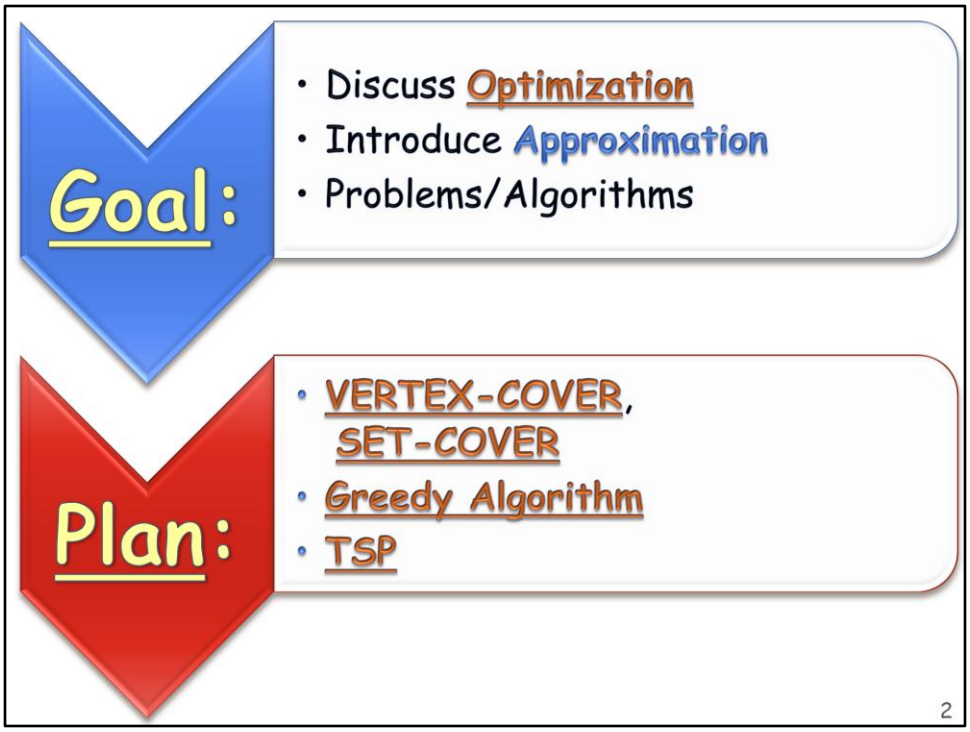
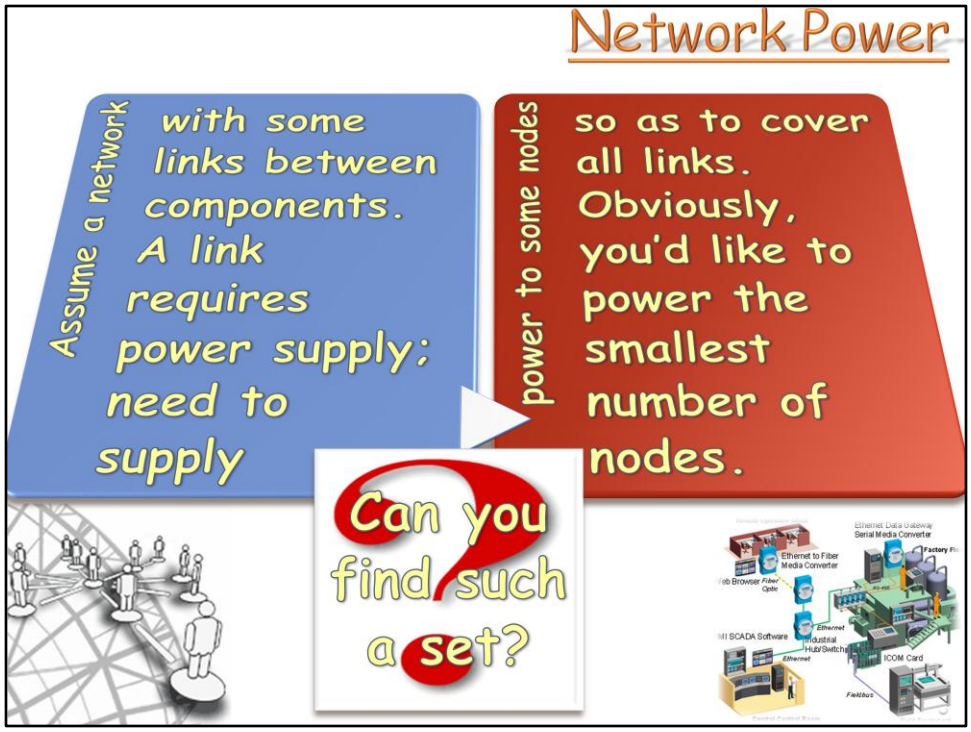
Approximation Problems

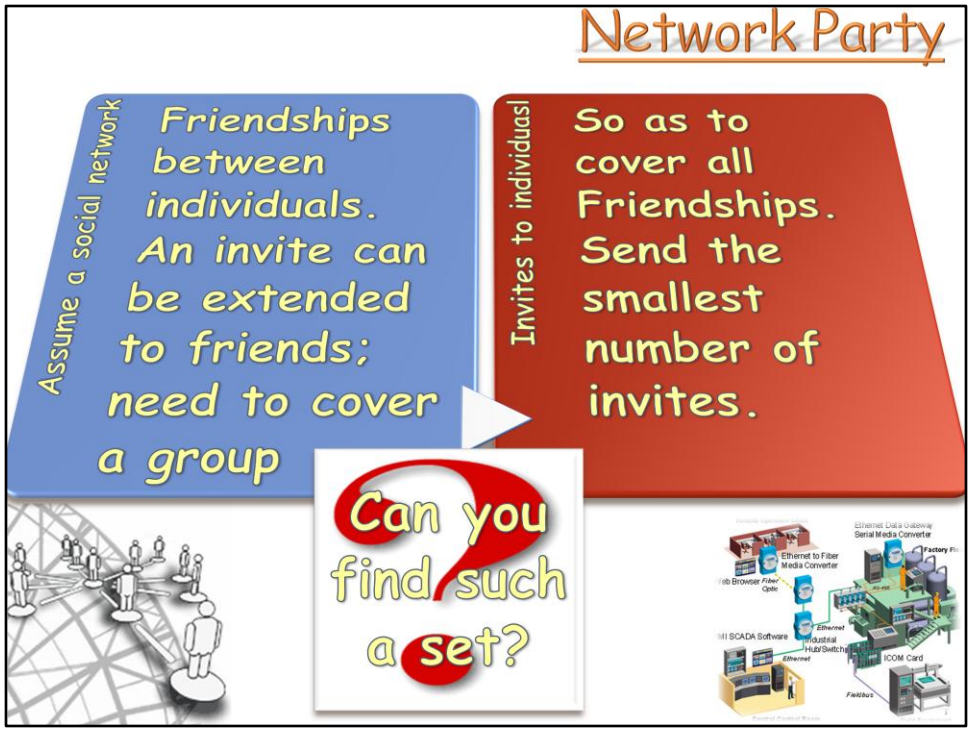
In this lecture we study approximation versions for optimization problems whose exact solution is NP hard.



We'll define optimization parameters and what it means to approximate them, via several concrete problems.



Consider a wired network, with various locations and some wiring. One needs to find a set of locations so that every wire has one of its ends covered. Naturally, one looks for the smallest possible set of locations.



