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A family of ordinal solutions to bargaining problems with many players ☆

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Abstract

A solution to bargaining problems is ordinal when it is covariant with respect to order-preserving transformations of utility. Shapley has constructed an ordinal, symmetric, efficient solution to three-player problems. Here, we extend Shapley's solution in two directions. First, we extend it to more than three players. Second, we show that this extension lends itself to the construction of a continuum of ordinal, symmetric, efficient solutions. The construction makes use of ordinal path-valued solutions that were suggested and studied by O'Neil et al. [Games Econ. Behav. 48 (2004) 139–153].
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1. Introduction

1.1. Ordinal solutions

A bargaining problem is described here, as in Nash's (1950) bargaining theory, by the set of all utility vectors that arise from possible agreements. A solution is a function that selects for each problem a vector of utilities.

☆ A PowerPoint presentation of this article is available at http://www.tau.ac.il/~samet.

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The utility functions in Nash's theory are assumed to be derived from the von Neumann–Morgenstern representation of preferences. This representation is determined up to linear positive transformations of the utility functions. Therefore, any two problems obtained from each other by such transformations should be considered equivalent. Thus, a solution in this theory must be covariant with respect to such transformation. Namely, it should assign to any two equivalent problems the same solution, up to the required transformation. Indeed, one of the axioms which characterizes Nash's solution spells explicitly this requirement.

Suppose, that contrary to Nash's theory, no assumption is made on the utility functions other than that they represent preferences (i.e. the more preferred outcome has a higher utility). In this case the presentation of preferences is determined up to order-preserving (i.e. monotonically increasing) transformations of utility functions. Hence, a solution in this bargaining theory should be covariant with respect to these transformations. We say that such a solution is *ordinal*.

1.2. Shapley's solution for three players

Shapley (1969) has shown that there is no ordinal, symmetric, and efficient solution for bargaining problems of two players. However, he has constructed such a solution for three-player problems (see Shubik, 1982).[1]

The construction is based on the following observation. Suppose that $a = (a_1, a_2, a_3)$ is the disagreement point of a bargaining problem with a Pareto surface S. Then there exists a unique point $\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3)$, such that the points,

$$(a_1, \bar{x}_2, \bar{x}_3), \qquad (\bar{x}_1, a_2, \bar{x}_3), \qquad \text{and} \qquad (\bar{x}_1, \bar{x}_2, a_3),$$

are all in S. In the terminology of Kalai and Smorodinsky (1975) the point a is the ideal point for $\bar{x}$.[2] Reversing the order we say that the point $\bar{x}$ is the *ground* point for a. (See Fig. 1.)

The relation between a point and its unique ground point is ordinal. Thus, assigning to each problem the ground point of its disagreement point is an ordinal solution. This solution is also symmetric, but it is not on the Pareto surface of the problem.

To fix this latter deficiency Shapley used this solution iteratively, applying it in each step to the problem with the same Pareto surface S, and a disagreement point which is the solution obtained in the previous step. The sequence of points generated this way can be shown to converge to a point on the Pareto surface, which is the desired solution.

The construction of Shapley's solution hinges on both the existence and the uniqueness of the ground point $\bar{x}$ for any given a. For more than three players the construction cannot be carried out since the uniqueness of a ground point is not guaranteed, as was shown by Sprumont (2000). However, Safra and Samet (2004) proved for any number of players the existence of at least one ground point for each point a. They used the *set* of ground

[1] Recently, Kıbrıs (2003) has proposed an axiomatization of the three-player Shapley solution.

[2] Kalai and Smorodinsky (1975) defined the ideal point for a feasible disagreement point. However, the feasibility assumption is not used in their definition, and therefore it can be applied also to infeasible points like $\bar{x}$ in this example.

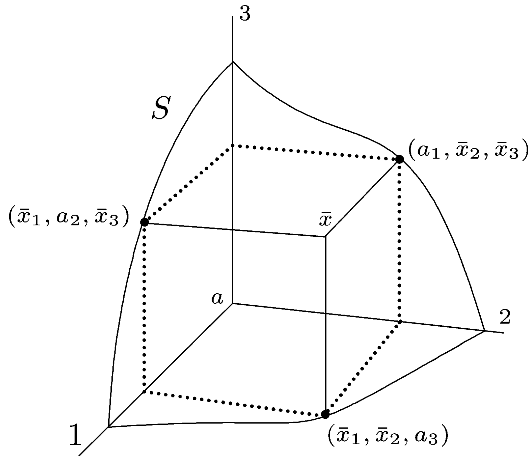

Fig. 1. The unique ground point for a three-player problem.

points for a given disagreement point to generate an auxiliary ordinal, symmetric solution. This solution was used iteratively, as in the three-player case, to define a solution with the desired properties. For the three-player case this construction coincides with Shapley's solution, since the auxiliary solution in this case yields the unique ground point of the disagreement point.

Here, we present another way to extend Shapley's solution to more than three players. This extension, which we call the basic extension, has the advantage of serving in turn as the basis for a further extension to a continuum of ordinal, symmetric, and efficient solutions.[3] Like Shapley's solution and its extension in Safra and Samet (2004), all the solutions proposed here are based on the construction of an auxiliary solution which is applied iteratively. We first describe in broad strokes and without proofs the basic extension of Shapley's solution to more than three players and then turn to the construction of a whole family of solutions.

Recently, Calvo and Peters (2002) presented another ordinal solution, suggested by Shapley, which does not require an iterative process like all the solutions studied here.

1.3. The basic extension of Shapley's solution

Consider a bargaining problem for the set of players $N = \{1, 2, 3\}$ with a Pareto surface S and disagreement point a. This problem induces a family of bargaining problems for players 1 and 2 as follows.

We consider the plane where 3's utility is fixed at her disagreement utility a_3. For any given utility level x_3 of 3 we consider the projection on the said plane, of the points in S where 3's utility is x_3. The projection of these points forms a Pareto line of bargaining problem for 1 and 2, as depicted in Fig. 2. Let $p^3(x_3)$ be the ideal point of a for this

[3] The extension here is also simpler than that of Safra and Samet (2004) in that it is more constructive and does not require the use of a fixed point argument.

