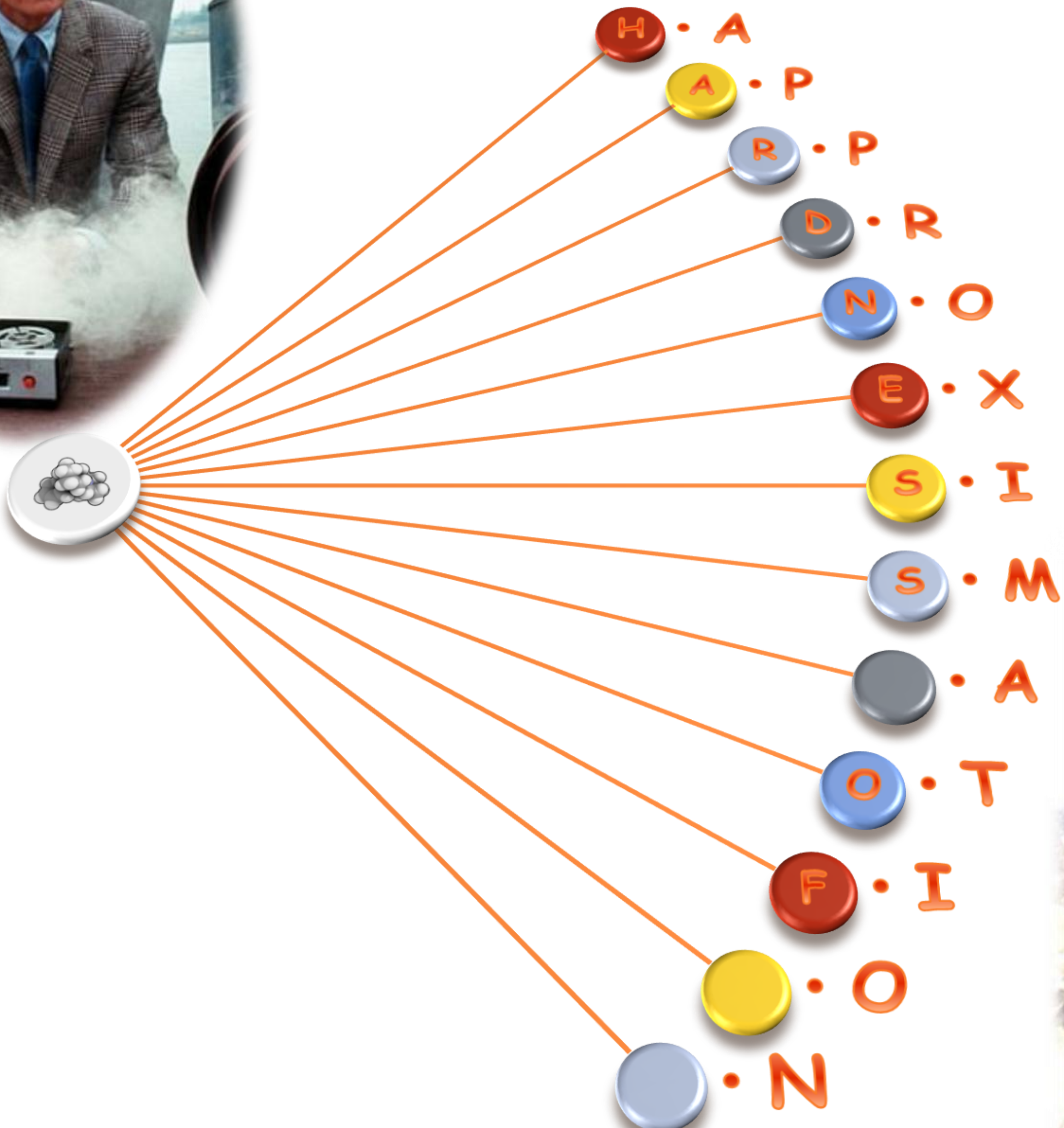

H · A
A · P
R · P
D · R
N · O
E · X
S · I
S · M
· A
O · T
F · I
· O
· N

Goal:

- Show some approximation problems NP-hard

Plan:

- How to show inapproximability?
- Probabilistically Checkable Proofs (PCP)
- CLIQUE, COLORING

Promise Problems – Ilustrated

EG Is there an assignment satisfying **90%** of a **3SAT**, **assuming** one can satisfy **80%**?