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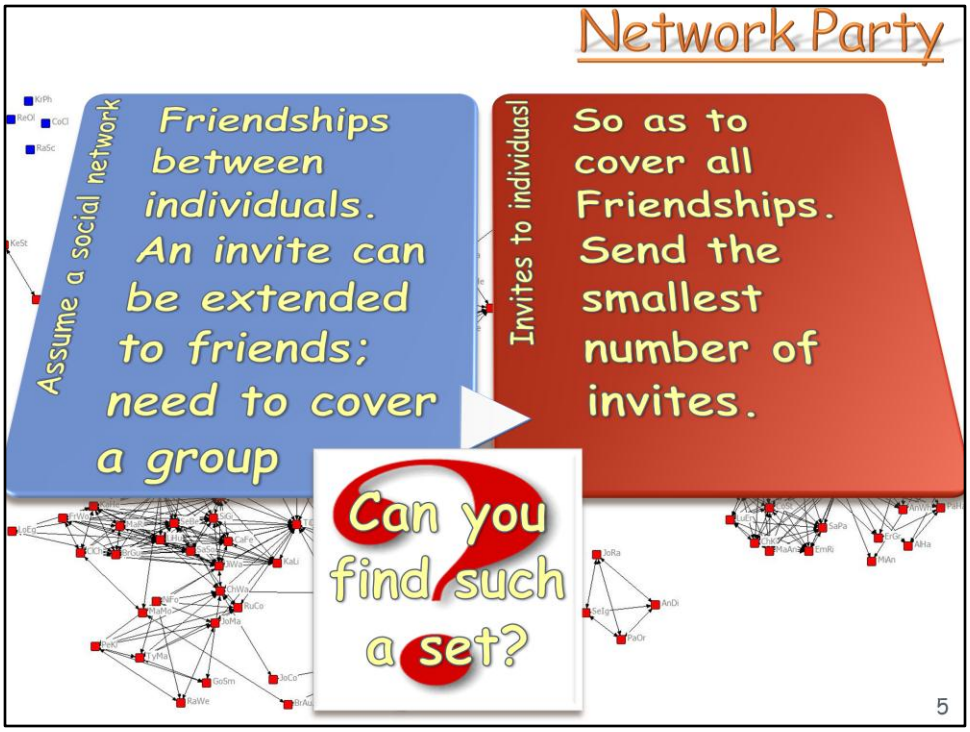

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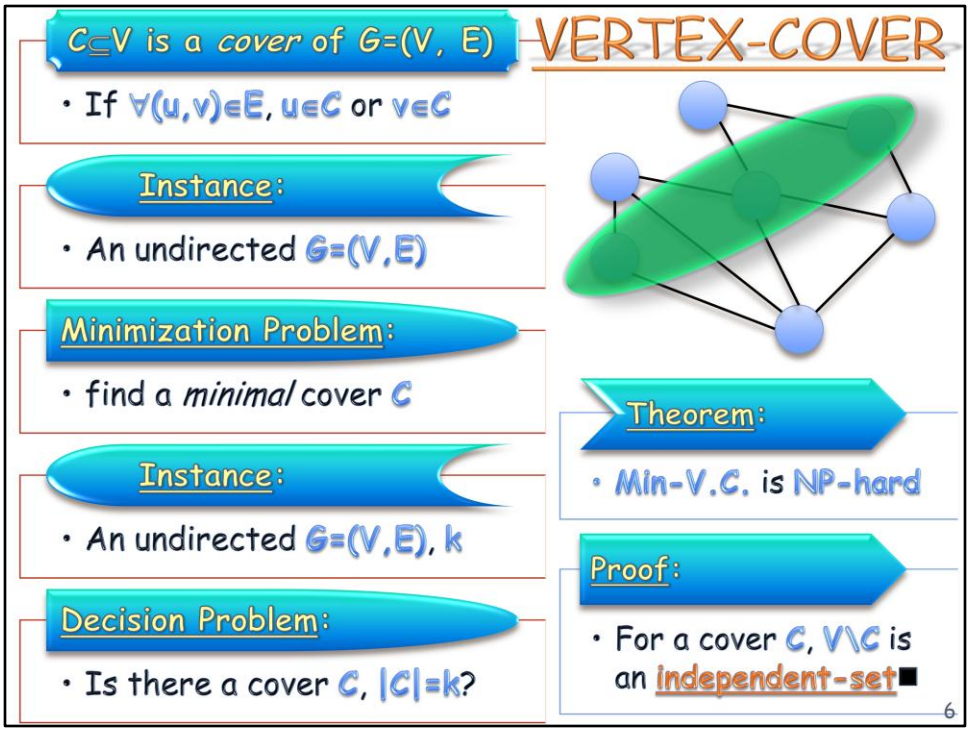
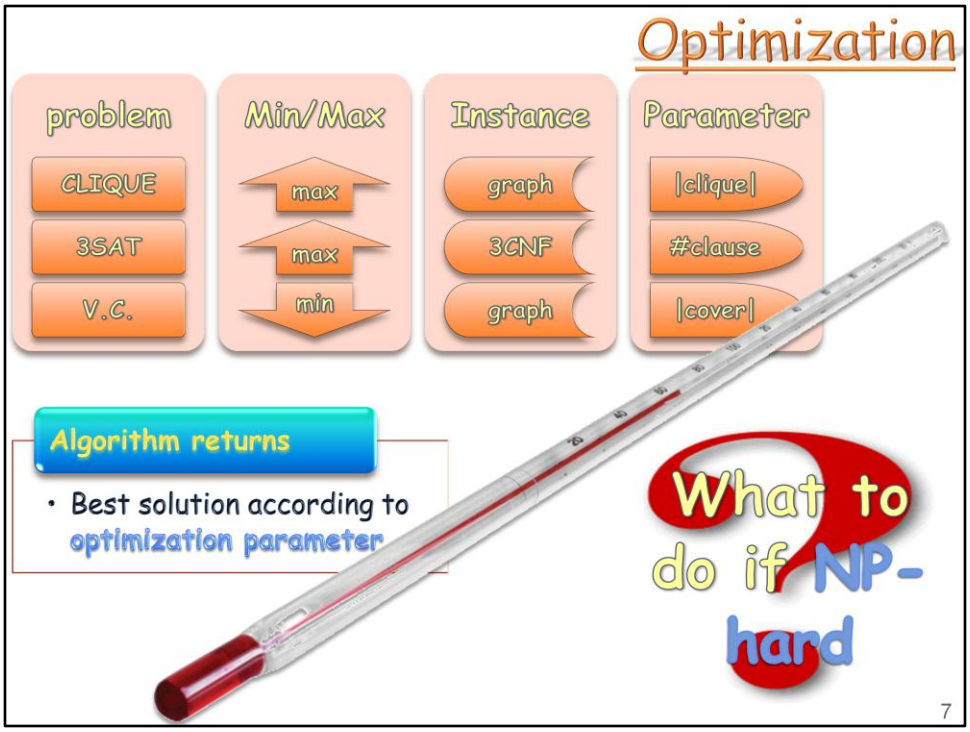
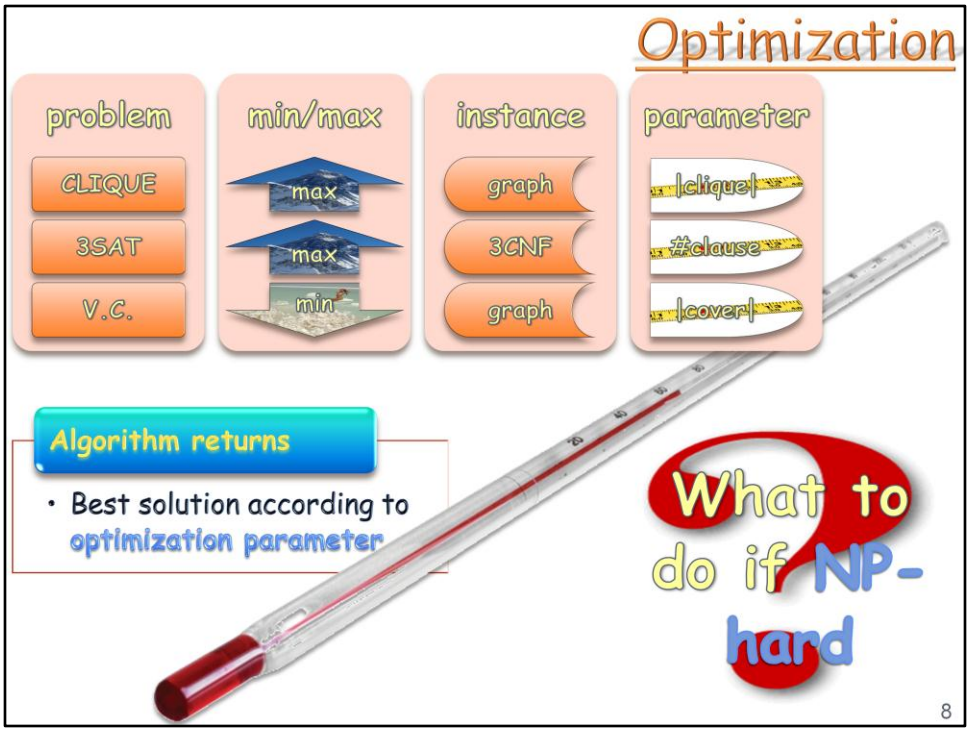


This problem is the Vertex Cover.
Given a graph, find a smallest set of nodes that covers all edges. We can define the corresponding decision problem, where the question is whether there is such a cover smaller than the given threshold.
This problem is clearly NP-hard: the complement of the Vertex Cover is an Independent Set, hence a solution to one implies a solution to the other.



Let us now recall some of the optimization problems we've discussed, both minimization and maximization, and identify the parameter optimized. Now, what do we do if it turns out that the problem is NP-hard?