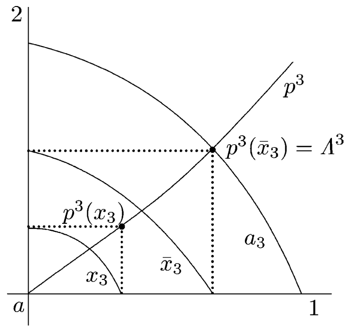

Fig. 2. Constructing Λ^3.

The set of players is $N = \{1, 2, 3\}$. The plane where 3's utility is fixed at a_3 is depicted. For a fixed x_3, the projection of all the points in S whose third coordinate is x_3 form a Pareto line of a bargaining problem of 1 and 2. Three such Pareto lines are drawn, for the values x_3, $\bar{x}_3$ and a_3. The latter line is the intersection of S with the plane. The ideal point for the problem defined by x_3 is $p^3(x_3)$. The path p^3 intersects S at Λ^3, for $\bar{x}_3$. The point Λ^3 is the projection of the ground point $\bar{x}$ on the plane, as depicted in Fig. 1.

bargaining problem of 1 and 2. When the utility level x_3 varies, we obtain a path p^3 in the plane, parameterized by x_3. The path intersects the Pareto surface S at a single point Λ^3, obtained for 3's utility level $\bar{x}_3$. The points Λ^2 and Λ^1 are similarly defined. It is easy to see that these three points are the projections $(a_1, \bar{x}_2, \bar{x}_3)$, $(\bar{x}_1, a_2, \bar{x}_3)$, and $(\bar{x}_1, \bar{x}_2, a_3)$, of the ground point $\bar{x}$ of a.

The same construction can be carried out for bargaining problems with larger number of players. For each i we consider a family of bargaining problems for the players in $N \setminus i$. This family is parameterized by i's utility x_i, and it is embedded in the hyperplane at which i's utility is fixed at her disagreement utility a_i. The Pareto surface of the problem associated with x_i is the projection on the said hyperplane of all the points in S at which i's utility is x_i. The ideal point of the bargaining problem associated with x_i is denoted by $p^i(x_i)$. The path p^i thus defined intersects S at one point denoted Λ^i. This construction is symmetric in all the players in $N \setminus i$ and being defined by ideal points and S it is ordinal.

For problems with more than three players, the points Λ^i are not necessarily the projection of a ground point for a. We use them to define a point Φ, where Φ_i is the minimum of i's payoffs at the points Λ^j for $j \neq i$. When the disagreement point a is on the other side of S, maximum, rather than minimum, is used in the definition of Φ.

In the three-player case the point Φ is obviously the ground point $\bar{x}$ for a.

The construction of Φ is symmetric in the players. Being constructed from the ordinally constructed points Λ^i, using the order-preserving functions min and max, Φ's construction is also ordinal. Using iteratively the auxiliary solution described by the construction of Φ, as in Shapley's solution, yields a solution with the required properties.

1.4. A family of solutions

The point $p^i(x_i)$, used in the construction of the basic solution, is the ideal point of some bargaining problem. We generalize the solution in the previous subsection by extending the

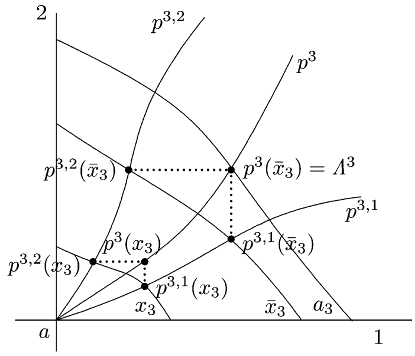

Fig. 3. Defining a path p^3 with respect to guidelines.

The set of players is $N = \{1, 2, 3\}$. The guidelines $p^{3,1}$ and $p^{3,2}$ are ordinal paths. The path p^3 consists of ideal points for the projected bargaining problems with respect to these guidelines.

notion of ideal point. Compare Fig. 3 and Fig. 2. Figure 3 includes two extra paths $p^{3,1}$ and $p^{3,2}$, which we call the guidelines for player 3. We interpret the point $p^{3,1}(x_3)$, which is on the Pareto line of the bargaining problem associated with x_3, as the best point player 1 can expect in this problem. The point $p^{3,2}(x_3)$ is similarly interpreted. Thus the point $p^3(x_3)$ provides both players 1 and 2 their best payoffs, and it is the ideal point with respect to the given guidelines. The guidelines in the basic solution described in the previous subsection are obviously the two axes that pass through a.

Given the paths p^i, the rest of the construction is the same as in the basic solution. Obviously, in order for this construction to be ordinal, the guideline themselves should be ordinal. Here we use a result of O'Neill et al. (2004) who studied bargaining situations in which players face a family of increasing bargaining problems ordered by time, rather than a single bargaining problem. A solution for such a problem is a path, parameterized by time, which specifies an agreement for each of the problems in the given family. O'Neill et al. (2004) characterized axiomatically such a solution which turns out to be covariant with respect to order-preserving transformations of both utility and time for any number of players.

We employ here ordinal path-valued solution for the family of bargaining problems for $N \setminus i$ which are parameterized by i's utility level, rather than time. A weighted version of the path-valued ordinal solution of O'Neill et al. (2004) provides us with a plethora of ordinal paths that can be used as guidelines.

1.5. The paper plan

In Section 2 we introduce Pareto surfaces, order-preserving transformations, and solutions to bargaining problems. We discuss path-valued solutions in Section 3. The construction of the family of ordinal solutions is carried out in Section 4. In Section 5 we focus on one solution from this family, the basic extension, which coincides with Shapley's solution to three-player problems. The construction of other families of solutions is proposed in Section 6. The proofs are in Section 7.

2. Preliminaries

2.1. Pareto surfaces

Consider a finite set N of n players, with $n \geqslant 2$. A point in R^N describes the utility levels of the players. For $x = (x_i)_{i \in N}$ and $y = (y_i)_{i \in N}$ in R^N we write $x \geqq y$ when $x_i \geqslant y_i$ for each $i \in N$, $x \geqslant y$ when $x \geqq y$ and $x \neq y$, and $x > y$ if $x_i > y_i$ for each $i \in N$. The inequalities $\leqq, \leqslant$ and $<$ are similarly defined. For each proper subset M of N, we denote by x_{-M} a generic point in $R^{N \setminus M}$. For $x = (x_i)_{i \in N}$ in R^N, the vector x_{-M} is the projection of x on $R^{N \setminus M}$, i.e., the vector $(x_i)_{i \in N \setminus M}$. When M is a singleton we omit the curly brackets and write x_{-i} and $N \setminus i$.