Good inputs

- Must **accept**

Bad inputs

- Must **reject**

Other inputs

- **Whatever**

Gap problems? A **special case**

Optimization

problem	min/max	instance	solution	parameter
CLIQUE	max	graph	clique	|clique|
3SAT	max	3CNF	assign.	#clause
S.C.	min	family	cover	|cover|

Objective

- Best solution according to **optimization parameter**

algorithm

optimal

Approximation

- Find a solution within some **factor** of **optimal**

Gap-A[C,hC] (minimization)

In optimal Solution

- Parameter is

<C
Yes

GAP

>hC
No

GAP

- Whatever

Observation:

- ∀C, efficient h-approximation for A resolves gap-A[C, hC]:

accept if <hC

Gap-3SAT

Instance:

- 3-CNF

Gap Problem:

[7/8+ε, 1]

- Maximum, over assignments, of fraction of satisfiable clauses:

Yes

- =1

No

- < 7/8+ε

EG
$(x_1 \vee x_2 \vee x_3)$
$(x_1 \vee \neg x_2 \vee \neg x_2)$
$(\neg x_1 \vee x_2 \vee x_3)$
$(\neg x_1 \vee \neg x_2 \vee \neg x_2)$
$(\neg x_1 \vee x_2 \vee \neg x_3)$
$(\neg x_3 \vee \neg x_3 \vee \neg x_3)$

$x_1 \rightarrow F$; $x_2 \rightarrow T$; $x_3 \rightarrow F$
satisfies 5/6 of clauses

Why 7/8?

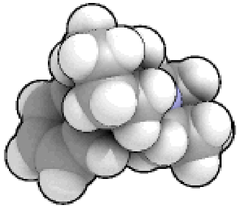
Claim:

- ∀E3CNF (clauses: exactly 3 independent literals) ∃assignment that satisfies ≥7/8 of clauses

Proof:

E	How many does an assignment satisfy expectedly?
Y_i	∀clause C_i, let Y_i be a 0/1 variable: "is C_i satisfied?"
E Y_i	Y_i's expectation: $E[Y_i] = 7/8$
E	$E[\Sigma Y_i] = \Sigma E[Y_i] = m7/8$ (m = #clauses)
∃	∃assignment with at least that number satisfied

PCP Characterization of NP



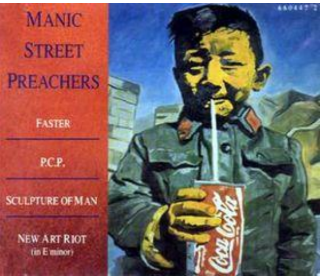
Theorem (PCP):

- $\forall\varepsilon>0$, Gap-3SAT[$7/8+\varepsilon$, 1] is NP-hard

i.e.: $\forall L\in NP$, Karp reduction f to 3CNF:
$x\in L \Rightarrow f(x)\in 3SAT$
$x\notin L \Rightarrow$ max satisfy $<7/8+\varepsilon$ of f(x)

Proof:

- Cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet 😐



Approximating max-3SAT to within which factor is thus proven NP-hard?

Probabilistic Checking of Proofs

Verify witness

- Choose only **O(1) bits** to read (**randomly**)

for **gap-3SAT**: pick **O(1)** clauses

Gap-CLIQUE

Instance:

- G=(V,E) (of m IS's each of size 3)

Gap Problem: [(7/8+ε)/3 , 1/3]

- Parameter: fractional size of largest CLIQUE:

Yes

- ≥ 1/3

No

- < (7/8+ε)/3

Theorem:

