Exercise No. 10: Identical Particles and Second Quantization

1. Let h_0 be the Hamiltonian of a particle. Assume that the operator h_0 acts only on the orbital variables and has three equidistant levels of energy 0, $\hbar\omega_0$ and $2\hbar\omega_0$ (where ω_0 is a real positive constant) which are non-degenerate in the orbital state space.
   (a) Consider a system of three independent electrons whose Hamiltonian is
   $$H = h_0(1) + h_0(2) + h_0(3) \ .$$
   Find the energy levels of H and their degree of degeneracy.
   (b) Do the same for a system of three identical bosons of spin 0.

2. Let $|\phi\rangle$ and $|\chi\rangle$ be two normalized orthogonal states belonging to the orbital state space of an electron and let $|+\rangle$ and $|-\rangle$ be the two spin eigenstates with respect to the z-axis.
   (a) Consider a system of two electrons, one in the state $|\phi, +\rangle$ and the other in the state $|\chi, -\rangle$. Let $\rho_2(\mathbf{r}, \mathbf{r}')d^3r d^3r'$ be the probability of finding one of them in the volume d^3r centered at the point $\mathbf{r}$ and the other in a volume d^3r' centered at $\mathbf{r}'$ (two-particle density function). Similarly, let $\rho_1(\mathbf{r})d^3r$ be the probability of finding one of the electrons in a volume d^3r centered at $\mathbf{r}$ (one-particle density function). Find $\rho_2(\mathbf{r}, \mathbf{r}')$ and $\rho_1(\mathbf{r})$. Show that the expressions you have obtained remain valid even if $|\phi\rangle$ and $|\chi\rangle$ are not orthogonal. Compare these results with those which would be obtained for a system of distinguishable particles (both spin $\frac{1}{2}$), one in the state $|\phi, +\rangle$ and the other in the state $|\chi, -\rangle$ when the device which measures the positions is assumed to be unable to distinguish between the two particles.
   (b) Now assume one electron is in a state $|\phi, +\rangle$ and the other one in the state $|\chi, +\rangle$. Compute $\rho_2(\mathbf{r}, \mathbf{r}')$ and $\rho_1(\mathbf{r})$ in this case. What happens to ρ_1 and ρ_2 if $|\phi\rangle$ and $|\chi\rangle$ are no longer orthogonal?

3. Construct explicit 4×4 matrices to represent the fermion creation and annihilation operators a_0, $a_0^\dagger$, a_1 and $a_1^\dagger$ for two one-particle states. Check the anti-commutation relations.