Definition 1. A subset $S \subset R^N$ is a *Pareto surface* (a *surface*, for short) when for all $x, y \in S$, $x \geqq y$ implies $x = y$, and for each i, the projection of S on $R^{N \setminus i}$ is $R^{N \setminus i}$.

The following properties of Pareto surfaces are proved in Safra and Samet (2004). If S is a Pareto surface, then for each i and $x \in R^N$ there is a unique number called i's *Pareto payoff* at x, and denoted by $\pi_i^S(x)$ such that $(x_{-i}, \pi_i^S(x)) \in S$. The function $\pi_i^S : R^N \to R^i$ thus defined is continuous, it is strictly decreasing in x_j for $j \neq i$, and constant with x_i. We omit the superscript S from π_i^S, when the surface S is clear from the context. Using the terminology of Kalai and Smorodinsky (1975), we call $\pi(x) = (\pi_i(x))_{i \in N}$ the *ideal point* of x. We say that x is a *ground point* for $\pi(x)$.

The relation between the points x and $\pi(x)$ defines the position of x with respect to S. If $x < \pi(x)$ we say that x is *below* S and denote it by $x \prec S$ or $S \succ x$; if $x > \pi(x)$ we say that x is *above* S and denote it by $x \succ S$ or $S \prec x$; if $x = \pi(x)$, then $x \in S$. These three possibilities are exhaustive. We write $a \succeq S$ when either $a \succ S$ or $a \in S$. The relation $\preceq$ is similarly defined.

We assume that the Pareto surfaces are smooth in the following sense.

Definition 2. A surface S is *smooth* if the following hold for each i:

(1) the function π_i is continuously differentiable;
(2) for each $j \neq i$, $\partial \pi_i / \partial x_j < 0$;
(3) $\nabla \pi_i$ is Lipschitz on any bounded subset of R^N.

2.2. Bargaining problems and solutions

A *bargaining problem* (a *problem*, for short) for a set of players N is a pair (a, S), where S is a smooth Pareto surface in R^N and $a \in R^N$. The point a is called the *disagreement point*. The set of all problems is denoted by $\mathcal{B}$. A *solution* is a function $\Psi : \mathcal{B} \to R^N$.

We are interested in solutions that depend on representation of preferences by utility functions, but not on any specific choice of these functions. Such a solution should be covariant with order-preserving transformations of utility functions, which we introduce next.

An *order-preserving transformation* is a vector of functions $\mu = (\mu_i)_{i \in N}$ such that for each i, μ_i is a function from R onto R, with strictly positive derivative. The vector μ defines a map from R^N onto R^N by $\mu(x) = (\mu_i(x_i))_{i \in N}$.

An order-preserving transformation μ preserves the properties of Pareto surfaces as we state now.

Observation 1 (Safra and Samet, 2004). If S is a Pareto surface and μ an order-preserving transformation, then the set $\mu(S) = \{\mu(x) \mid x \in S\}$ is also a Pareto surface; $a \prec S$ ($a \succ S$) iff $\mu(a) \prec \mu(S)$ ($\mu(a) \succ \mu(S)$); and

$$\pi^{\mu(S)}\big(\mu(x)\big) = \mu\big(\pi^S(x)\big). \tag{1}$$

Definition 3. A solution to bargaining problems is *ordinal* if for each problem (a, S) and order-preserving transformation μ,

$$\Psi\big(\mu(a), \mu(S)\big) = \mu\big(\Psi(a, S)\big).$$

Here we construct a family of ordinal solutions which are also symmetric and efficient. A solution Ψ is *symmetric* when it is covariant with respect to any permutation of players. It is *efficient* when it satisfies $\Psi(a, S) \in S$ for each problem (a, S). Finally, the solution is *individually rational* if for each problem (a, S) such that a is below S or on S, $\Psi(a, S) \geqq a$.

3. Path-valued solutions

3.1. Gradual bargaining

We associate with each bargaining problem (a, S) and player i a family of bargaining problems for the players in $N \backslash i$, parameterized by i's utility. Following O'Neill et al. (2004) we call such a family a *gradual bargaining problem*.

Consider the hyperplane $H = \{x \mid x_i = a_i\}$ where i's utility is fixed at a_i. Since i's utility is fixed in H, it can be viewed as representing possible agreements between the players other than i. Each value $\bar{x}_i$ of i's utility defines a Pareto surface $S_{\bar{x}_i}$ in H of all the points x in H such that $\pi_i^S(x) = \bar{x}_i$.[4] In other words, $S_{\bar{x}_i}$ is the projection on H of the set $S \cap \{x \mid x_i = \bar{x}_i\}$. The gradual bargaining problem consists of all bargaining problems $(a, S_{\bar{x}_i})$ in H.

A *path* for this gradual bargaining problem is a function $p : R^i \to H$, which assigns a point in H to each value x_i of i's utility. A *path-valued solution* assigns to each problem (a, S) a path $p(a, S)$.[5] For our purpose we need *ordinal* path-valued solutions which we define next.

[4] We are a little bit sloppy here, because Pareto surfaces were defined in full dimensional spaces. The precise statement is that when we omit the fixed i coordinate these sets are Pareto surfaces in $R^{N \backslash i}$.

[5] Here, unlike O'Neill et al. (2004), the term path-valued solution is used to describe a solution for bargaining problems rather than gradual bargaining problems.

Definition 4. A path-valued solution $(a, S) \to p(a, S)$ is *ordinal* if for each smooth problem (a, S), order-preserving transformation μ, and x_i,

$$p\big(\mu(a), \mu(S)\big)\big(\mu_i(x_i)\big) = \mu\big(p(a, S)(x_i)\big).$$

In the next subsection we describe a family of ordinal path-valued solutions which are defined by certain differential equations. These solutions are weighted versions of the solution characterized axiomatically in O'Neill et al. (2004).

3.2. *A family of ordinal path-valued solutions*

In Fig. 4 we illustrate the meaning of the differential equation that defines the ordinal path-valued solutions.

Consider a bargaining problem (a, S) for $N = \{1, 2, 3\}$. We depict in this figure the plane H in R^N, where player 3 is bound to her disagreement utility a_3. The curve $S_{\bar{x}_3}$ consists of all the points x in H where $\pi_3(x) = \bar{x}_3$. Thus, it is the projection on H of the points in S where player 3's utility is fixed at level $\bar{x}_3$.