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Approximation

Definition:

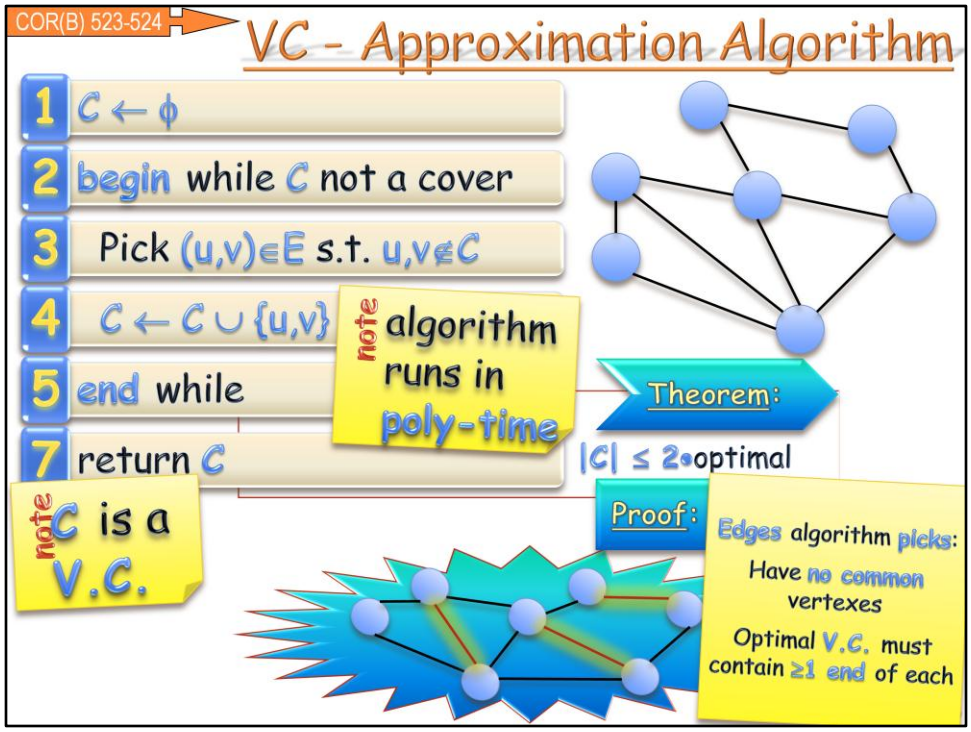
- An *α-approximation* algorithm for a maximization/minimization problem with an optimum solution O, returns a solution that is $\geq \alpha O / \leq \alpha O$

Note

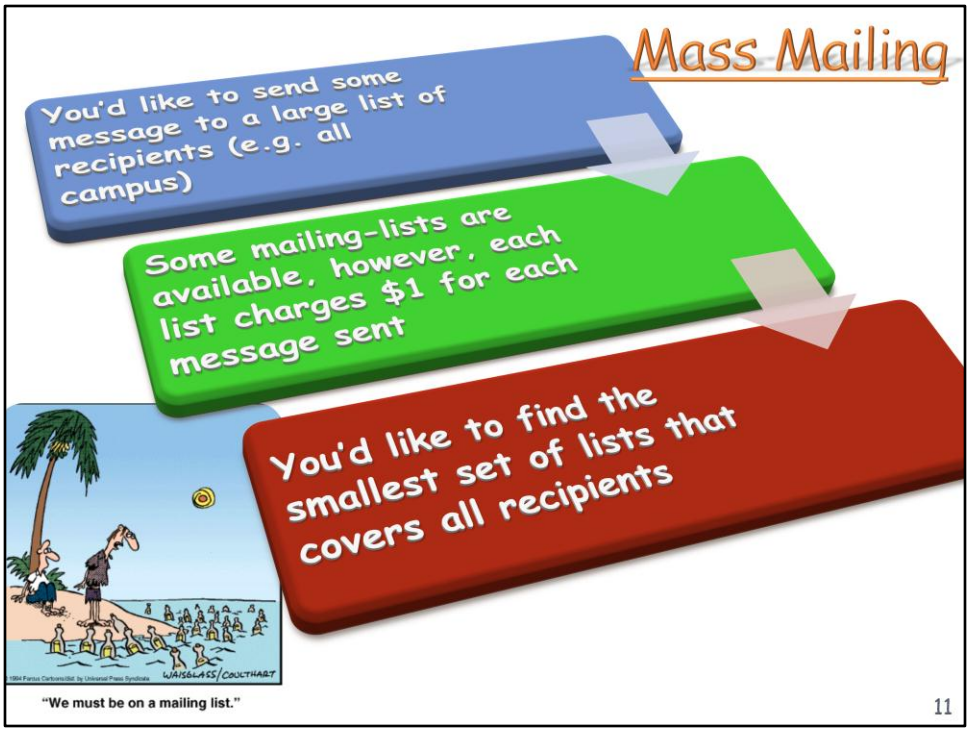
- α and O may depend on the size of the input

9

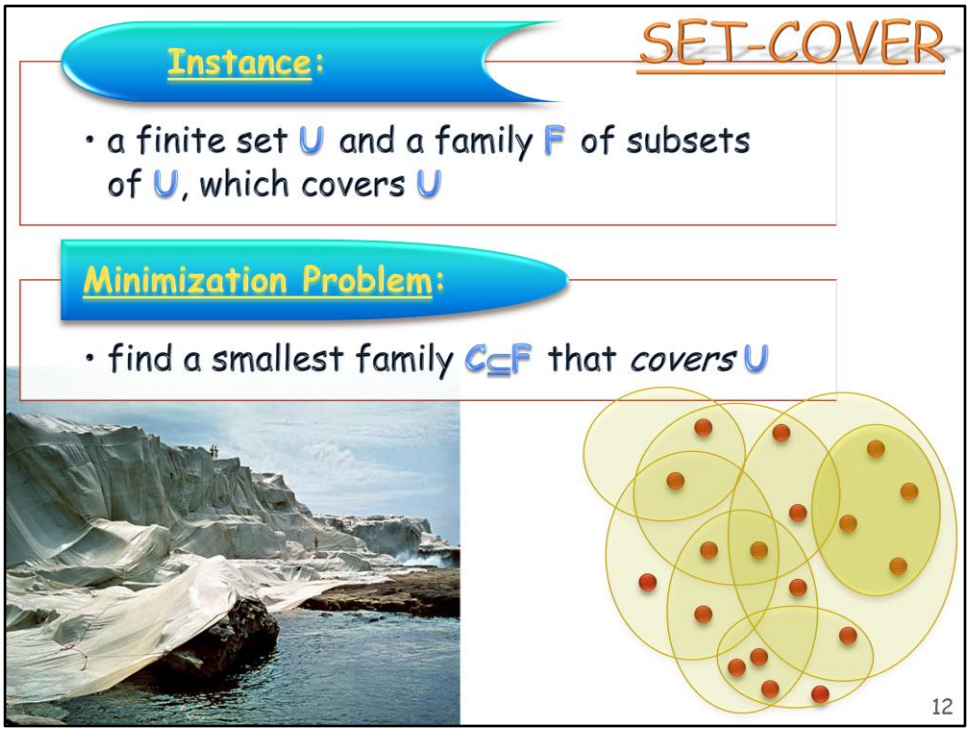
Many times it suffices to come up with an approximated solution, namely one which is not optimal, however deviates from optimal by some known factor. This factor is called the approximation factor; it may be a constant or depend on the input size.



