Exercise No. 3: Coherent States and The Density Operator

1. Consider a harmonic oscillator, whose Hamiltonian is given by $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$, in thermal equilibrium with a reservoir at temperature T.
   (a) Find the average energy of the oscillator.
   (b) Compare what you obtained in 1a with the classical result (take the classical limit).
   (c) What is the thermal expectation value of $N = a^\dagger a$?
2. (a) Prove that the density operator ρ satisfies the inequality $\mathrm{Tr}\rho^2 \leq 1$ and that the equality holds only for pure states.
   (b) Show that $\mathrm{Tr}\rho^2$ is time-independent.
3. A spin $\frac{1}{2}$ in equilibrium with a reservoir at temperature T is placed in a magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$.
   (a) Compute the average magnetization $\langle M \rangle$.
   (b) What is the uncertainty in the magnetization?
4. In a harmonic oscillator we define the dimensionless coordinates

$$Q = \frac{1}{\sqrt{2}}\left(a^\dagger + a\right) \ , \quad P = \frac{i}{\sqrt{2}}\left(a^\dagger - a\right) \ .$$

   The squeezing operator $S(\xi)$ is given by

$$S(\xi) = \exp\left[\frac{1}{2}\xi^*(a^\dagger)^2 - \frac{1}{2}\xi a^2\right] \ .$$

   (a) Compute $\langle Q \rangle$, $\langle P \rangle$ and their uncertainties for the state $S(\xi)|\alpha\rangle$, where $|\alpha\rangle$ is a coherent state and ξ is real.
   (b) How do $\langle Q \rangle$ and ΔQ evolve with time? (Here also assume that α is real.)