(E^j). \tag{21}$$

Thus,

$$\mu(\Omega') = \sum_j \mu(E^j \cap G^j) \geq (2p - 1) \sum_j \mu(E^j) = (2p - 1)(1 - \delta).$$

This proves B-1. To prove B-2 note that

$$|F_i(\sigma) - \sum_j p^j H_i^j(s^j)|$$
$$= |E(H_i^J(\sigma)) - E(H_i^J(s^J))| \leq 2M[1 - (2p - 1)(1 - \delta)]$$

since $\sigma = s^J$ on Ω'.

We now define σ_i on Ω_i^c. Define a new game Γ_0 as follows. The players' set is $I' = \{i \in I: \Omega_i^c \neq \emptyset\}$. A strategy for player i is a Π_i-measurable function $\tau_i: \Omega_i^c \to \Delta_i$. Given a strategy profile $\tau = (\tau_i)_{i \in I'}$ the payoff for player i is $F_i(\bar{\tau})$, where for $k \in I$, $\bar{\tau}_k(\omega) = \sigma_k(\omega)$ for $\omega \in \Omega_k$, and $\bar{\tau}_k(\omega) = \tau_k(\omega)$ for $k \in \Omega_k^c$. Let τ be an equilibrium point in the game Γ_0 (such an equilibrium necessarily exists because the strategy sets are convex compact metric spaces, and the payoff functions are multilinear). Extend σ_i to Ω_i^c by letting $\sigma_i(\omega) = \tau_i(\omega)$ for all $\omega \in \Omega_i^c$. We now proceed to prove B-3:

If $\omega \in \Omega_i^c$, then σ_i is a b.r.s. of player i at ω (not merely ε-b.r.s.). Suppose $\omega \in \Omega_i$ and let $s_i \in \Delta_i$. For $u \in E^j \cap G^j$ we have

$$H^{J(\omega)}(\sigma_i(u), \sigma_{-i}(u)) - H^{J(\omega)}(s_i, \sigma_{-i}(u)) = H^j(s^j) - H^j(s_i, s_{-i}^j) \geq 0$$

because s^j is an equilibrium point in the game $\mathcal{G}^j$.

As $\mu(\Omega'|\Pi_i(\omega)) \geq 2p - 1$ (because $\omega \in B_i^p(E^j)$ and therefore $\omega \in B_i^p(G^j)$, which implies that $\omega \in B_i^{2p-1}(E^j \cap G^j)$), we have

$$E_u(H_i^J(\sigma)|\Pi_i(\omega)) \geq E_u(H_i^J(s_i, \sigma_{-i})|\Pi_i(\omega)) - 2M(1 - (2p - 1)). \quad \blacksquare$$

Alternatively, we can state Theorem B as follows:

THEOREM B*. *Let $\varepsilon > 0$ and let $M > 0$. Then there exist $p^0 < 1$ and $\delta^0 > 0$ such that for all $p > p^0$ and for all $\delta < \delta^0$ the following holds: For every $n \geq 1$ and a set of n players I, for every $m \geq 1$ and m n-person games $\mathcal{G}^1, \mathcal{G}^2, \ldots, \mathcal{G}^m$ with $\max_{i,j,s} |H_i^j(s)| \leq M$, for every selection of equilibrium points (s^j) in the games $(\mathcal{G}^j)$, for every probability space (Ω, Σ, μ) and a random variable $J: \Omega \to \{1, 2, \ldots, m\}$, and for every n information structures (Π_i), if*

$$\mu\{\omega \in \Omega: \text{For some } j, G^j \text{ is common } p\text{-belief at } \omega\} > 1 - \delta,$$

then there exists a strategy profile $\sigma \in \bar{\Delta}$ such that:

B*-1: *$\mu(\sigma(\omega) = s^{J(\omega)}) > 1 - \varepsilon$.*

B*-2: *$|F_i(\sigma) - \sum_j p^j H_i^j(s^j)| < \varepsilon$ for all $i \in I$.*

B*-3: *σ is an ex post ε-equilibrium point.*

We say that *the game* $\mathcal{G}^j$ *(or the event* G^j*) is p-believed by the players at* ω, if G^j is p-believed at ω by each $i \in I$.

THEOREM C*. *Let* $\varepsilon > 0$ *and let* $M > 0$. *Then there exist* $p^0 < 1$ *and* $\delta^0 > 0$ *such that for all* $p > p^0$ *and for all* $\delta < \delta^0$ *the following holds: For every* $n \geq 1$ *and a set of n players I, for every* $m \geq 1$ *and m n-person games* $\mathcal{G}^1, \mathcal{G}^2, \ldots, \mathcal{G}^m$ *with* $\max_{i,j,s} |H_i^j(s)| \leq M$, *for every selection of equilibrium points* (s^j) *in the games* $(\mathcal{G}^j)$, *for every probability space* (Ω, Σ, μ) *and a random variable* $J: \Omega \to \{1, 2, \ldots, m\}$, *and for every n information structures* (Π_i), *if*

$$\mu\{\omega \in \Omega: \text{For some } j, G^j \text{ is } p\text{-believed by the players at } \omega\} > 1 - \delta,$$

then there exists a strategy profile $\sigma \in \bar{\Delta}$ *such that:*

B*-1: $\mu(\sigma(\omega) = s^{J(\omega)}) > 1 - \varepsilon$.

