Exercise No. 12: Second Quantization and Cooper Pairs

1. Calculate $\langle r^2 \rangle$ (where $\mathbf{r}$ is the relative coordinate) for a $\mathbf{K} = 0$ bound pair.
2. To see the role played by the Fermi sea in the Cooper pair problem, suppose that $k_f = 0$. What is then the exact condition on v_0 that there be a bound state ($E < 0$) for $\mathbf{K} = 0$?
3. Consider a one-dimensional lattice of spin $\frac{1}{2}$ spins.
   (a) Express the quantum spin operators $\mathbf{S}_i$ (where i is the index of the lattice site) of such a single spin using the fermionic annihilation and creation operators a_i and $a_i^\dagger$ (*hint*: think of the spin raising and lowering operators). Show that your construction indeed satisfies the required commutation relations.
   (b) Define the operator
   $$\phi_i = \pi \sum_{j<i} a_j^\dagger a_j$$
   and show that $e^{i\phi_i}$ commutes with the fermionic operators at sites to the right of it and anticommutes with those at sites to the left of it.
   (c) Spins at different sites are commutative. Use the operators defined above to modify your definition of the spin operators so that spins at different sites commute as required. (This definition of the spin operators in terms for fermionic creation and annihilation operators is called the Wigner–Jordan transformation.)
   (d) The one-dimensional Heisenberg model is a one dimensional lattice of spin $\frac{1}{2}$ interacting spins with the following Hamiltonian
   $$H = -J \sum_i \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y \right) - J_z \sum_i S_i^z S_{i+1}^z \,,$$
   where J and J_z are the coupling constants. Express this Hamiltonian in terms of the fermionic creation and annihilation operators.
   (e) Find the eigenstates and their energies of the model for the case $J_z = 0$ (the so-called x-y model).