Exercise No. 5: Quantum Fields

1. The equations of motion of a complex scalar field $\phi(x)$ are

$$\eta^{\mu\nu}\frac{\partial^2\phi}{\partial x^\mu \partial x^\nu} = -\frac{m^2c^2}{\hbar^2}\phi \ , \quad \eta^{\mu\nu}\frac{\partial^2\phi^*}{\partial x^\mu \partial x^\nu} = -\frac{m^2c^2}{\hbar^2}\phi^* \ .$$

   (a) Write the Lagrangian density that reproduces these equations of motion.

   (b) Perform the quantization of the field.

   (c) What is the classical Hamiltonian of this field?

   (d) Find the quantum Hamiltonian.

2. Compute the following commutators for the electromagnetic field:

   (a) $[A_i(\mathbf{r},t), \pi_i(\mathbf{r}',t)]$,

   (b) $[B_i(\mathbf{r},t), E_j(\mathbf{r}',t)]$.

3. Show that the propagator of a driven harmonic oscillator whose Lagrangian is $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2x^2 + f(t)x$ is

$$U(x_b,t_b;x_a,t_a) = \left(\frac{m\omega}{2\pi i\hbar \sin\omega T}\right)^{1/2} e^{iS_{\rm cl}/\hbar} \ ,$$

where

$$\begin{aligned} S_{\rm cl} \ = \ & \frac{m\omega}{2\pi i\hbar \sin\omega T}\Big[(x_a^2+x_b^2)\cos\omega T - 2x_a x_b \\ & + \frac{2x_b}{m\omega}\int_{t_a}^{t_b} f(t)\sin\omega(t-t_a)dt + \frac{2x_a}{m\omega}\int_{t_a}^{t_b} f(t)\sin\omega(t_b-t)dt \\ & - \frac{2}{m^2\omega^2}\int_{t_a}^{t_b} dt\int_{t_a}^{t} ds f(t)f(s)\sin\omega(t_b-t)\sin\omega(s-t_a)\Big] \ , \end{aligned}$$

and $T = t_b - t_a$.