Suppose now, that a point x on this curve is on the path assigned by the solution to this problem. The path is parameterized by 3's utility. When we increase it, the resulting curve gets closer to the point a. The arrow in this figure describes the direction of the path at x. The slope of this arrow is the ratio of the marginal losses of 1 and 2 as a result of the increase in 3's utility. This ratio is required to be the rate of exchange of 1 and 2's utilities at the point x on the given curve. This rate is the negative of the slope of the tangent to the curve at x, namely,

$$\frac{\partial \pi_3^S}{\partial x_1}(x) \Big/ \frac{\partial \pi_3^S}{\partial x_2}(x).$$

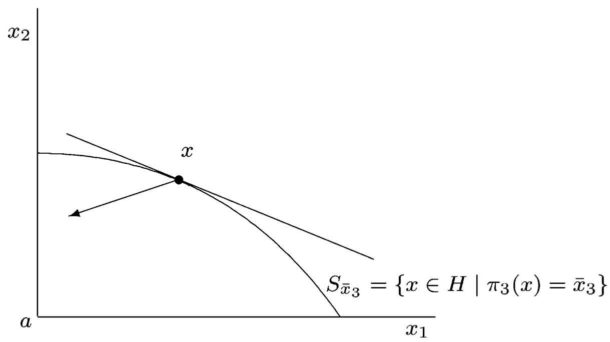

Fig. 4. The direction of the path at point x.

The plane H, where $x_3 = a_3$, is depicted. The Pareto surface $S_{\bar{x}_3}$, corresponding to the utility $\bar{x}_3$ of player 3, is drawn. The arrow indicates the direction of the path at a point x on it. The ratio of the marginal losses of players 1 and 2 in this direction is the rate of utility exchange at x along the curve $S_{\bar{x}_i}$. Thus, the slope of the direction is the negative of the slope of the tangent to $S_{\bar{x}_i}$ at x.

The generalization to more than three players is straightforward. The direction of the path at x is such that the ratio of marginal losses of *any* two players is defined by the rate of their utility exchange.

The differential equation below defines a weighted version of this path. It assumes a weighted rate of utility exchange, where each player j is assigned a weight w_j, and the ratio of players j and k's marginal losses at x is required to be

$$\left(\frac{\partial \pi_i^S}{\partial x_k}(x)/w_k\right) \Big/ \left(\frac{\partial \pi_i^S}{\partial x_j}(x)/w_j\right).$$

We now define formally this path-valued solution. Fix player i and a vector of weights $w_{-i} \in R^{N \setminus i}$ such that $\sum_{j \neq i} w_j = 1$. For each smooth problem (a, S) consider the path $p = p(a, S)$ defined by the following differential equations and initial condition:

$$\begin{aligned} & p_j'(x_i) = w_j \left[\frac{\partial \pi_i^S}{\partial x_j}\big(p(x_i)\big)\right]^{-1}, \quad j \in N \setminus i, \\ & p_i'(x_i) = 0, \\ & p\big(\pi_i^S(a)\big) = a. \end{aligned} \tag{2}$$

By condition 2 in Definition 2, the right-hand side of (2) is well defined. By conditions 1 and 3, it is continuous and satisfies the Lipschitz condition on any bounded subset. Therefore there exists a unique solution to these equations (see Hartman, 1982). Note that the initial condition in the last line of the equation says that the path passes at the point a. Also, the equation $p_i' = 0$ implies that $p_i = a_i$, that is, i's utility is fixed at a_i at all the points on this path, and hence the path is in the hyperplane H. Finally,

Proposition 1. *The path-valued solution* $(a, S) \to p(a, S)$ *defined by* (2) *is ordinal.*

3.3. Guidelines

In constructing the ordinal path-valued solutions above we singled out a player i whose utility remains fixed at her disagreement utility in all the points on the path. This is obviously an obstacle to symmetry of all the players. The symmetry of the players in $N \setminus i$ depends on the weight vector w_{-i}. If all the components of this vector are the same, then all the players other than i are treated symmetrically. We say that a solution that treats equally $n - 1$ of the players is *almost symmetric*. We show now how we can construct a continuum of almost symmetric ordinal path-valued solutions.

Fix a in $[0, 1]$ and a player $j \neq i$. Consider the weight vector w_{-i} for which $w_j = \alpha$, and $w_m = (1 - \alpha)/(n - 2)$ for all players $m \in N \setminus \{i, j\}$. We denote by $p^{i,j}(a, S)$ the path defined by (2) for this weight vector and problem (a, S). We call the solution $(a, S) \to p^{i,j}(a, S)$ the ij*-guideline*. In this solution the players in $N \setminus \{i, j\}$ are treated symmetrically.

To achieve almost symmetry we use the ij-guidelines for $j \in N \setminus i$ to construct a new ordinal path-valued solution $(a, S) \to p^i(a, S)$. For a problem (a, S) the path $p^i(a, S)$ is defined such that for each player k,

$$p_k^i(x_i) = \begin{cases} \max_{j \neq i} p_k^{i,j}(x_i) & \text{if } x_i < \pi_i(a), \\ a_k & \text{if } x_i = \pi_i(a), \\ \min_{j \neq i} p_k^{i,j}(x_i) & \text{if } x_i > \pi_i(a). \end{cases} \tag{3}$$

By construction, all players other than i are treated symmetrically in p^i, as required. Note that since each of the guidelines is in H so is also p^i. That is, $p_i^i(x_i) = a_i$ for all x_i. The construction of p^3 for three players is depicted in Fig. 3.

Finally, being defined from ordinal path-valued solutions by the order preserving functions min and max, the construction of p^i preserves ordinality.

Proposition 2. *The path-valued solution $(a, S) \to p^i(a, S)$ is ordinal.*

4. From path-valued solutions to solutions

Using the n almost symmetric path-valued solutions p^i, which were defined in the previous section for α in $[0, 1]$, we construct now an ordinal, efficient, and symmetric solution Ψ^α in three steps.

