Exercise No. 2: The Harmonic Oscillator and Coherent States

1. Compute explicitly using the commutation relations the following commutators:
   (a) $[N_{tot}, (a_i^\dagger)^n]$,
   (b) $[N_{tot}, a_i^n]$,
   (c) $[N_{tot}, A_{ij}^\dagger]$,
   (d) $[A_{ij}^\dagger, A_{kl}^\dagger]$,
   (e) $[A_{ij}, A_{kl}^\dagger]$,

   where $A_{ij}^\dagger = a_i^\dagger a_j^\dagger$ and $N_{tot} = \sum_i a_i^\dagger a_i$ and the indices i and j refer to different uncoupled harmonic oscillators.

2. Show the following relations
   (a) $[a, f(a^\dagger)] = \frac{\partial f(a^\dagger)}{\partial a^\dagger}$ and $[a^\dagger, f(a)] = -\frac{\partial f(a)}{\partial a}$,
   (b) $e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!} C_n$, where $C_n = [A, C_{n-1}]$ and $C_0 = B$.

3. Compute the following for the displacement operator $D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$:
   (a) $D^\dagger(\alpha) a D(\alpha)$,
   (b) $D^\dagger(\alpha) a^\dagger D(\alpha)$,
   (c) $D^\dagger(\alpha) f(a, a^\dagger) D(\alpha)$,
   (d) $D(\alpha) D(\beta)$.

4. Consider the two time-dependent operators $A(t)$ and $B(t)$ whose commutation relations are

   $$[A(t), A(t')] = [B(t), B(t')] = 0\ , \quad [A(t), B(t')] = f(t, t')\ .$$

   $f(t, t')$ commutes with $A(t'')$ and $B(t'')$ for all t, t' and t''.

   (a) Solve the differential equation

   $$\frac{dF(t)}{dt} = [A(t) + B(t)]\, F(t)\ .$$

   Hint: Use your knowledge of the equation $\frac{dG(t)}{dt} = B(t)G(t)$ and follow the same approach used in the interaction picture.

(b) Express the solution obtained in 4a as a function of $A(t) + B(t)$ and $f(t, t')$.

5. (a) Find the time evolution of a driven harmonic oscillator whose Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \sqrt{2m\hbar\omega} f(t)x \ ,$$

i.e., express $|\psi(t)\rangle$ given the initial state $|\psi(0)\rangle$. Use the interaction picture but remember that the initial and final states should be in the Schrödinger picture.

(b) The system is known to be in the ground state at $t = 0$, what will its state at any later time t be? Express your answer in the Schrödinger picture.