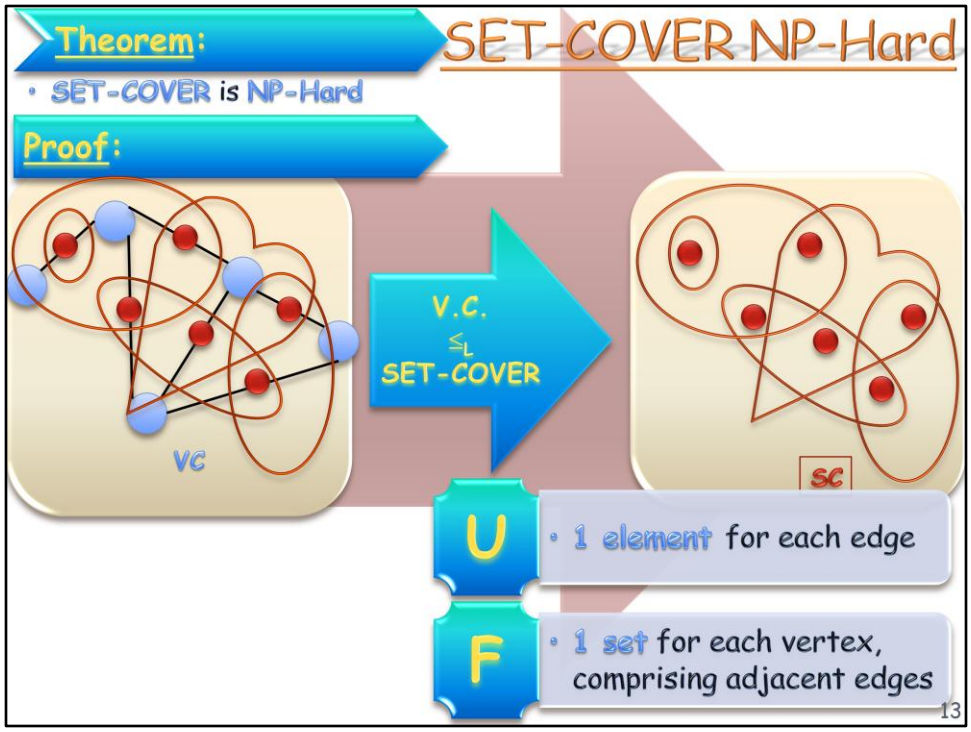
Let us now describe a very simple Approximation Algorithm for Vertex Cover: at every stage, pick an edge not yet covered and add both its ends to the cover.
The cover returned by this algorithm is at most twice as large as the optimal. To see that, note that the edges picked by the algorithm have no common vertexes. The optimal solution must contain one vertex for each of these edges, while we added both.



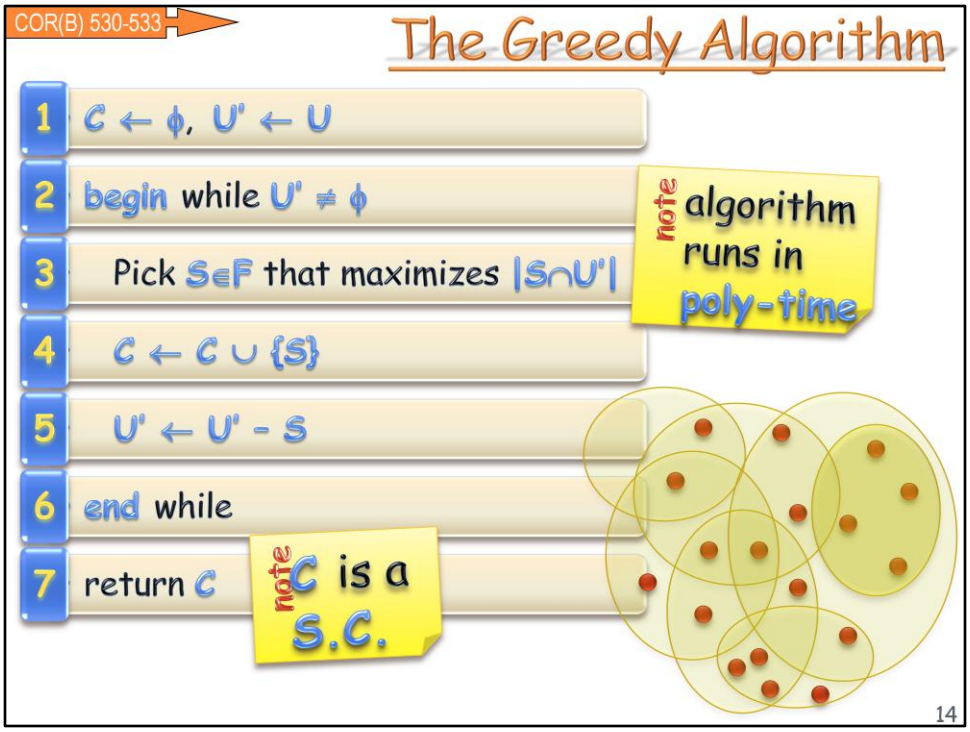
Now consider a different problem: you have a set of mail recipients, and some mailing lists. These mailing lists charge $1 for each use. Hence, when sending a message, one tries to find the minimal number of mailing lists that cover all recipients.



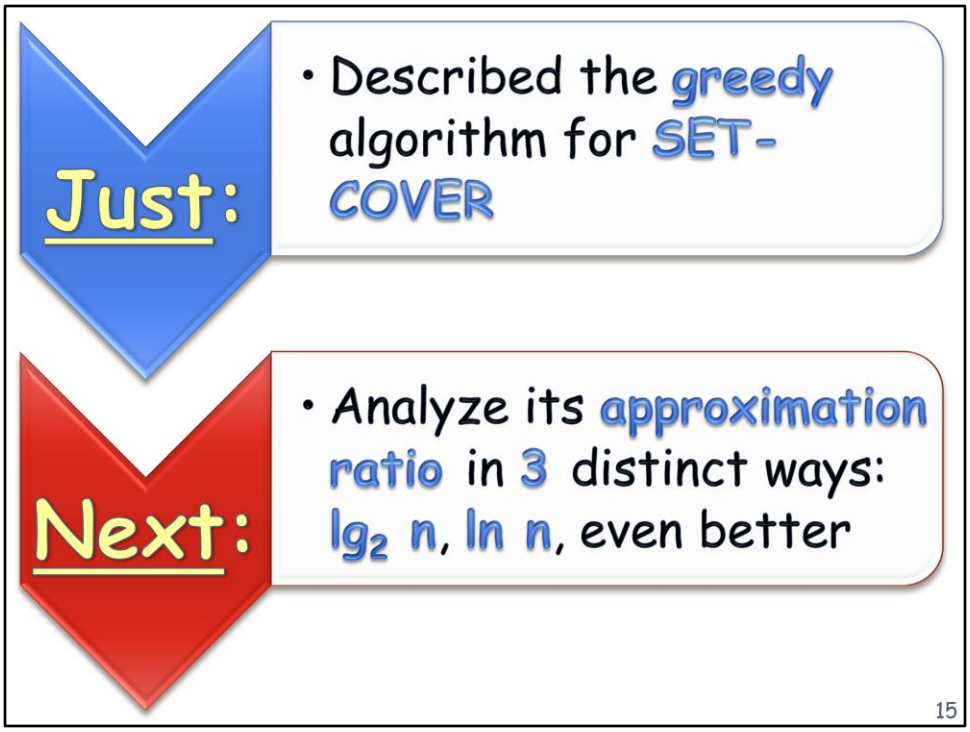
Formally, in the Set Cover problem, we have the universe U and a family F of subsets of U. The goal is to find the smallest number of subsets in the family that still cover the entire universe.



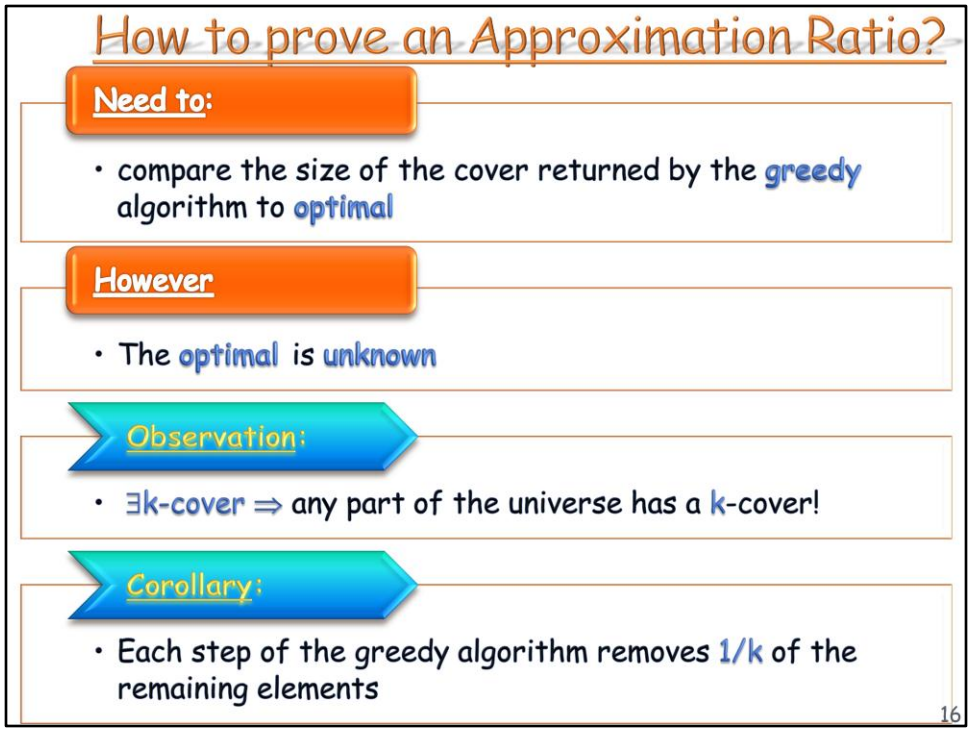
Set Cover is NP-hard. We prove this via direct reduction from Vertex Cover. Each edge becomes an element of the universe, while each vertex becomes a subset comprising all of the edges that touch it.



