Quasicrystals

Homework assignment No. 3 (2 pages) March 7, 2000

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1. **External (or outer) direct products of groups:** Let G and H be two (unrelated) groups. The *direct product* $G \times H$, consisting of all ordered pairs

$$G \times H = \{(g,h) | g \in G, h \in H\}, \tag{1}$$

is a group with the multiplication rule

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, h_1 h_2). \tag{2}$$

a) Show that $G \times \{e_H\}$ is a normal subgroup of $G \times H$ which is isomorphic to G, and that $\{e_G\} \times H$ is a normal subgroup of $G \times H$ which is isomorphic to H, where e_G and e_H are the identity elements of G and H respectively.

b) Show that if G and H are both abelian then $G \times H$ is abelian as well.

2. **Internal (or inner) direct products of groups:** If G and H are both subgroups of a common group K one can multiply their elements. We say that a group K is the direct product of two of its subgroups, $K = G \times H$, if the mapping

$$\begin{array}{rccc} f: & G \times H & \to & K \\ & (g,h) & \to & gh \end{array} \tag{3}$$

is an isomorphism. It is easy to see that for f to be an isomorphism, in addition to the requirement that every element $k \in K$ be uniquely expressed as a product $k = gh$ with $g \in G$ and $h \in H$, one needs also to require that $gh = hg$.

a) Alternatively, show that f is an isomorphism if and only if

1. as sets $K = GH = \{gh | g \in G, h \in H\}$;
2. $G \cap H = \{e\}$; and
3. G and H are both *normal* subgroups of K.

b) Show that the cyclic group of order 6, C_6, is isomorphic to $C_2 \times C_3$, but that C_4 is not isomorphic to $C_2 \times C_2$. Show that, in general, $C_{nm} = C_n \times C_m$ if and only if n and m are co-prime, *i.e.* they have no common divisor except 1. You may need to use the fact that if n and m are co-prime then there exist two integers h and k such that $hn + km = 1$.

c) Show that any *finite abelian group* can be expressed as the direct product (also called direct sum in this case) of cyclic groups.

3. **Semidirect products of groups:** If H is a group, then the set $\mathcal{A}$ of all its automorphisms

$$\begin{array}{rccc} \alpha: & H & \to & H \\ & h & \to & \alpha(h) \end{array} \tag{4}$$

forms a group. If G is any subgroup of $\mathcal{A}$, then the set of ordered pairs (h, α) where $h \in H$ and $\alpha \in G$ is called the semidirect product $H \circledS G$.

a) Show that $H \circledS G$ forms a group with the multiplication rule

$$(h_1 \alpha_1) \cdot (h_2, \alpha_2) = (h_1 \alpha_1(h_2), \alpha_1 \alpha_2). \tag{5}$$

b) Show that the ordered pairs (h, ϵ) where ϵ is the identity of G and $h \in H$ form a normal subgroup of $H \circledS G$, isomorphic to H.

c) Show that the ordered pairs (e, α) where e is the identity of H and $\alpha \in G$ form a subgroup of $H \circledS G$, isomorphic to G. This subgroup is not necessarily normal.

4. **Inner semidirect products of groups:** As a special case, K is called the semidirect product of a normal subgroup H and a subgroup G, $K = H \circledS G$, if the mapping

$$\begin{array}{rccc} f': & H \circledS G & \to & K \\ & (h, g) & \to & hg \end{array} \tag{6}$$

is an isomorphism, where it is understood that the way in which elements of G act as automorphisms of H is by conjugation. Thus $g(h) = ghg^{-1}$.

a) Let N be a normal subgroup of G. When is it true that $G = N \times (G/N)$?

b) When is it true that $G = N \circledS (G/N)$?

c) What happens in the case of a space group $\mathcal{S}$ and its subgroup of pure translations T?