Exercise No. 4: The Density Operator and Propagators

1. A harmonic oscillator is brought to thermal equilibrium at temperature T and then is disconnected from the reservoir and coupled to a two-state system in such a way that the two-state system is in a $\sigma_3 = +1$ state if the level of the oscillator is even, and $\sigma_3 = -1$ if it is odd. Write the reduced density matrix if one is interested only in the two-state system. Use the density operator to compute $\langle \sigma_3 \rangle$.
2. (a) Compute the polarization density matrix in the linear polarization basis for the following cases:
   i. unpolarized light,
   ii. right circularly-polarized light.

   (b) Give a quantitative criterion for discerning between the two and demonstrate it for the above two cases.
3. (a) Compute the propagator of a free particle in momentum space $U(\mathbf{p}_f, t_f; \mathbf{p}_i, t_i)$ (i.e., the amplitude for a particle with momentum $\mathbf{p}_i$ at the time t_i to be found with momentum $\mathbf{p}_f$ at t_f) by means of a Fourier transform of the propagator in coordinate space.

   (b) Compute the same propagator directly.