- Gap-CLIQUE [(7/8+ε)/3, 1/3] is NP-hard

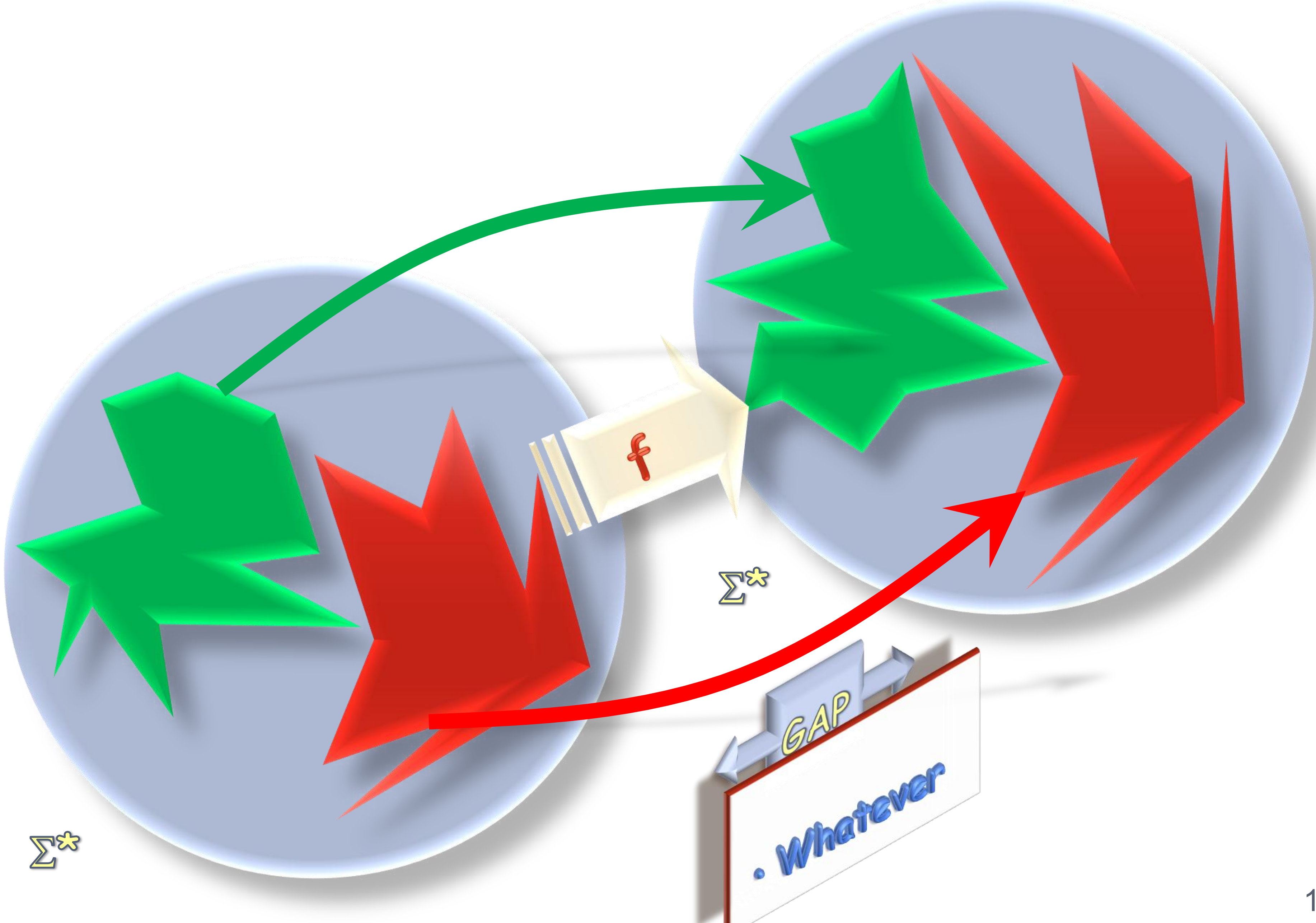
Corollary:

- It is NP-hard to approximate MAX-CLIQUE to within 7/8+ε

How does one prove such an NP-hardness theorem?

Reduce from gap-3SAT?

