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On Extreme Points of the Dual Ball of a Polyhedral Space

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Abstract: We prove that every separable polyhedral Banach space X is isomorphic to a polyhedral Banach space Y such that, the set $\operatorname{ext} B_{Y^*}$ cannot be covered by a sequence of balls $B(y_i, \epsilon_i)$ with $0 < \epsilon_i < 1$ and $\epsilon_i \to 0$. In particular $\operatorname{ext} B_{Y^*}$ cannot be covered by a sequence of norm compact sets. This generalizes a result from [7] where an equivalent polyhedral norm $|||\cdot|||$ on c_0 was constructed such that $\operatorname{ext} B_{(c_0,|||\cdot|||)^*}$ is uncountable but can be covered by a sequence of norm compact sets.

Key words: Polyhedral Banach space, boundary, extreme points.

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In [8] V. Klee introduced the following definition of a polyhedral Banach space.

Definition 1. A Banach space X is called *polyhedral* if the unit ball of every finite dimensional subspace of X is a polytope.

Recall that a subset $B \subseteq S_{X^*}$ of the unit sphere of the dual Banach space X^* is called a *boundary* of X if for any $x \in X$ there is $f \in B$ with $f(x) = \|x\|$. In [3] (see also [5] and [10]), it was proved that any separable polyhedral space has a countable boundary. The converse is true under a suitable renorming (see [2]).

By the Krein-Milman Theorem, the set $\operatorname{ext} B_{X^*}$ is a boundary for any Banach space X. In [7], a separable polyhedral Banach space X was constructed (actually X is isomorphic to c_0) such that $\operatorname{ext} B_{X^*}$ is uncountable. Of course, being separable polyhedral, X admits a countable boundary. However, it is easily seen from the construction in [7] that the set $\operatorname{ext} B_{X^*}$ can be covered by a sequence of norm compact sets, i.e. although $\operatorname{ext} B_{X^*}$ is uncountable it is in a sense "close" to a countable set.

Definition 2. Let L be a Banach space and $C \subset E$. We say that C has property (A) if for each sequence $\epsilon_i \to 0$, $0 < \epsilon_i < 1$ and any sequence of balls $B(z_i, \epsilon_i) = \{x \in L : ||x - z_i|| \leq \epsilon_i\}$, we have $C \nsubseteq \bigcup_{i=1}^{\infty} B(z_i, \epsilon_i)$.

Clearly, if C has (A) then C cannot be covered by a sequence of norm compact sets.

The main result of this paper is the following

Theorem 1. *Let Y be a separable polyhedral Banach space. Then Y is isomorphic to a polyhedral Banach space Z such that the set* $\operatorname{ext} B_{Z^*}$ *has property* (A).

Remark. It follows from Theorem 3 [4], that if a Banach space Y is not isomorphic to a polyhedral space then $\operatorname{ext} B_{Y^*}$ has property (A) in any equivalent norm on Y.

We prove Theorem 1 in two steps. First we prove Theorem 1 for $Y = c_o$. Here we use some ideas from [7]. Then, by using that any polyhedral space contains an isomorphic copy of c_0 (see [3]) we finish the proof.

Theorem 2. *There exists a separable polyhedral Banach space X, isomorphic to c_0, such that the set* $\operatorname{ext} B_{X^*}$ *has property* (A).

Proof. Let $\{e_i\}_{i=1}^{\infty}$ be the natural basis of c_0 and $\{e_i^*\}_{i=1}^{\infty}$ be its biorthogonal sequence in $l_1 = c_0^*$. Fix $\varrho \in (0, \frac{1}{2})$ and denote

$$\lambda_i = \frac{1}{2^i}, \quad i = 1, 2, \ldots, \quad a = \frac{1}{\lambda_1}, \quad a_n = \frac{a \sum_{i=1}^{n} \lambda_i}{1 - \varrho \sum_{i=n+1}^{\infty} \lambda_i}, \quad n = 1, 2, \ldots .$$

Let $\mathcal{G}_m$ be the family of all injective, non-decreasing mappings from $\{1, \ldots, m\}$ to $\mathbb{N}$ and $\mathcal{G}_\infty$ be the family of all injective, non-decreasing mappings from $\mathbb{N}$ to $\mathbb{N}$. Next define:

$$A_m = \left\{ a_m \left(\sum_{i=1}^{m} \lambda_i \right)^{-1} \sum_{k=1}^{m} \epsilon_k \lambda_k e_{g(k)}^* : \ \epsilon_k = \pm 1, \ g \in \mathcal{G}_m \right\}.$$

Clearly, each A_m is countable. Denote

$$B = \bigcup_{m=1}^{\infty} A_m, \quad U^* = \overline{\operatorname{conv}}^{w^*} B,$$

and define a new norm on c_0 as follows

$$|||x||| = \sup\{ f(x) : \ f \in U^* \}, \quad x \in c_0.$$

It is easily seen that the norm $|||\cdot|||$ on c_0 is equivalent to the original one (note that $A_1 = \{\pm a_1 e_k^* : k = 1, 2 \ldots\}$). Put $X = (c_0, |||\cdot|||)$. Also a standard argument shows that $B_{X^*} = U^*$.