(1) For each i we use the ordinal path-valued solution p^i to define an ordinal solution Λ^i.
(2) Using $(\Lambda^i)_{i \in N}$ we define a symmetric ordinal solution Φ.
(3) Applying Φ repeatedly, using in each stage the agreement of the previous stage as a disagreement point, we construct a converging sequence of points. The solution that assigns to each problem the limit point constructed for the problem is the required solution Ψ^α.

Step 1. We first observe that the path $p^i(a, S)$ intersects the Pareto surface S at a single point.

Proposition 3. *For each problem (a, S) there exists a unique efficient point on the path $p^i(a, S)$. That is, there exists a unique $\bar{x}_i$, such that $p^i(a, S)(\bar{x}_i) \in S$.*

For each i define a solution Λ^i by letting $\Lambda^i(a, S)$ be the unique efficient point on $p^i(a, S)$.

Observation 2. The solution Λ^i is ordinal.

Indeed, suppose $\Lambda^i(a, S) = p^i(a, S)(\bar{x}_i) \in S$. Then

$$\mu\big(\Lambda^i(a, S)\big) = \mu\big(p^i(a, S)(\bar{x}_i)\big) \in \mu(S).$$

By the ordinality of p^i, $\mu(\Lambda^i(a, S)) = p^i(\mu(a), \mu(S))(\mu_i(\bar{x}_i)) \in \mu(S)$. By the uniqueness of the efficient point on $p^i(\mu(a), \mu(S))$, this point must be $\Lambda^i(\mu(a), \mu(S))$.

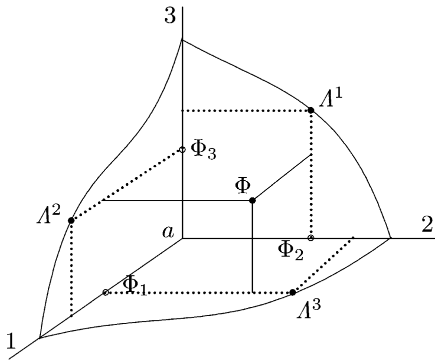

Fig. 5. Constructing Φ from the points Λ^i.

Figure 3 illustrates part of the paths $p^{3,1}$, $p^{3,2}$, and p^3, and the solution $\Lambda^i(a, S)$, for a problem (a, S) with $N = \{1, 2, 3\}$.

We now have n ordinal solutions Λ^i. Each is almost symmetric. Using them we construct a symmetric ordinal solution.

Step 2. Define a solution Φ as follows. For each player j,

$$\Phi_j(a, S) = \begin{cases} \min_{i \neq j} \Lambda^i_j(a, S) & \text{if } a \prec S, \\ a_j & \text{if } a \in S, \\ \max_{i \neq j} \Lambda^i_j(a, S) & \text{if } a \succ S. \end{cases}$$

Proposition 4. *Φ is ordinal and symmetric.*

Figure 5 illustrates the construction of $\Phi(a, S)$ from $\Lambda^i(a, S)$, for a three-player bargaining problem with $a \prec S$.

The solution Φ may fail the efficiency axiom, which we correct in the next step.

Step 3. For each problem (a, S) define a sequence of points $(a^k)_{k \geqslant 0}$, by $a^0 = a$, and for $k \geqslant 0$, $a^{k+1} = \Phi(a^k, S)$.

Proposition 5. *For each problem (a, S), the sequence $(a^k)_{k \geqslant 0}$ converges to a point in S.*

Theorem 1. *The solution $\Psi^\alpha(a, S) = \lim a^k$ is ordinal, efficient, symmetric, and individually rational.*

5. The basic extension of Shapley's solution: Ψ^1

We now show that the solution Ψ^1 is the basic extension of Shapley's solution, described in Section 1.3.

For $\alpha = 1$, $p^{i,j}$ is the solution of the differential equations with the initial condition

$$p'_j(x_i) = \left[\frac{\partial \pi_i^S}{\partial x_j}\big(p(x_i)\big)\right]^{-1}, \tag{4}$$

$$p'_k(x_i) = 0, \quad k \in N \setminus j, \tag{5}$$

$$p\big(\pi_i^S(a)\big) = a.$$

By (5) and the initial condition, all the coordinates of $p^{i,j}$ other than j are fixed at the disagreement point. That is, $p^{i,j}(x_i) = (a_{-j}, p_j^{i,j}(x_i))$ for each x_i. Hence, Eq. (4) is reduced to

$$p'_j(x_i) = \left[\frac{\partial \pi_i^S}{\partial x_j}\big(a_{-j}, p_j(x_i)\big)\right]^{-1}. \tag{6}$$

Proposition 6. *The solution of* (6) *is*

$$p_j(x_i) = \pi_j(a_{-i}, x_i). \tag{7}$$

Thus, when $\alpha = 1$, $p^{i,j}(x_i) = (a_{-j}, \pi_j(a_{-i}, x_i))$. That is, at the utility level x_i, player j receives her Pareto payoff at the point (a_{-i}, x_i), while all other players are bound to their disagreement point. The image of the path $p^{i,j}$ coincides with the j-axis that passes through a.

To find p^i, as defined in (3), consider first the case $x_i \leqslant \pi_i(a)$. In this case $(a_{-i}, x_i) \leqq (a_{-i}, \pi_i(a)) \in S$. Hence, $\pi_j(a_{-i}, x_i) \geqslant \pi_j(a_{-i}, \pi_i(a)) = a_j$. Thus, the largest value of $p_k^{i,j}(x_i)$ over all $j \neq i$ is $\pi_k(a_{-i}, x_i)$. Analogously, when $x_i \geqslant \pi_i(a)$ the smallest value of $p_k^{i,j}(x_i)$ over all $j \neq i$ is $\pi_k(a_{-i}, x_i)$. Thus, by (3),

Observation 3. For $\alpha = 1$, $p^i(x_i) = (a_i, \pi_{-i}^S(a_{-i}, x_i))$.

The utility vector $p_{-i}^i(x_i)$ is the ideal point of the bargaining problem for the players $N \setminus i$ when i's payoff is bound to x_i. Equivalently, the point $p^i(x_i)$ can be viewed as the ideal point of the projection of this problem on the hyperplane at which i's payoff is a_i. This is the extension of Shapley's solution described in the introduction.