Gap Preserving Reduction



SAT $\leq_p$ CLIQUE

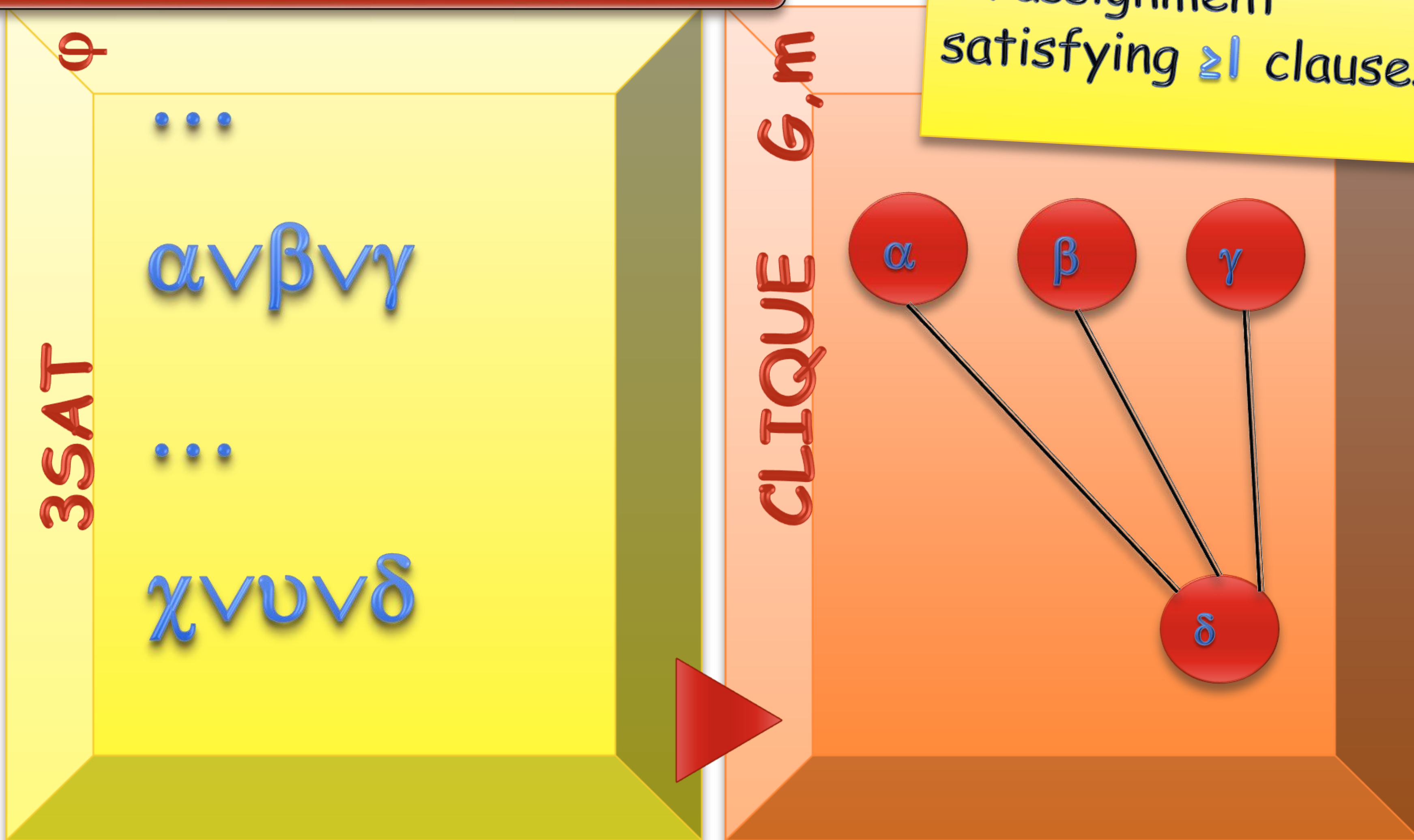
1 vertex for 1 occurrence

inconsistency ⇔ non-edge

- within triplets, $\alpha = \neg\delta$.

m= number of clauses

A **clique** of size $l \Rightarrow$ an assignment satisfying $\geq l$ clauses



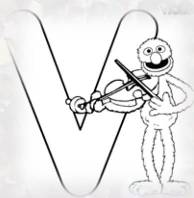
Color Coordination

Party

- A set of partygoers; each pair of friends agree on possible pairs of colors

Solution: MAX_V

- Invite some partygoers --- set colors s.t. all pairs OK



Optimize:

- How many invited

Solution: MAX_E

- Assign a color to every partygoer

Optimize:

- Number of OK pairs

Constraints` Graph

CSG Input:

- $U = (V, E, \Sigma, \phi)$ - $\phi: E \rightarrow P[\Sigma^2]$

Solution: MAX_V

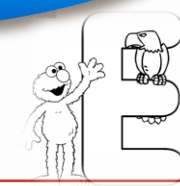
- A: $V \rightarrow \Sigma \cup \{\perp\}$ **s.t.** $\forall (u,v) \in E$: $A(u), A(v) \in \Sigma$ $\Rightarrow (A(u), A(v)) \in \phi(u,v)$

Optimize:

- $Pr[A(u) \in \Sigma]$

EG
CLIQUE
IS
3SAT
V.C.
$\chi(G)$

Solution: MAX_E



- A: $V \rightarrow \Sigma$

Optimize:

- $Pr[(A(u), A(v)) \in \phi(u,v)]$

EG
MAX-CUT
$\chi(G)$

gap$_V$-kCSG[α, β]

Instance:

- U=(V,E, [k], ϕ)

Gap Problem: [α, β]

- Fractional size of largest all-consistent assignment:

Yes

- ≥ β

No

- < α

How does one prove kCSG is NP-hard?

