Exercise No. 9: Identical Particles and Second Quantization

1. Let h_0 be the Hamiltonian of a particle. Assume that the operator h_0 acts only on the orbital variables and has three equidistant levels of energy 0, $\hbar\omega_0$ and $2\hbar\omega_0$ (where ω_0 is a real positive constant) which are non-degenerate in the orbital state space.
   (a) Consider a system of three independent electrons whose Hamiltonian is
   $$H = h_0(1) + h_0(2) + h_0(3) \ .$$
   Find the energy levels of H and their degree of degeneracy.
   (b) Do the same for a system of three identical bosons of spin 0.
2. Let $|\phi\rangle$ and $|\chi\rangle$ be two normalized orthogonal states belonging to the orbital state space of an electron and let $|+\rangle$ and $|-\rangle$ be the two spin eigenstates with respect to the z-axis.
   (a) Consider a system of two electrons, one in the state $|\phi, +\rangle$ and the other in the state $|\chi, -\rangle$. Let $\rho_2(\mathbf{r}, \mathbf{r}')d^3r d^3r'$ be the probability of finding one of them in the volume d^3r centered at the point $\mathbf{r}$ and the other in a volume d^3r' centered at $\mathbf{r}'$ (two-particle density function). Similarly, let $\rho_1(\mathbf{r})d^3r$ be the probability of finding on of the electrons in a volume d^3r centered at $\mathbf{r}$ (one-particle density function). Find $\rho_2(\mathbf{r}, \mathbf{r}')$ and $\rho_1(\mathbf{r})$. Show that the expressions you have obtained remain valid even if $|\phi\rangle$ and $|\chi\rangle$ are not orthogonal. Compare these results with those which would be obtained for a system of distinguishable particles (both spin $\frac{1}{2}$), one in the state $|\phi, +\rangle$ and the other in the state $|\chi, -\rangle$ when the device which measures the positions is assumed to be unable to distinguish between the two particles.
   (b) Now assume one electron is in a state $|\phi, +\rangle$ and the other one in the state $|\chi, +\rangle$. Compute $\rho_2(\mathbf{r}, \mathbf{r}')$ and $\rho_1(\mathbf{r})$ in this case. What happens to ρ_1 and ρ_2 if $|\phi\rangle$ and $|\chi\rangle$ are no longer orthogonal?
3. Consider the Hamiltonian
   $$H = \epsilon a^\dagger a + \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + a^\dagger a \sum_{\mathbf{k}} M_{\mathbf{k}}(a_{\mathbf{k}} + a^\dagger_{\mathbf{k}}) \ ,$$
   where ϵ is the energy of a particular *fermionic* state, $\epsilon_{\mathbf{k}}$ is the energy of a phonon (which is a boson) with a wave-vector $\mathbf{k}$ and $M_{\mathbf{k}}$ are $\mathbf{k}$-

dependent coupling constants. Define the following transformation for any operator A:

$$\bar{A} = e^S A e^{-S} \ ,$$

where S is an operator. In particular we take $S = a^\dagger a \sum_{\mathbf{k}} \frac{M_{\mathbf{k}}}{\epsilon_{\mathbf{k}}}(a^\dagger_{\mathbf{k}} - a_{\mathbf{k}})$.

(a) Explain the possible physical meaning of each term in the Hamiltonian.

(b) Show that $\bar{a} = aX$, $\bar{a}^\dagger = a^\dagger X^\dagger$, $\bar{a}_{\mathbf{k}} = a_{\mathbf{k}} - \frac{M_{\mathbf{k}}}{\epsilon_{\mathbf{k}}} a^\dagger a$, $\bar{a}^\dagger_{\mathbf{k}} = a^\dagger_{\mathbf{k}} - \frac{M_{\mathbf{k}}}{\epsilon_{\mathbf{k}}} a^\dagger a$, where $X = \exp\left[-\sum_{\mathbf{k}} \frac{M_{\mathbf{k}}}{\epsilon_{\mathbf{k}}}(a^\dagger_{\mathbf{k}} - a_{\mathbf{k}})\right]$ (note that $X^\dagger = X^{-1}$).

(c) Show that $\bar{H} = a^\dagger a(\epsilon - \Delta) + \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} a^\dagger_{\mathbf{k}} a_{\mathbf{k}}$, where $\Delta \equiv \sum_{\mathbf{k}} \frac{M^2_{\mathbf{k}}}{\epsilon_{\mathbf{k}}}$. Verify that this form agrees with your initial interpretation of H.