Obviously, the definition of p^i for the case $\alpha = 1$ does not require the smoothness of the problem in Definition 2. Also, the uniqueness of the efficient point on this path (Proposition 3), and its continuity in the disagreement point follow from the continuity properties of the functions π_i, which hold for all Pareto surfaces, even when they are not smooth. Finally, the three last steps in the construction do not use the smoothness either. Thus, the solution Ψ^1 can be defined for the set of bargaining problems without the smoothness requirement.

5.1. Three-player problems

For three-player problems Ψ^1 is the Shapley's solution. The following properties of Ψ^1 are peculiar to the three-player case.

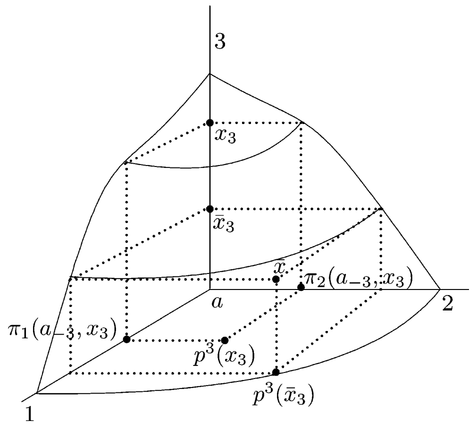

Fig. 6. Constructing the path p^3 and the point $\bar{x}$ for three players.

The set of players is $N = \{1, 2, 3\}$. The utility levels x_3 and $\bar{x}_3$ are marked on player 3's axis. The Pareto payoffs $\pi_1(a_{-3}, x_3)$ and $\pi_2(a_{-3}, x_3)$ are marked on the corresponding axes of these players. The resulting point $p^3(x_3)$ is depicted. The point $p^3(\bar{x}_3)$ is in S. The point $\bar{x}$ satisfies $\pi(\bar{x}) = a$.

Proposition 7. (1) *For each three-player problem (a, S) there exists a unique point $\bar{x}$ such that for each i, $(\bar{x}_{-i}, a_i) \in S$.*

(2) *Moreover, $\bar{x} = \Phi(a, S)$, and for each i, $(\bar{x}_{-i}, a_i) = \Lambda^i(a, S)$, where Λ^i and Φ are the solutions constructed in steps* 1 *and* 2 *for* $\alpha = 1$.

(3) *The sequence $(a^k)_{k \geqslant 0}$ alternates between the two sides of S.*

Figure 6 describes the construction of $\bar{x} = \Phi(a, S)$ for a three-player problem (a, S), using Observation 3. It is a more detailed version of the construction in Fig. 2.

6. More solutions

The family of solutions Ψ^α is a long way from exhausting the set of ordinal, efficient, symmetric, and individually rational solutions. We delineate here more such solutions.

Extending the family (Ψ^α). Consider a sequence $A = (\alpha^k)_{k \geqslant 0}$ of numbers in $[0, 1]$. We define for each sequence A a solution Ψ^A as follows. For a given problem (a, S), define a sequence $(a^k)_{k \geqslant 0}$ in R^N by $a^0 = a$ and for $k \geqslant 0$, $a^{k+1} = \Psi^{\alpha^k}(a^k, S)$. The sequence (a^k) converges and the solution Ψ^A defined by $\Psi^A(a, S) = \lim a^k$ is ordinal, efficient, symmetric, and individually rational. The solution Ψ^α is a special case of this construction for the constant sequence $\alpha^k = \alpha$.

Another way to define the solutions Λ^i. This construction is also based on the path-valued solutions $p^{i,j}$ which are defined for a given α. Given a problem (a, S), there exists a unique value $\bar{x}^{i,j}$ such that $p^{i,j}(a, S)(\bar{x}^{i,j})$ is efficient. Define for each k,

$$\Lambda_k^i(a, S) = \begin{cases} \min_{j \neq i} \ p_k^{i,j}(\bar{x}^{i,j}) & \text{if } a \prec S, \\ a_k & \text{if } a \in S, \\ \max_{j \neq i} \ p_k^{i,j}(\bar{x}^{i,j}) & \text{if } a \succ S. \end{cases}$$

Unlike the construction in the previous sections, the solutions Λ^i here need not be efficient. Nevertheless, they fail the test of efficiency in the "right" direction, and Lemma 2 in Section 7, which guarantees the convergence of the sequence (a^k), still holds as long as $\alpha < 1$. When $\alpha = 1$, $\Lambda^i(a, S) = a$ and (a^k) is the constant point a.

The two constructions of Λ^i coincide in the case $\alpha = 1/(n-1)$. In this case all the paths $p^{i,j}(a, S)$ for all $j \neq i$ coincide and therefore they are the same as $p^i(a, S)$. Also all the points $\bar{x}^{i,j}$ coincide with $\bar{x}_i$, the unique efficient point on $p^i(a, S)$.

7. Proofs

We first make the observation that if p solves the differential equation (2), then for each utility level x_i, the point $p(x_i)$ is on the Pareto surface in H defined by $\pi_i^S = x_i$.

Observation 4. For each x_i, $\pi_i(p(x_i)) = x_i$, or equivalently, $(p_{-i}(x_i), x_i) \in S$.

To see this, denote $f(x_i) = \pi_i(p(x_i))$. Then

$$f'(x_i) = \sum_j \frac{\partial \pi_i}{\partial x_j}\big(p(x_i)\big) p_j'(x_i).$$

By (2) this sum is $\sum_{j \neq i} w_j = 1$. By initial condition,

$$f\big(\pi_i(a)\big) = \pi_i\big(p\big(\pi_i(a)\big)\big) = \pi_i(a).$$

Thus, $f(x_i) = x_i$.