Reduce from gap-3SAT?

Observe:

- Gap$_V$-3CSG[(7/8+ε),1] is NP-hard

3SAT is a 'special case'

Theorem:

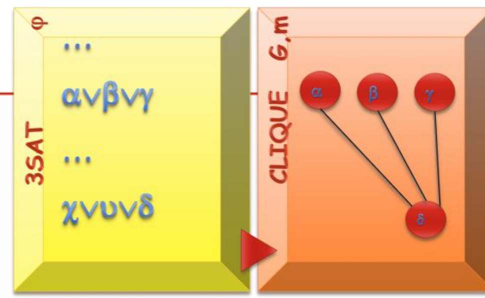
- ∀δ>0 ∃k=k(δ) gap-IS[δ/k, 1/k] NP-hard

Proof: via kCSG

Claim [CSG→IS]:

- gap_V-kCSG[δ, 1] $\leq_L$ gap-IS[δ/k, 1/k]

Proof:



- $V' = V \times \Sigma$,
- $i \neq j \Rightarrow ((u,i),(u,j)) \in E'$
- $(i,j) \notin \phi(u,v)\} \Rightarrow ((u,i),(v,j)) \in E'$

Gap_V-3CSG[(7/8+ε),1] is NP-hard

gap_V-kCSG[δ, 1] $\leq_L$ gap_V-k'CSG[δ', 1]

gap_V-kCSG[δ, 1] $\leq_L$ gap-IS[δ/k, 1/k]

Yes → • I = (u, A(u))

1 in each clique

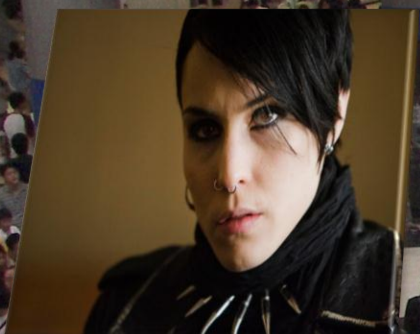
No ← • A(u) = i | (u, i) ∈ I

∀u there is ≤1 such i

Party on

Whatever constraints, I can always invite 1% of partygoers

In that case, I can do 99%, whatever the constraints are!



How?

'Invent' a guest for every set of l real guests

Theorem:

- gap_V-kCSG[δ, 1] $\le_L$ gap_V-k^lCSG[δ^l, 1]

Proof:

- $V' = V^l$, $\Sigma' = \Sigma^l$
- E': (u', v') s.t. ∃ u∈u', v∈v' s.t. u=v or (u, v)∈E
- φ' forbids: $A(u')_{\downarrow u} \neq A(v')_{\downarrow u}$ or $(A(u')_{\downarrow u}, A(v')_{\downarrow v}) \notin \phi(u, v)$

Yes

- Natural, consistent assignment

No

- Color u∈u' by $A(u')_{\downarrow u}$ if any such u'∈V' is colored. u'∈V' is colored only if all u∈u' are colored

Amplify

MIND THE GAP

Gap_V-3CSG[(7/8+ε),1] is NP-hard

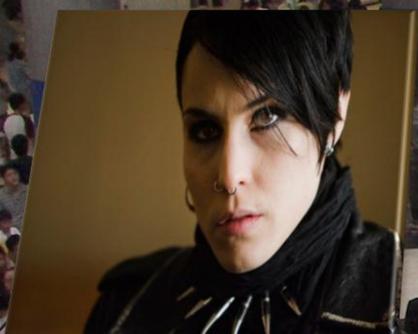
gap_V-kCSG[δ, 1] $\le_L$ gap_V-k^lCSG[δ^l, 1]

gap_V-kCSG[δ, 1] $\le_L$ gap-IS[δ/k, 1/k]

Settle differences

Constraints depend only on difference ⇒ I can invite δ of partygoers

In that case, I can do the same for general constraints



And what does it get you?