B*-2: $|F_i(\sigma) - \Sigma_j\, p^j H_i^j(s^j)| < \varepsilon$ *for all* $i \in I$.

C*-3: σ *is an ex ante* ε*-equilibrium point.*

Proof. For all $i \in I$ let $\Omega_i = \bigcup_j B_i^p(G^j)$. Define $\sigma_i(\omega) = s_i^j$ whenever $\omega \in B_i^p(G^j)$ and define σ_i arbitrarily for $\omega \in \Omega_i^c$.

It is obvious that $\sigma(\omega) = s^{J(\omega)}$ for $\omega \in \Omega^*$, where

$$\Omega^* = \bigcup_j \left[G^j \cap \left(\bigcap_i B_i^p(G^j)\right)\right].$$

If $1 - p$ and δ are sufficiently small, then $\mu(\Omega^*)$ is close to 1, which simultaneously proves B*-1, B*-2, and C*-3. ∎

As there are only finitely many players, Theorem C* ensures that if the partition to games is p-believed by each of the players at a large set, where p is sufficiently close to 1, then the players (using *ex ante* ε-equilibrium strategies) can almost correlate their strategies according to a given selection of equilibrium points in the games $\mathcal{G}^j$, $j \in \mathcal{J}$.

Theorem B* ensures that if the partition to games is *common* p-belief at a large set, then the players can almost correlate their strategies using *ex post* ε-equilibrium strategies.

We conclude this section by pointing out the obvious but interesting fact that we can dispose of the assumption of common priors in Theorems B, B*, and C*. First note that one can define the belief operators B_i^p by using a different prior for each agent i. Then one can use the results of Section 3 to define common p-belief. Finally, instead of requiring that with high probability the games be common p-belief, we can require that each of the players, using his own priors, assigns a high probability to the event that the games are common p-belief.

To illustrate the results of this section we adopt (with minor changes) an example of Rubinstein (1987).

EXAMPLE 2. Nature chooses in equal probabilities one of two two-person games $\mathcal{G}^1$ and $\mathcal{G}^2$ with the same strategies sets. Player 1 is informed which game is to be played. If the game chosen is $\mathcal{G}^2$ he sends a message to player 2 telling him that $\mathcal{G}^2$ is to be played. They then send messages back and forth acknowledging, each in his turn, the previously received message. Each message has a probability $\tau < 0.5$ of being lost, in which case the whole process stops, and the players knowing the number of messages they sent choose their strategies. Note that the process stops with probability 1 after a finite number of steps and that the whole process is not part of the strategic decisions of the players, but is given to them. We now analyze the game Γ_τ just described with

$$\mathcal{G}^1 = \begin{bmatrix} (1, 1) & (1, 0) \\ (0, 0) & (0, 0) \end{bmatrix} \quad \text{and} \quad \mathcal{G}^2 = \begin{bmatrix} (0, 0) & (0, -2) \\ (-2, 0) & (1, 1) \end{bmatrix}.$$

The set of states is $\Omega = \{0, 1, 2, \ldots\}$, where each k in Ω represents the event "k messages had been sent until the process stopped." Therefore, $\mu(0) = 0.5$ and, for $k \geq 1$, $\mu(k) = 0.5\tau(1 - \tau)^{k-1}$. Also, $\mathcal{G}^1$ is played iff $k = 0$. That is, the partition to games is $G = \{G^1, G^2\}$, where $G^1 = \{0\}$ and $G^2 = \{1, 2, 3, \ldots\}$ (i.e., $J(0) = 1$, and, for $k \geq 1$, $J(k) = 2$).

When the process stops neither of the players knows whether it was his message that got lost or his partner's acknowledgment. Therefore the information structures are

$$\Pi_1 = \{\{0\}, \{1, 2\}, \{3, 4\}, \ldots\}$$

and

$$\Pi_2 = \{\{0, 1\}, \{2, 3\}, \{4, 5\}, \ldots\}.$$

Let $p_\tau = 1/(1 + \tau)$. It is easily verified that

$$B_1^{p_\tau}(G^1) \cap B_2^{p_\tau}(G^1) = \{0\}$$

and that

$$B_1^{p_\tau}(G^2) \cap B_2^{p_\tau}(G^2) = \{2, 3, \ldots\}.$$

Therefore the game played is p_τ-believed by the players at each $k \neq 1$. Let $s^1 = (R1, C1)$ and $s^2 = (R2, C2)$, where Ri and Cj stand for row i and column j, respectively. Obviously s^1 and s^2 are equilibrium points in the games $\mathcal{G}^1$ and $\mathcal{G}^2$, respectively. As both $p_\tau \to 1$ and $\mu(\{1\}) = 0.5\tau \to 0$ when

$\tau \to 0$, Theorem C* ensures (for any $\varepsilon > 0$ and sufficiently small τ) the existence of an *ex ante* ε-equilibrium point σ with

$$\mu(\{k \in \Omega: \sigma(k) = s^{J(k)}\}) > 1 - \varepsilon. \tag{22}$$

Since $\mu(\{0\}) = 0.5$, (22) implies that $\sigma(\{0\}) = s^1$ and that $\sigma(k) = s^2$ for $k \in \Omega' \subseteq \{1, 2, 3, \ldots\}$, with $\mu(\Omega') > 0.5 - \varepsilon$.