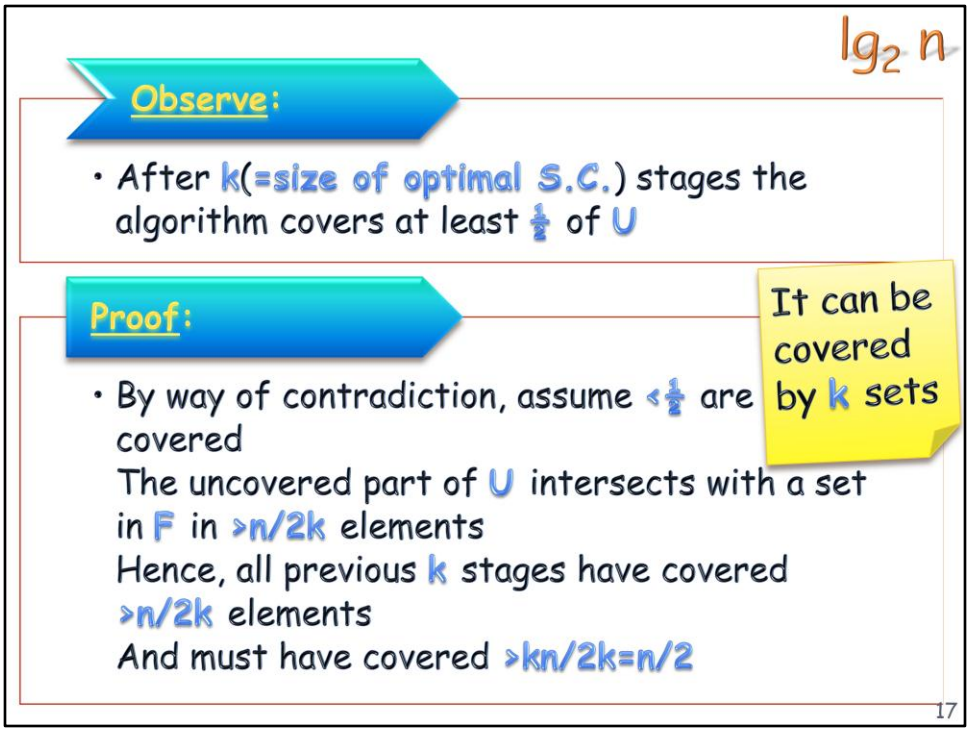
To approximate Set Cover let us introduce a general scheme for simple approximation algorithms: the greedy strategy, which in every stage picks locally the most advantageous option.
For Set Cover, this means choosing a subset that covers the most out of the yet uncovered elements.



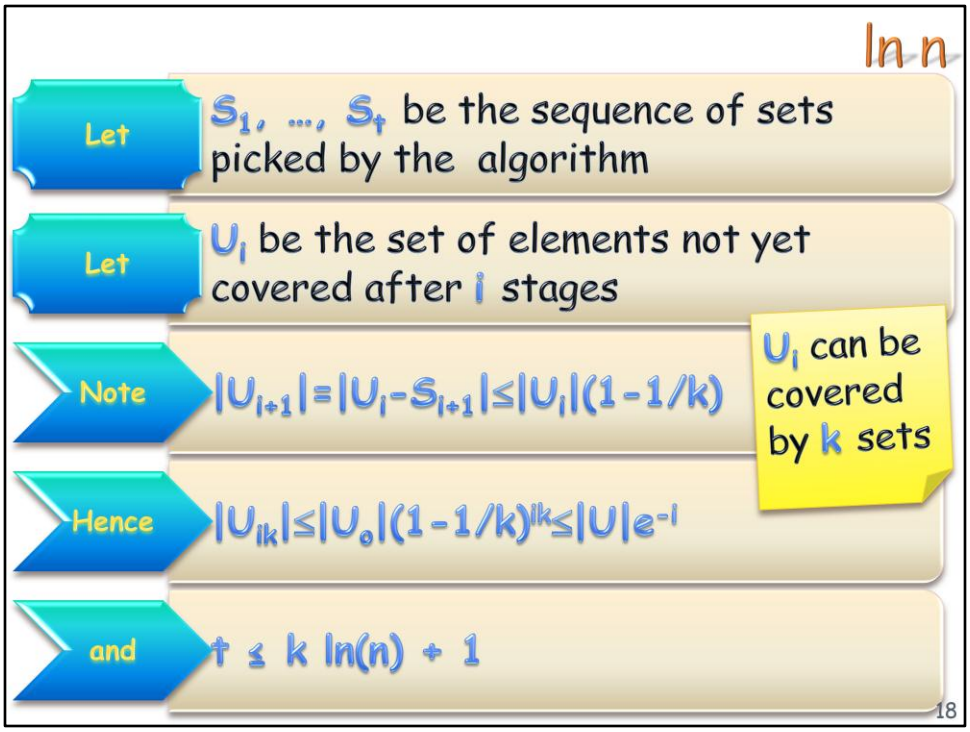
In the next few slides we look at the greedy algorithm, and analyze it in three different ways.



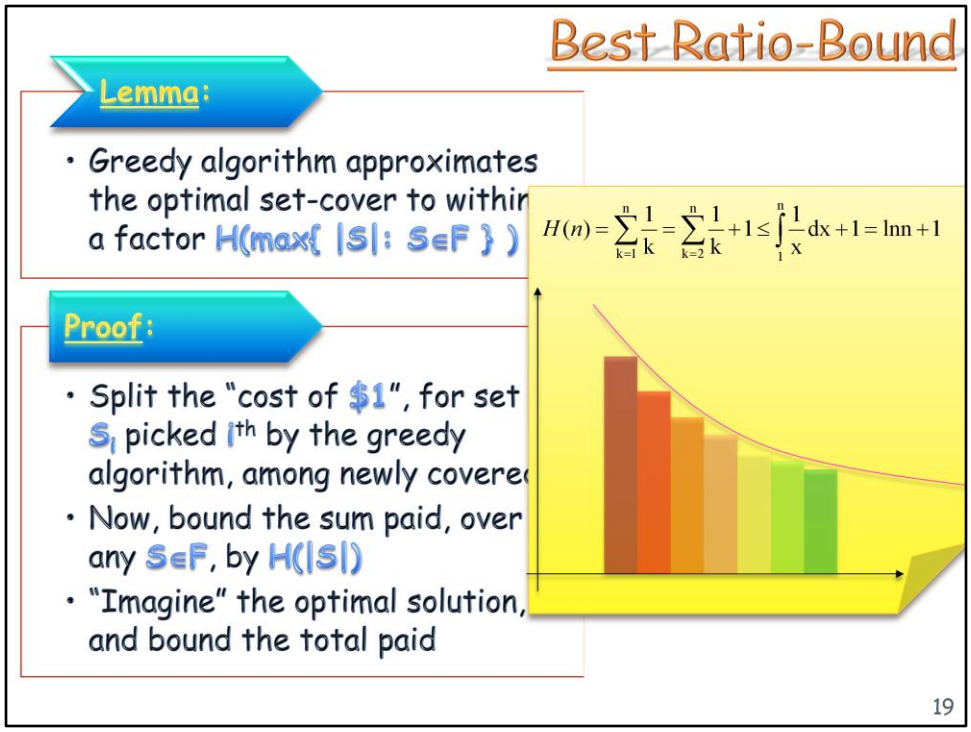
The first point we need to realize is that we are going to compare the solution given by the Greedy Algorithm to the optimal solution, however, without any information about the optimal solution.
We will use the fact that if the optimal solution contains K subsets, any part of the universe ---in particular, the set of yet uncovered elements at each stage--- can be covered by K subsets.
Hence, at each stage, the algorithm covers 1/K fraction of the remaining elements.