Show approximate coloring of a graph is NP-hard

If colors are (mod q)

∀ constraint $\phi(u, v)$ depends only on $A(u)-A(v) \pmod q$

⇒ Many symmetric colorings

$qCSG_\Delta$ Instance:

- $U = (V, E, \Sigma, \phi)$

  $\phi: E \to P[\Sigma^2]$ is define by

  $\Delta: E \to P\{[q]\}$ thus

  $\phi(u, v) = \{(i,j) \mid i-j \bmod q \in \Delta(u, v)\}$

$P\{[q]\}$ = all subsets of $\{0..q-1\}$

Theorem:

- $gap_V\text{-}kCSG[\delta, 1] \leq_L gap_V\text{-}qCSG_\Delta[\delta, 1]$

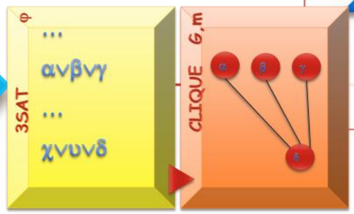
  $q = (nk)^5$

q-Coloring

Corollary:

- gap-$\chi[q, q/\delta]$ is NP-hard

Proof:



- Apply same CSG→IS reduction

$qCSG_\Delta$ Instance:

- $U = (V, E, \Sigma, \phi)$
  $\phi: E \rightarrow P[\Sigma^2]$ is define by
  $\Delta: E \rightarrow P\{[q]\}$ thus
  $\phi(u, v) = \{(i,j) \mid i-j \bmod q \in \Delta(u, v)\}$

Theorem:

- $gap_V\text{-}kCSG[\delta, 1] \leq_L gap_V\text{-}qCSG_\Delta[\delta, 1]$
  $q = (nk)^5$

q disjoint IS covering all

Yes → Consistent A ⇒ $\forall d\ A^d(v)=A(v)+d \pmod q$ consistent

No ← vertexes partitioned into IS's of fractional size $\leq\delta/q$ - how many?

Triplets` Unique T

Lemma:

- For $q=|X|^5$, can efficiently construct $T:X\rightarrow[q]$ s.t.
  $\forall x_1,x_2,x_3,x_4,x_5,x_6,$
  $T(x_1)+T(x_2)+T(x_3) \equiv T(x_4)+T(x_5)+T(x_6) \pmod q$
  $\Rightarrow \{x_1,x_2,x_3\}=\{x_4,x_5,x_6\}$

Proof:

- Incrementally assign values to members of U

Corollary:

- T is unique for pairs and for single vertexes as well

Proof:

- $T(x)+T(u)\equiv T(v)+T(w) \pmod q \Rightarrow \{x,u\}=\{v,w\}$
- $T(x)\equiv T(y) \pmod q \Rightarrow x=y$

Proof:

$qCSG_\Delta$

- $q=(|V|\cdot|\Sigma|)^5$
- $\Delta(u, v) = \{T(u,i)-T(v,j) \bmod q \mid (i,j)\in\phi(u,v)\}$

$qCSG_\Delta$ Instance:

- $U = (V, E, \Sigma, \phi)$
  $\phi: E\to P[\Sigma^2]$ is define by
  $\Delta: E\to P\{[q]\}$ thus
  $\phi(u, v) = \{(i,j) \mid i-j \bmod q \in \Delta(u, v)\}$

Theorem:

- $gap_V\text{-}kCSG[\delta, 1] \le_L gap_V\text{-}qCSG_\Delta[\delta, 1]$
  $q = (nk)^5$

Yes

- for good A: $A'(u) = T(u, A(u))$ good

No

- for good A': $\exists! d$ $A(u)=T^{-1}(u,A'(u)-d)$ good

Assume a complete graph – add trivial constraints

Pairs: assuming A' colors vertexes u,v

denote: $d_{A'}(u,v)$ = that unique d

- $\exists! d$ s.t. $A'(u)=T(u,i)+d$ & $A'(v)=T(v,j)+d$ for $(i,j)\in\phi(u,v)$

Triplets: assuming A' colors vertexes u,v,w

- $d_{A'}(u,v)=d_{A'}(v,w)$ as $A'(u)-A'(v) + A'(v)-A'(w) + A'(w)-A'(u)\equiv 0 \pmod q$

