Exercise No. 14: The Dirac Equation

1. Show the following relations where $\gamma^0 = \beta$ and $\gamma^i = \beta\alpha_i$:
   (a) $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$,
   (b) $\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu$,
   (c) $\gamma^\mu\gamma^\nu\gamma^\lambda\gamma_\mu = 4g^{\nu\lambda}$,
   (d) $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$,
   (e) $(a_\mu\gamma^\mu)(b_\nu\gamma^\nu) = a_\mu b^\mu - i\sigma^{\mu\nu}a_\mu b_\nu$, where a_μ and b_μ are four-vectors.
2. Find the transformation properties of the objects bellow under Lorentz transformations:
   (a) $\bar{\psi}\psi$,
   (b) $\bar{\psi}\gamma_5\psi$,
   (c) $\bar{\psi}\gamma^\mu\gamma_5\psi$,
   (d) $\bar{\psi}\sigma^{\mu\nu}\psi$.
3. An electron is subjected to a uniform magnetic field in the z-axis direction. The Dirac equation in this case takes the form

$$(c\vec{\alpha}\cdot\vec{\pi} + \beta mc^2)\psi = E\psi \ ,$$

   where $\vec{\pi} = \vec{p} - \frac{e}{c}\vec{A}$ and $\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$.
   (a) Use a gauge in which $\vec{A} = \frac{1}{2}B(x\hat{y} - y\hat{x})$. What is the exact equation satisfied by the upper two components φ?
   A useful identity is $(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B}) = \vec{A}\cdot\vec{B} + i\vec{\sigma}\cdot(\vec{A}\times\vec{B})$.
   (b) Show that $P = \pi_y$ and $Q = \frac{c\pi_x}{eB}$ are canonical variables and use this to compute the energy spectrum of the electron.
   (c) What is the energy spectrum in the non-relativistic limit?