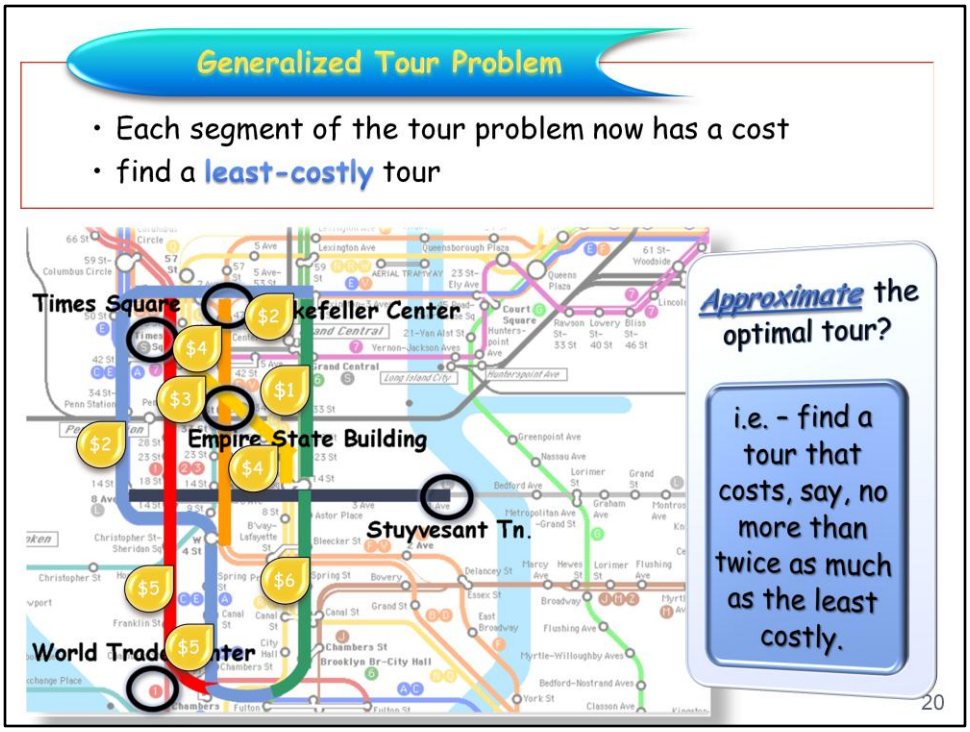
To prove the Greedy Algorithm approximates the optimal Set Cover to within a logarithmic factor (log base 2), we show that every K stages cover half of the remaining elements.
A fraction of 1/K of the remaining elements must be covered in the next stage, and thus all previous stages must've covered at least that number of elements.



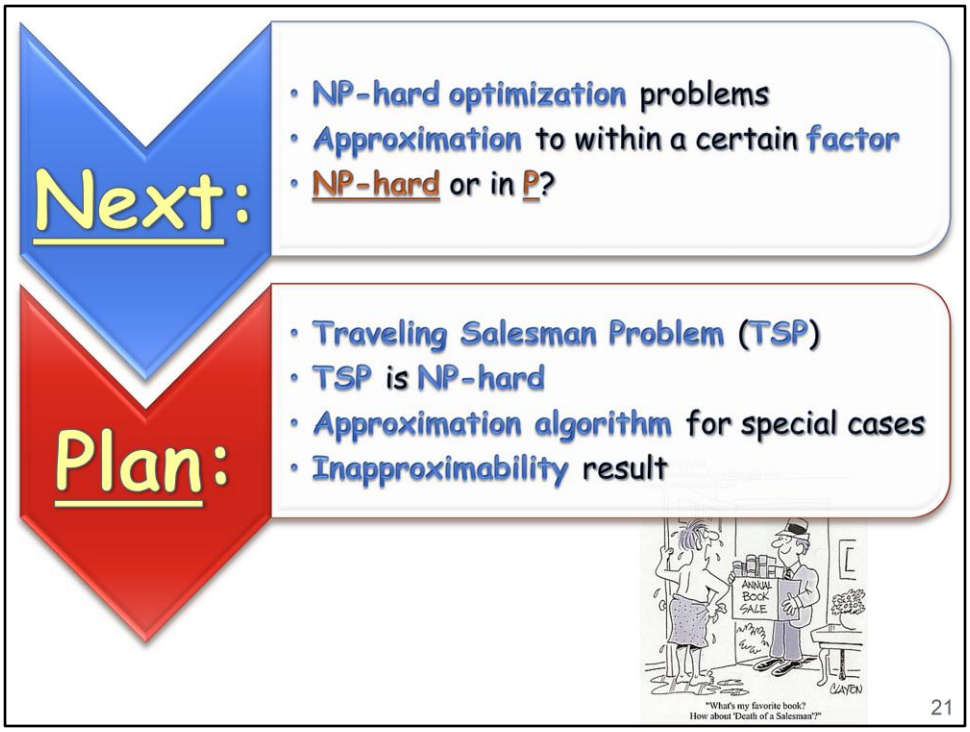
To prove a tighter bound, where the log is the natural log, we notice that at every stage a fraction 1/K of the remaining elements is covered. Hence, the number of remaining elements after K times ln |U| stages is 1.



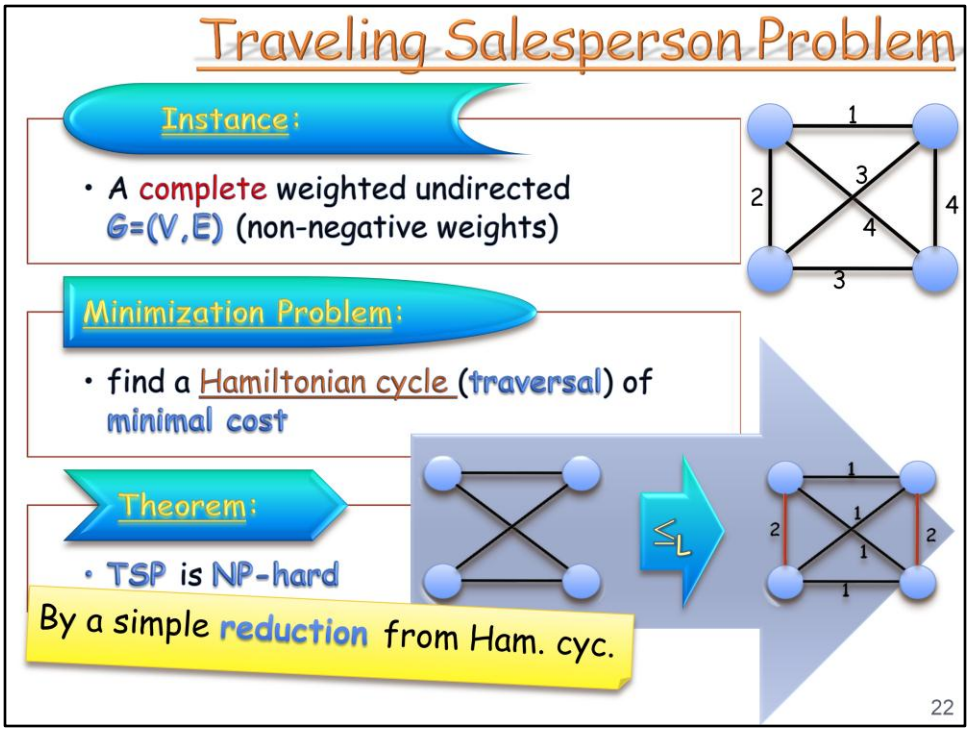
The best bound, which is ln of the size of the largest set in F, is obtained as followed:
Let us think of a price of the use of each mailing list being split among the recipients in the following manner. At each stage of the algorithm an additional $1 has to be paid, and it is split between those recipients just covered. We will later bound the total amount paid, which will give us a bound on the number of subsets the greedy algorithm outputs.
We now look at an arbitrary subset S of F, and examine how much is the total amount its member will pay, and next we'll prove it is bounded by ln |S|.
This will end the proof, as we can consider the optimal cover of K subsets (the one we can't find) and see that each of its subsets will pay at most ln of its size --- altogether at most K ln|S| for the largest S in the optimal cover.
Now for an arbitrary set, in whichever stage of the algorithm, if m of its members are not yet covered, the greedy algorithm chooses a subset that covers at least m elements, hence each will pay at most 1/m $1.
The members of the set S will pay the most, if they are covered one by one, and in each stage the set chosen covers exactly m elements, which will result with the harmonic series, which sums up to ln of S's size.