General: assuming A' any partial coloring

- If $d_{A'}$ is not everywhere the same, $\exists$ inconsistent triplet (u,v,w)

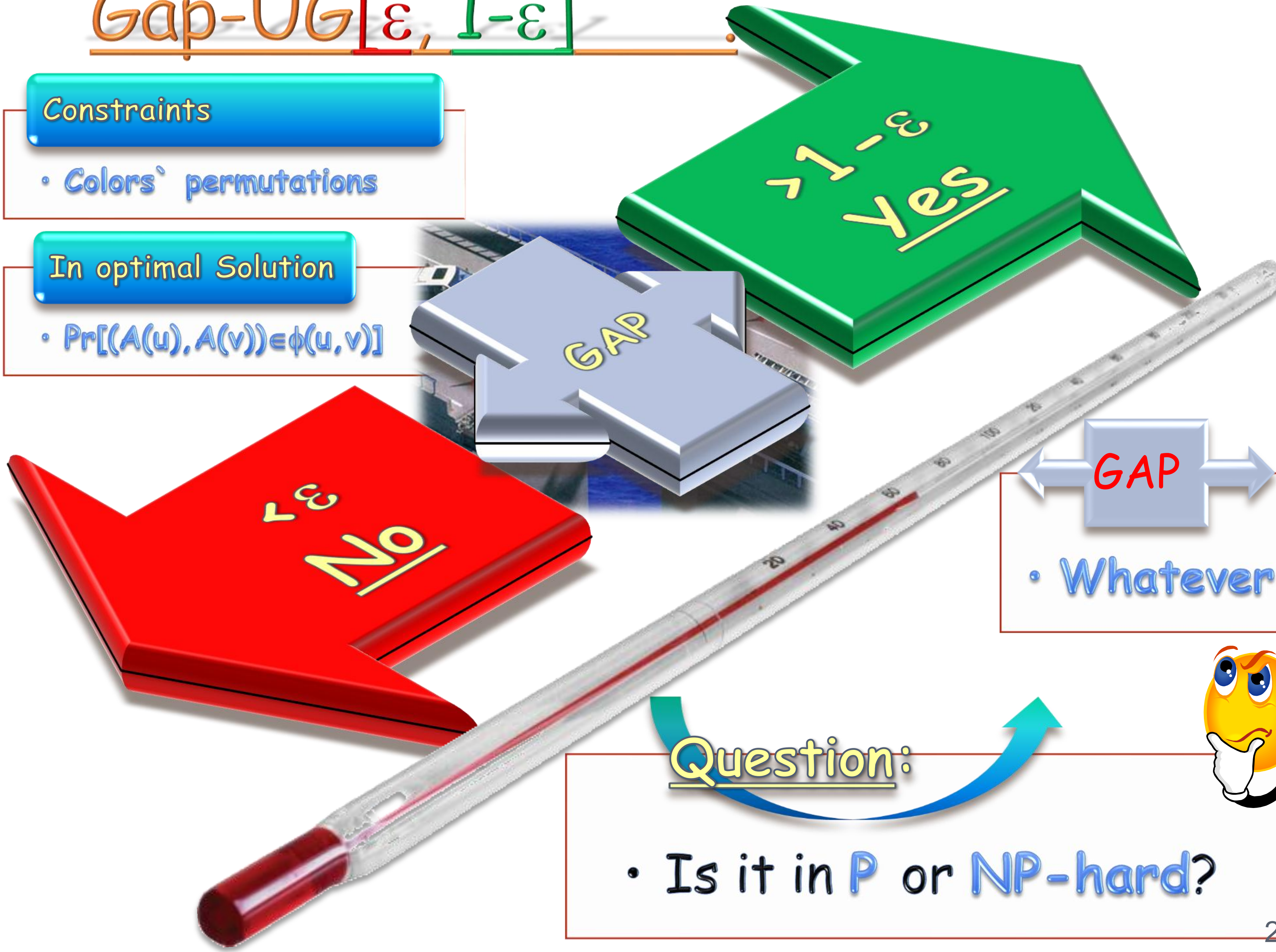
Gap-UG[ε, 1-ε]