Proof of Proposition 1. We need to show that the path q defined by $q(\mu_i(x_i)) = \mu(p(x_i))$ solves (2) for the problem $(\mu(a), \mu(S))$. Differentiating q_j with respect to x_i we have

$$q_j'\big(\mu_i(x_i)\big)\mu_i'(x_i) = \mu_j'\big(p_j(x_i)\big)p_j'(x_i). \tag{8}$$

Differentiating both sides of (1) with respect to x_j we obtain

$$\frac{\partial \pi_i^{\mu(S)}}{\partial x_j}\big(\mu(x)\big)\mu_j'(x_j) = \mu_i'\big(\pi_i^S(x)\big)\frac{\partial \pi_i^S}{\partial x_j}(x).$$

Plugging $p(x_i)$ for x, and remembering that $\pi_i^S(p(x_i)) = x_i$ by Observation 2, and $\mu(p(x_i)) = q(\mu_i(x_i))$ by definition, we get

$$\frac{\partial \pi_i^{\mu(S)}}{\partial x_j}\big(q\big(\mu_i(x_i)\big)\big)\mu_j'\big(p_j(x_i)\big) = \mu_i'(x_i)\frac{\partial \pi_i^S}{\partial x_j}\big(p(x_i)\big). \tag{9}$$

Plugging the expression for $\frac{\partial \pi_i^S}{\partial x_j}(p(x_i))$ from (9) into the right-hand side of (2), and plugging the expression for $p_j'(x_i)$ from (8) into the left-hand side of (2), we get

$$q_j'\big(\mu_i(x_i)\big) = w_j\left[\frac{\partial \pi_i^{\mu(S)}}{\partial x_j}\big(q\big(\mu_i(x_i)\big)\big)\right]^{-1}. \tag{10}$$

It remains to check the initial condition. Indeed,

$$q\big(\pi_i^{\mu(S)}\big(\mu(a)\big)\big) = q\big(\mu_i\big(\pi_i^S(a)\big)\big) = \mu\big(p\big(\pi_i^S(a)\big)\big) = \mu(a).$$

Thus q is the solution of the differential equations for $(\mu(a), \mu(S))$. □

Proof of Proposition 2. We need to show that for each k, $p_k^i(\mu(a), \mu(S))(\mu_i(x_i)) = \mu_k(p_k^i(a, S)(x_i))$. This is obvious for $k = i$ since p_i^i is fixed at i's disagreement payoff. Suppose $k \neq i$. Assume

$$\mu_i(x_i) < \pi_i^{\mu(S)}\big(\mu(a)\big). \tag{11}$$

In this case $p_k^i(\mu(a), \mu(S))(\mu_i(x_i)) = \max_{j\neq i} p_k^{i,j}(\mu(a), \mu(S))(\mu_i(x_i))$. By the ordinality of $p^{i,j}$ and since the maximum function commutes with μ, the right-hand side is $\mu_k(\max_{j\neq i} p_k^{i,j}(a, S)(x_i))$. Also the condition (11) is equivalent, by Observation 1, to $x_i < \pi_i^S(a)$ in which case $\mu_k(\max_{j\neq i} p_k^{i,j}(a, S)(x_i)) = \mu_k(p_k^i(a, S)(x_i))$. The proof for the other two cases in the definition of p^i are similar. □

Proof of Proposition 3. For any path p that solves (2),

$$p_k(x_i) = a_k + w_k \int_{\pi_i(a)}^{x_i} \left[\frac{\partial \pi_i^S}{\partial x_k}\big(p(t)\big)\right]^{-1} \mathrm{d}t$$

for all $k \neq i$. The integrand is strictly negative and therefore for any k such that $w_k > 0$, p_k is strictly decreasing. For k such that $w_k = 0$, p_k is fixed at a_k. Thus, for fixed i and k such that $k \neq i$, each of the functions $p_k^{i,j}$ is either strictly decreasing or constant, and they all have the same value, a_k, at $\pi_i(a)$. Moreover, for at least one of the js, the coordinate k has a positive weight (if $\alpha > 0$ then it holds for $j = k$, if $\alpha = 0$ then it holds for any $j \neq k$). Therefore, for $x_i < \pi_i(a)$, $\max_{j\neq i} p_k^{i,j}$ is strictly decreasing and is above a_k, and for $x_i > \pi_i(a)$, $\min_{j\neq i} p_k^{i,j}$ is strictly decreasing and is below a_k. Thus the two "branches" of the function p_k^i form a strictly decreasing function such that for each $k \neq i$,

$$\begin{cases} p_k^i(x_i) > a_k & \text{if } x_i < \pi_i(a), \\ p_k^i(x_i) = a_k & \text{if } x_i = \pi_i(a), \\ p_k^i(x_i) < a_k & \text{if } x_i > \pi_i(a). \end{cases} \tag{12}$$

Therefore, there can be at most one point of S on the path p^i. To see that there exists such a point consider first the case $a \in S$. Then $\pi_i(a) = a_i$, and by (12), $p^i(a) = a$.

Assume that $a \prec S$. Consider the path p^i in the interval $I = [a_i, \pi_i(a)]$. By (12), $p^i(\pi_i(a)) = a \prec S$. At the other edge of I,

$$p^i(a_i) \geqq p^{i,j}(a_i) \tag{13}$$

for any $j \neq i$, by the definition of p^i. But $p^{i,j}(a_i) = (p_{-i}^{i,j}(a_i), a_i)$, and by Observation 2 this point is on S. Thus, by (13) and Observation 3 in Safra and Samet (2004), $p^i(a_i) \succeq S$.

Consider the function $D(x_i) = p^i_j(x_i) - \pi_j(p^i(x_i))$. Then $D(\pi_i(a)) < 0$, and $D(a_i) \geqslant 0$. Since D is continuous there exists $\bar{x}_i$ in the interval $[a_i, \pi_i(a))$ such that $D(\bar{x}_i) = 0$. The vanishing of D implies $p^i(\bar{x}_i) \in S$. We record the following inequality which we use later:

$$p^i_k(\bar{x}_i) > p^i_k\big(\pi_i(a)\big) = a_k \tag{14}$$

for each $k \neq i$.

The proof of the existence of an efficient point on the path p^i for the case $a \succ S$ is analogous. In this case the inequality in (14) should be reversed. □

Proof of Proposition 4. The symmetry follows from the symmetry of p^i, and hence of Λ^i, with respect to permutations of $N \setminus i$, and the symmetry of the vector $(\Lambda^i)_{i \in N}$ with respect to all permutations. The ordinality follows that of Λ^i, the covariance of the minimum and maximum functions with order-preserving transformations. □

For the proof of Proposition 5 we need the following lemmas.

Lemma 1. *For any Pareto surface S, the solution $\Phi(a, S)$ is continuous as a function of a.*

Proof. We observe first that solutions to the differential equation (2) are continuous in the initial condition a (see Hartman, 1982). Thus, for any S, if $p(a, S)$ is the solution of (2) then for any fixed x_i, $p(a, S)(x_i)$ is continuous in a. It easily follows that for each i, $p^i(a, S)(x_i)$ is also continuous in a.