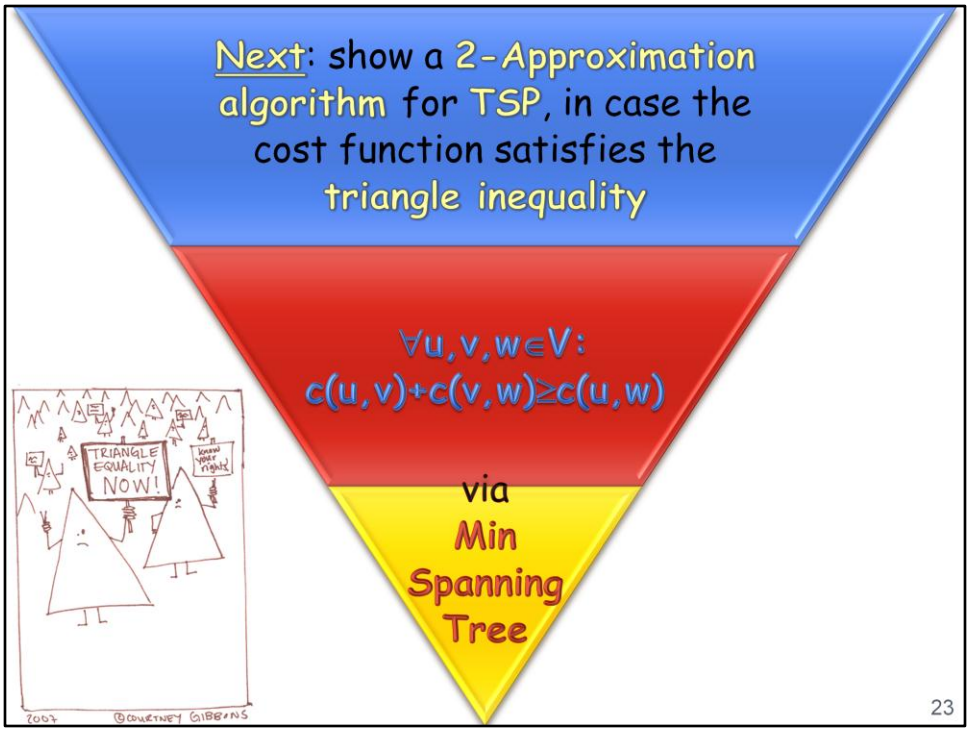
Now we go back to a problem we mentioned early on, namely, the tour problem where segments have a price that may vary.



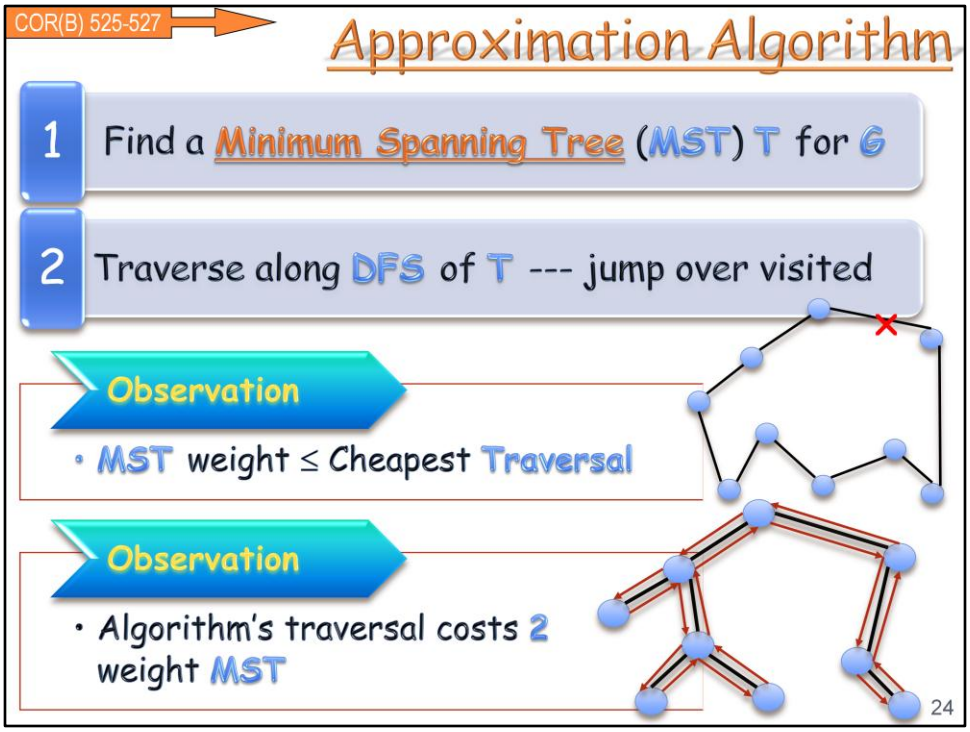
This is the Traveling Salesperson Problem (TSP): we will analyze the complexity of the general variant as well as a more restricted case. We will show an approximation algorithm in the special case, and prove an NP-hardness result, introducing a technique that will help us prove such results for approximation problems later on.



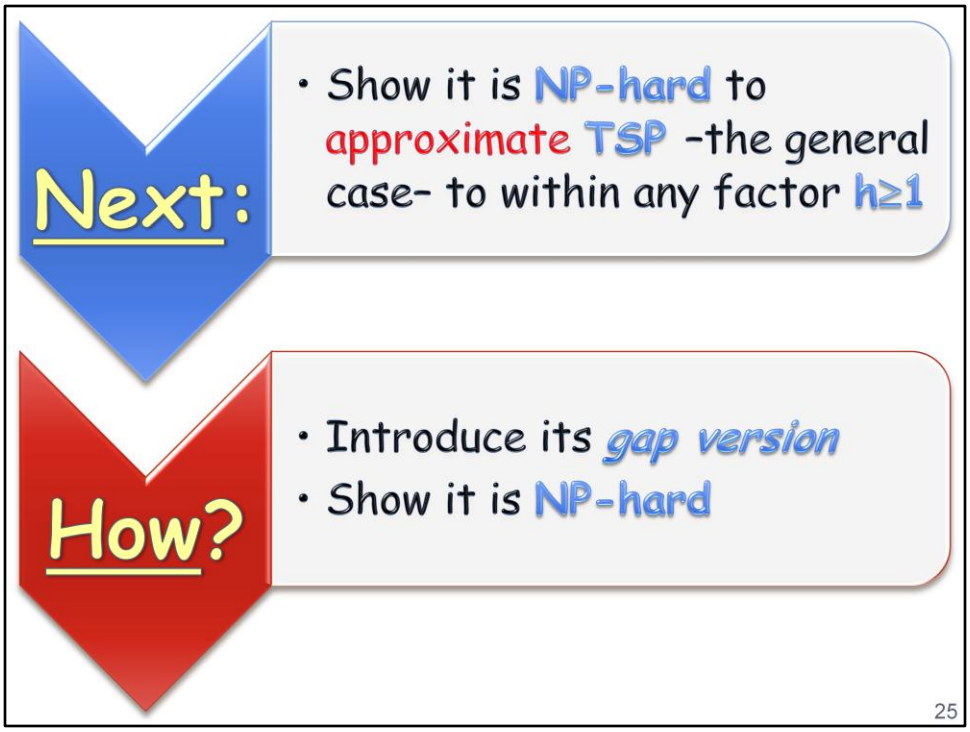
Formally, TSP instance is a graph with non-negative weights, and the goal is to find a traversal (a Hamiltonian cycle) that is of the smallest sum of weights.
We immediately see that the optimization exact solution is NP-hard, via a simple reduction from Hamiltonian cycle (edges become edges of weight 1; non-edges become edges of larger weight).



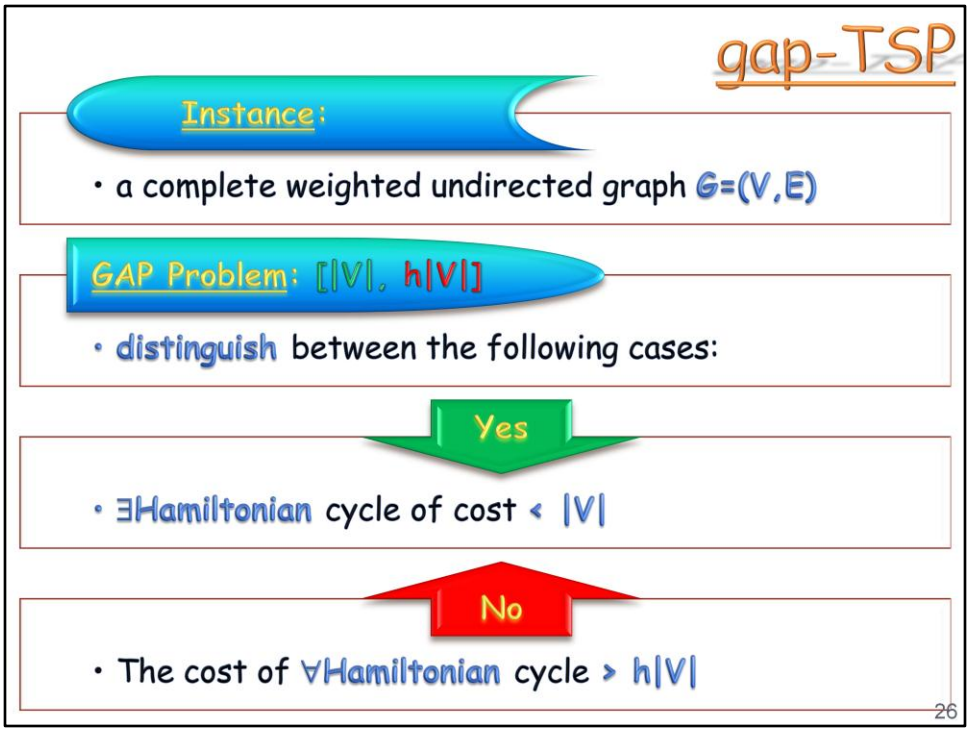
Now, we move to a special ---more interesting--- case of TSP, where the weights satisfy the triangle inequality.
In this case we are able to come up with an Approximation Algorithm, utilizing a procedure for finding the Minimum Spanning Tree (MST).



