Exercise No. 13: The BCS State

1. (a) Prove that if $|G\rangle$ is the Bogoliubov ground state, the true vacuum of the system is given by

$$|0\rangle = \prod_{k>0} \left(u_k + v_k \alpha^\dagger_{-k} \alpha^\dagger_k\right) |G\rangle \ .$$

   (b) Show that the state $|G\rangle$ is normalized.

2. Explicitly perform the Bogoliubov transformation and normal-ordering in order to obtain $K = H - \mu \hat{N}$ neglecting normal-ordered four-operator interaction terms.

3. Compute the condensation energy, i.e. the energy difference between the BCS and the normal ground states, neglecting terms of order G.