Constraints

- Colors` permutations

In optimal Solution

- $\Pr[(A(u),A(v))\in\phi(u,v)]$



- Whatever

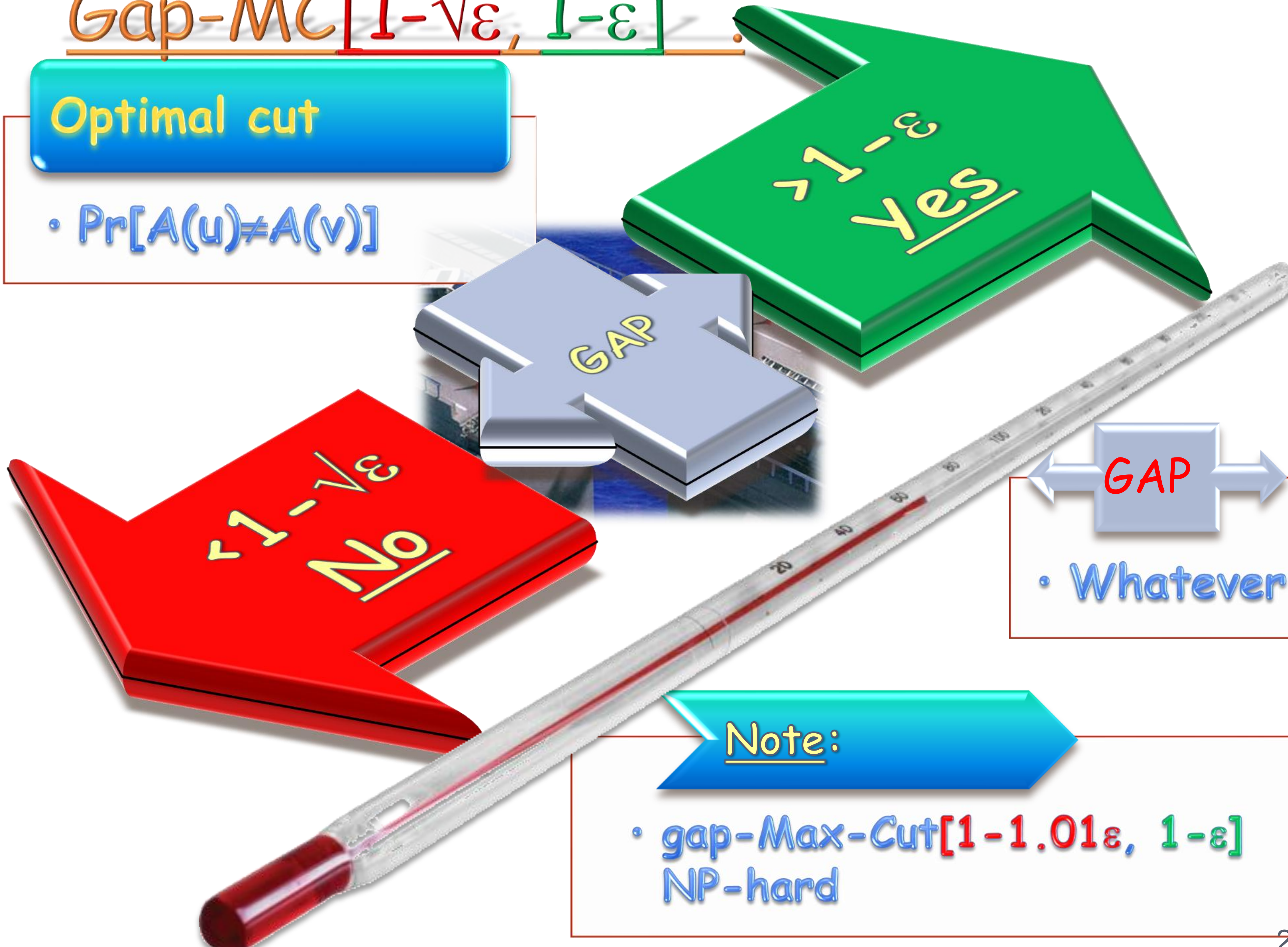
Question:

- Is it in P or NP-hard?

Gap-MC[1-$\sqrt{\varepsilon}$, 1-ε]

Optimal cut

- Pr[A(u)$\neq$A(v)]



GAP

- Whatever

Note:

- gap-Max-Cut[1-1.01ε, 1-ε] NP-hard

WWindex

PCP

Clique

3SAT

Coloring

NP-Hard

PCP Theorem

SAT

Vertex Cover

CNF

Interactive Proof System