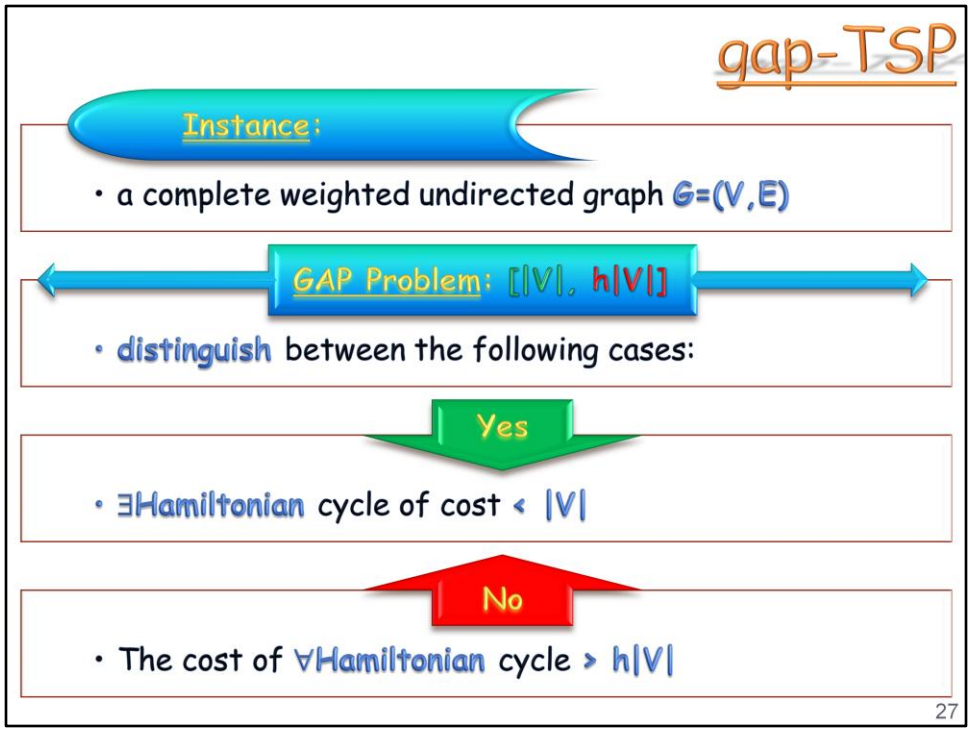
The algorithm starts by finding a MST of the graph, then follows a DSP to traverse the tree. While doing so some vertexes have already been visited, but due to the triangle inequality the traverse can simply jump to the next yet unvisited vertex, paying at most the sum of edges along the way.
To see this approximates TSP to within a factor of 2, note first that the minimal traversal bounds from above the weight of the MST, while the algorithm returns a traversal which is at most twice the MST weight.



We are now prepared to show our first NP-hardness result for an approximation problem. We will define for that purpose Gap Problems (which give us the closest version to a decision problem) and see how to prove NP-hardness of a Gap Problem, and how to obtain from that an NP-hardness result for the corresponding approximation problem.



A Gap Problem takes an optimization problem and defines two thresholds. For a minimization problem, the good instances are those whose optimization parameter is below the low threshold, while the bad instances are those for which this parameter turns out to be higher than the high parameter. The instances whose parameter is in between are the "don't care" part.
In TSP case, we define the Gap Problem to have the number of vertexes as the low threshold, while h times that to be the high threshold --- this for an arbitrary h.



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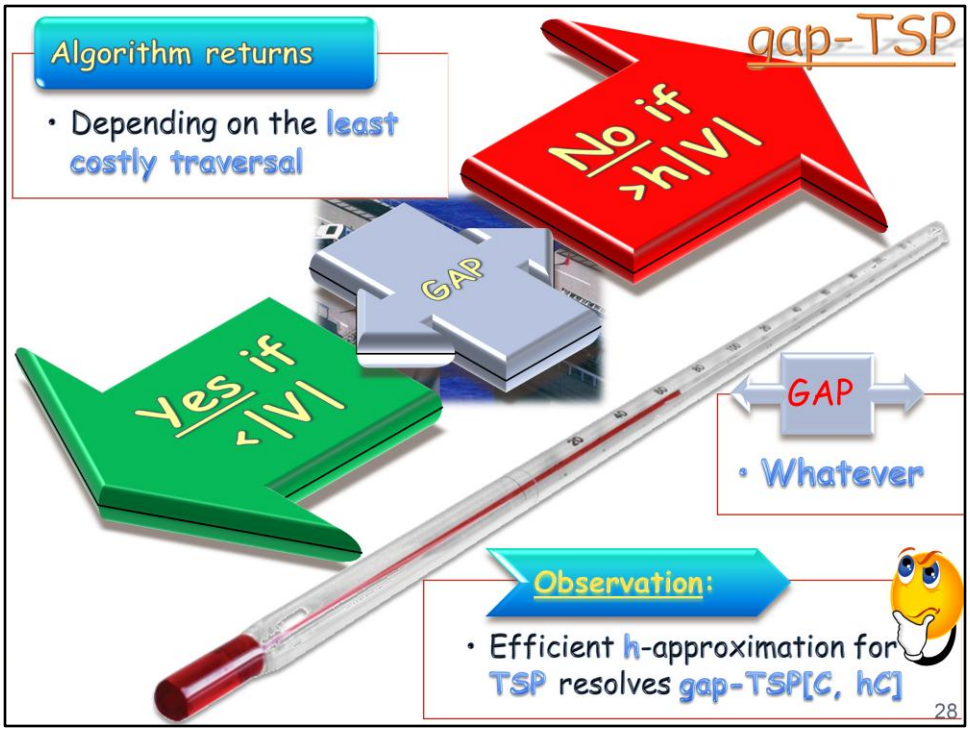

An algorithm for a Gap Problem must accept all good instances, must reject all bad instances, and can accept/reject all the "don't care" instances.
An efficient algorithm that approximates the problem to within a factor h, can be used to efficiently solve the Gap Problems where the thresholds are C and hC, for any C: simply apply it and accept all instances for which the solution returned lets the optimization parameter be smaller than hC. No bad instances can be accepted, while no good instance will be rejected as the approximation factor is h.

gap-TSP is NP-hard

Theorem:

- For any $h \geq 1$, HAM-CYCLE $\leq_L$ gap-TSP[$|V|$, $h|V|$]

Proof:

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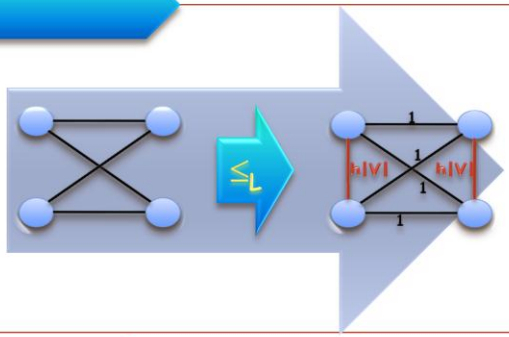
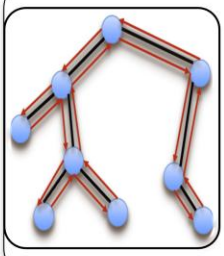


Corollary:

- Approximating TSP to within any factor is NP-hard

29

Synopsis



Some NP-hard optimization problems can be efficiently approximated:

- VERTEX-COVER (2)
- SET-COVER (ln n)
- TSP[triangle ineq.] (2)

For some factors it may still be NP-hard

We've introduced GAP problems for that purpose

30

WWindex