We show that $\Lambda^i(a, S)$ is continuous in a, which implies this lemma. Let $a^\nu \to a$ and suppose that $\Lambda^i(a, S) = p^i(a, S)(\bar{x})$ and for each ν, $\Lambda^i(a^\nu, S) = p^i(a^\nu, S)(\bar{x}^\nu_i)$. Assume that $p^i(a^\nu, S)(\bar{x}^\nu_i) \to y$. Then y is in S. Assume also that for each ν, $\bar{x}^\nu_i \geqslant \bar{x}_i$. Since for each ν $p^i(a^\nu, S)$ is decreasing, it follows that $p^i(a^\nu, S)(\bar{x}^\nu_i) \leqq p^i(a^\nu, S)(\bar{x}_i)$. The left-hand side of this inequality converges to y, while the right-hand side converges, as noted above, to $p^i(a, S)(\bar{x}_i)$. Thus, $y \leqq p^i(a, S)(\bar{x}_i)$. Since both are in the Pareto surface S, they must coincide. □

Lemma 2. *If $a \prec S$ then $a < \Phi(a, S) < \pi(a)$ and $\pi(\Phi(a, S)) \geqq a$.*
If $a \succ S$ then $a > \Phi(a, S) > \pi(a)$ and $\pi(\Phi(a, S)) \leqq a$.

Proof. Suppose $a \prec S$. Then, by (14) for all $i \neq k$,

$$\Lambda^i_k(a, S) > a_k. \tag{15}$$

Thus, for each k, $\Phi_k(a, S) > a_k$.

Since $n \geqslant 3$, we can choose k different from i and j. As π_j is decreasing in x_k, we conclude from (15) that $\pi_j(a) > \pi_j(\Lambda^i(a, S))$. Since $\Lambda^i(a, S) \in S$, $\pi_j(\Lambda^i(a, S)) = \Lambda^i_j(a, S) \geqslant \Phi_j(a, S)$, which proves that second inequality in the lemma.

To prove the last inequality, we note that by definition, $\Phi_{-i}(a, S) \leqq \Lambda^i_{-i}(a, S)$. As π_i is independent of x_i, $\pi_i(\Phi(a, S)) \geqq \pi_i(\Lambda^i(a, S)) = a_i$.

The proof for $a \succ S$ is analogous. □

Proof of Proposition 5. In the proof of Proposition 5 in Safra and Samet (2004) it is shown that Lemma 2 implies the following claim.

Claim 1. *The subsequence of* $(a^k)_{k\geqslant 0}$ *of all the points below* S (*above* S) *is strictly increasing* (*decreasing*) *and bounded, and therefore if it is infinite it converges to a point* b (c).

Thus, it remains to show that each of b and c, when defined, are in S, and if both are defined then they coincide. Indeed, suppose that there are infinitely many elements $(a^{k_l})_{l=1}^{\infty}$ below S. Then, taking the limits on both sides of $a^{k_{l+1}} = \Phi(a^{k_l}, S)$ we conclude by Lemma 1 that $b = \Phi(b, S)$. By Lemma 2, this can be the case only when $b \in S$. Similarly, when the limit point c is defined, then $c = \Phi(c, S)$ and therefore $c \in S$.

If both subsequences are infinite, then there exist an infinite sequence $(a^{k_l})_{l=1}^{\infty}$ of points below S such that all the points in $(a^{k_l+1})_{l=1}^{\infty}$ are above S. Taking the limits on both sides of $a^{k_l+1} = \Phi(a^{k_l}, S)$ we conclude that $c = \Phi(b, S)$ and therefore, by what was shown previously, $b = c$. □

Proof of Theorem 1. The theorem follows from Propositions 4 and 5. The details are the same as in the proof of Theorem 1 in Safra and Samet (2004).

Proof of Proposition 6. Differentiating both sides of (7) we get

$$p_j'(x_i) = \frac{\partial \pi_j}{\partial x_i}(a_{-i}, x_i).$$

Thus we need to show that

$$\frac{\partial \pi_j}{\partial x_i}(a_{-i}, x_i)\frac{\partial \pi_i^S}{\partial x_j}\big(a_{-i}, \pi_j(a_{-i}, x_i)\big) = 1. \tag{16}$$

Note that by definition, $(a_{-\{i,j\}}, x_i, \pi_j(a_{-i}, x_i)) \in S$. Therefore, applying π_i to this efficient point results in $\pi_i(a_{-\{i,j\}}, x_i, \pi_j(a_{-i}, x_i)) = x_i$. As π_i is independent of x_i, $\pi_i(a_{-j}, \pi_j(a_{-i}, x_i)) = x_i$. Differentiating both sides of this equation yields (16). □

Proof of Proposition 7. To prove the first part of the proposition, suppose $(a_1, \bar{x}_2, \bar{x}_3)$, $(\bar{x}_1, a_2, \bar{x}_3)$, $(\bar{x}_1, \bar{x}_2, a_3) \in S$. From the first two inclusions we infer $\bar{x}_2 = \pi_2(a_{-3}, \bar{x}_3)$ and $\bar{x}_1 = \pi_1(a_{-3}, \bar{x}_3)$.[6] Thus, by Observation 3, $(\bar{x}_1, \bar{x}_2, a_3) = p^3(\bar{x}_3)$. From the third inclusion it follows that $p^3(\bar{x}_3) \in S$. Thus $\Lambda^3(a, S) = (\bar{x}_1, \bar{x}_2, a_3)$. Similar equalities hold for players 1 and 2. Hence $\bar{x}$ is uniquely determined.

The second part of the proposition follows from the definition of Φ.

To prove the third part, assume first that $a \prec S$. By Lemma 2, $\Phi(a, S) > a$. Thus, $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \geqslant (a_1, \bar{x}_2, \bar{x}_3) \in S$. Hence by Observation 3 in Safra and Samet (2004), $\Phi(a, S) \succ S$. The proof for the case $a \succ S$ is similar. □

[6] When the projection of $\bar{x}$, $(\bar{x}_{-i}, a_i)$, is in S, then for each $j \neq i$, $\bar{x}_j = \pi_j(\bar{x}_{-i}, a_i)$. The claim is true for any number of players. It is only in the case $n = 3$ that this claim implies $\pi_j(x_{-i}, a_i) = \pi_j(a_{-k}, x_k)$ for $k \notin \{i, j\}$.

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