Exercise No. 11: Second Quantization II

1. Consider the Hamiltonian
$$H = \epsilon a^\dagger a + \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + a^\dagger a \sum_{\mathbf{k}} M_{\mathbf{k}} (a_{\mathbf{k}} + a^\dagger_{\mathbf{k}}) \ ,$$
where ϵ is the energy of a particular *fermionic* state, $\epsilon_{\mathbf{k}}$ is the energy of a phonon (which is a boson) with a wave-vector $\mathbf{k}$ and $M_{\mathbf{k}}$ are $\mathbf{k}$-dependent coupling constants. Define the following transformation for any operator A:
$$\bar{A} = e^S A e^{-S} \ ,$$
where S is an operator. In particular we take $S = a^\dagger a \sum_{\mathbf{k}} \frac{M_{\mathbf{k}}}{\epsilon_{\mathbf{k}}} (a^\dagger_{\mathbf{k}} - a_{\mathbf{k}})$.
   (a) Explain the possible physical meaning of each term in the Hamiltonian.
   (b) Show that $\bar{a} = aX$, $\bar{a}^\dagger = a^\dagger X^\dagger$, $\bar{a}_{\mathbf{k}} = a_{\mathbf{k}} - \frac{M_{\mathbf{k}}}{\epsilon_{\mathbf{k}}} a^\dagger a$, $\bar{a}^\dagger_{\mathbf{k}} = a^\dagger_{\mathbf{k}} - \frac{M_{\mathbf{k}}}{\epsilon_{\mathbf{k}}} a^\dagger a$, where $X = \exp\left[-\sum_{\mathbf{k}} \frac{M_{\mathbf{k}}}{\epsilon_{\mathbf{k}}} (a^\dagger_{\mathbf{k}} - a_{\mathbf{k}})\right]$ (note that $X^\dagger = X^{-1}$).
   (c) Show that $\bar{H} = a^\dagger a(\epsilon - \Delta) + \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} a^\dagger_{\mathbf{k}} a_{\mathbf{k}}$, where $\Delta \equiv \sum_{\mathbf{k}} \frac{M^2_{\mathbf{k}}}{\epsilon_{\mathbf{k}}}$. Verify that this form agrees with your initial interpretation of H.
2. Suppose that the wave-function of an N-fermion system is a Slater determinant of orthonormal functions ϕ_i. Using the second quantization formalism show that the pair correlation function of the state factors as for plane waves.
   Hint: consider the operators $a_i = \int d^3r \phi^*_i(\mathbf{r}) \psi(\mathbf{r})$.
3. (a) Compute the pair correlation function for a system of free non-interacting spinless bosons in a state $|\Phi\rangle$.
   (b) Find the pair correlation function if the boson occupation numbers are given by
$$n_{\mathbf{p}} = c e^{-\alpha(\mathbf{p}-\mathbf{p}_0)^2/2} \ ,$$
   where $\mathbf{p}$ is the momentum of the bosons.