Indeed, such an equilibrium is obtained for $\tau < \varepsilon$ if both players choose s^j whenever they p_τ-believe G^j. That is, $\sigma_1(0) = \sigma_2(0) = \sigma_2(1) = s^1$, $\sigma_1(1) = s^2$, and $\sigma(k) = (s^2, s^2)$ for all $k \geq 2$.

However, for $\varepsilon < \frac{1}{3}$ the above selection of strategies σ is not an *ex post* ε-equilibrium point. To verify this consider the information set $\{1, 2\}$ of player 1. If σ was an *ex post* ε-equilibrium point, then

$$H_1^2(R2, C1)\frac{1}{2-\tau} + H_1^2(R2, C2)\frac{1-\tau}{2-\tau}$$
$$> H_1^2(R1, C2)\frac{1}{2-\tau} + H_1^2(R1, C2)\frac{1-\tau}{2-\tau} - \varepsilon.$$

Therefore $(1 - \tau)/(2 - \tau) < \varepsilon$, and as $\tau < 0.5$ we get $\varepsilon > \frac{1}{3}$.

Moreover, it can be shown (using similar arguments) that if $\varepsilon < \frac{1}{24}$, then for every *ex post* ε-equilibrium σ, the players will (at any state k) choose row 2 and column 2, respectively, with a probability of at most $\frac{1}{8}$. That is, if for all $k \geq 0$

$$\sigma(k) = ((p_1^k, p_2^k), (q_1^k, q_2^k)),$$

then for all $k \geq 0$, $p_2^k \leq \frac{1}{8}$ and $q_2^k \leq \frac{1}{8}$. So, by Theorem B*, there exists $p_0 < 1$ and $\alpha_0 < 1$ such that no matter how small τ is, for $p > p_0$, the amount of common p-belief at the game Γ_τ is at most α_0. Indeed, for $p > \frac{1}{2}$ it is easily verified that $E^p(G^2) = \emptyset$ and $E^p(G^1) \subseteq \{0\}$, and therefore

$$\mu\{k \in \Omega: \text{For some } j, G^j \text{ is common } p\text{-belief at } k\} \leq 0.5.$$

However, if player 2 is not supposed to acknowledge, then the situation is totally different. In this case $\Omega = \{0, 1NR, 1R\}$, where $1R$ stands for "player 1 sends a message and it is received by player 2," and $1NR$ stands for the case that the message has not been received. Hence, $\mu(0) = 0.5$, $\mu(1NR) = 0.5\tau$, $\mu(1R) = 0.5(1 - \tau)$, and the information structures are $\Pi_1 = \{\{0\}, \{1NR, 1R\}\}$ and $\Pi_2 = \{\{0, 1NR\}, \{1R\}\}$. It is easily checked that $E^{p_\tau}(G^1) = \{0\}$ and that $E^{p_\tau}(G^2) = \{1R\}$, and hence

$$\mu\{k \in \Omega: \text{For some } j, G^j \text{ is common } p\text{-belief at } k\} = 1 - 0.5\tau \to 1$$

when $\tau \to 0$.

Thus, by Theorem B*, for any $\varepsilon > 0$, for sufficiently small τ we can find an *ex post* ε-equilibrium point σ in which the players play according to s^1 and s^2 in a probability of at least $1 - \varepsilon$. Indeed, such a strategy profile σ is given by $\sigma(0) = s^1$ and $\sigma(1R) = s^2$, which uniquely defines a strategy profile.

6. Conclusion

We proposed the notion of common p-belief as an approximation to common knowledge and showed that theories that use common knowledge can be approximated when common p-belief replaces common knowledge. The concept of common p-belief can be considered a formal definition of almost common knowledge. A priori there are two possible ways to define it. We have chosen to weaken the "knowledge" part of common knowledge, requiring that it be only a belief that is shared by the agents. Thus we defined almost common knowledge to be common almost-knowledge, leaving intact the "common," that is, the way agents share what they share. We could instead weaken the "common" part of common knowledge, defining almost common knowledge to be almost-common knowledge. Viewing common knowledge from the iterative viewpoint suggests that this can be done by truncating the hierarchy of knowledge. The coordinating attack game shows that as far as equilibrium theory is concerned this approach fails. The reason seems to be that a complete analysis cannot be carried out on the basis of truncated hierarchies. Two situations in which truncation takes place after the same number of iterations may differ dramatically because different beliefs are held beyond the truncation point. One is necessarily led to consider an infinite hierarchy of beliefs which results in the approach we adopted.

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