Exercise No. 3: The Density Operator

1. Consider a harmonic oscillator, whose Hamiltonian is given by $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$, in thermal equilibrium with a reservoir at temperature T.
   (a) Find the average energy of the oscillator.
   (b) Compare what you obtained in 1a with the classical result (take the classical limit).
   (c) What is the thermal expectation value of $N = a^\dagger a$?
2. (a) Prove that the density operator ρ satisfies the inequality $\mathrm{Tr}\rho^2 \leq 1$ and that the equality holds only for pure states.
   (b) Show that $\mathrm{Tr}\rho^2$ is time-independent.
3. A spin $\frac{1}{2}$ in equilibrium with a reservoir at temperature T is placed in a magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$.
   (a) Compute the average magnetization $\langle M \rangle$.
   (b) What is the uncertainty in the magnetization?
4. A harmonic oscillator is brought to thermal equilibrium at temperature T and then is disconnected from the reservoir and coupled to a two-state system in such a way that the two-state system is in a $\sigma_3 = +1$ state if the level of the oscillator is even, and $\sigma_3 = -1$ if it is odd. Write the reduced density matrix if one is interested only in the two-state system. Use the density operator to compute $\langle \sigma_3 \rangle$.
5. (a) Compute the polarization density matrix in the linear polarization basis for the following cases:
      i. unpolarized light,
      ii. right circularly-polarized light.
   (b) Give a quantitative criterion for discerning between the two and demonstrate it for the above two cases.