For every subset A of X^*, denote A' the set of all w^*-limit points of the set A.

Claim 1. Every $f \in B'$ with $|||f||| = 1$ (if any) does not attain its norm $|||f|||$ at an element of the unit ball of X.

Proof. Take $f \in B'$, $f \neq 0$. We first assume that $f \in A'_m$ for some $m \geq 2$. Since $e_n^* \to^{w^*} 0$ we get

$$f = a_m \left(\sum_{i=1}^{m} \lambda_i\right)^{-1} \sum_{k=1}^{n} \epsilon_k \lambda_k e_{g(k)}^*,$$

for some $n < m$ and $g \in \mathcal{G}_n$.

$$\begin{aligned} |||f||| &= \left|\left|\left| a_m \left(\sum_{i=1}^{m} \lambda_i\right)^{-1} \sum_{k=1}^{n} \epsilon_k \lambda_k e_{g(k)}^* \right|\right|\right| \\ &= \left|\left|\left| \frac{a_m \big(\sum_{i=1}^{m} \lambda_i\big)^{-1}}{a_n \big(\sum_{i=1}^{n} \lambda_i\big)^{-1}} a_n \left(\sum_{i=1}^{n} \lambda_i\right)^{-1} \sum_{k=1}^{n} \epsilon_k \lambda_k e_{g(k)}^* \right|\right|\right| < 1. \end{aligned}$$

Next assume that $f \in B'$ and $f \notin A'_m$, $m = 1, 2, \ldots$. It is easy to see that either f is of the form

$$f = a \sum_{k=1}^{\infty} \epsilon_k \lambda_k e_{g(k)}^*, \quad \epsilon_k = \pm 1, \ g \in \mathcal{G}_\infty, \tag{1}$$

or

$$f = a \sum_{k=1}^{n} \epsilon_k \lambda_k e_{g(k)}^*, \quad \epsilon_k = \pm 1, \ g \in \mathcal{G}_n \tag{2}$$

If f satisfies (2) then $|||f||| < 1$. So we assume that f satisfies (1). Assume to the contrary, that there is $x \in c_0$, $|||x||| = 1$, such that $f(x) = 1$. Choose s so large that $a \cdot \max\{|x_{g(k)}|\}_{k=s+1}^{\infty} < \frac{\varrho}{2}$. Then the definition of $|||\cdot|||$ implies

$$
\begin{aligned}
1 = f(x) &= a \sum_{k=1}^{s} \epsilon_k \lambda_k x_{g(k)} + a \sum_{k=s+1}^{\infty} \epsilon_k \lambda_k x_{g(k)} \\
&\leq \frac{a}{a_s} \left[a_s \left(\sum_{i=1}^{s} \lambda_i \right)^{-1} \sum_{k=1}^{s} \lambda_k |x_{g(k)}| \right] \sum_{i=1}^{s} \lambda_i + \Big(a \cdot \max_{k>s} |x_{g(k)}| \Big) \sum_{k=s+1}^{\infty} \lambda_k \\
&< \frac{a}{a_s} \cdot \sum_{i=1}^{s} \lambda_i + \frac{\varrho}{2} \sum_{i=s+1}^{\infty} \lambda_i < 1.
\end{aligned}
$$

The last inequality follows from the following equality:

$$
\frac{a}{a_s} \sum_{i=1}^{s} \lambda_i + \varrho \sum_{i=s+1}^{\infty} \lambda_i = 1.
$$

∎

Claim 2. B is a countable boundary for X and X is polyhedral.

Proof. Since each A_m is countable and $B = \bigcup_{m=1}^{\infty} A_m$, it follows that B is countable. The rest of the claim is a direct result of Claim 1 and Proposition 6.11 from [6]. We give a proof for the sake of completeness. Since $U^* = \overline{conv}^{w^*} B$, $\overline{B}^{w^*} = B \cup B'$ is a boundary for X. As a result of Claim 1, none of the elements in B' attain their norm at B_X hence B is a boundary for X. Now let F be a finite dimensional subspace of X and assume F^* has infinitely many extreme points, By Milman's theorem, these would be restrictions to F of elements of $\overline{B}^{w^*}$. Since F is finite-dimensional, any w^*-cluster point of the set of the extreme points of B_{F^*} attains its norm at an element of B_F. But this contradicts Claim 1. Hence F^* has only finitely many extreme points, and F is polyhedral. ∎

Claim 3. For any $g \in \mathcal{G}_\infty$ and $\{\epsilon_i\}_{i=1}^{\infty}$ a sequence of signs, we have $f = a \sum_{k=1}^{\infty} \epsilon_k \lambda_k e^*_{g(k)} \in \operatorname{ext} U^*$.

Proof. First note that from the definition of the norm $|||\cdot|||$ (the supremum over the set B) follows that

$$
\left|\left|\left| \sum_{i=1}^{n} \epsilon_i e_{g(i)} \right|\right|\right| \leq 2
$$

Next the series $\sum_{i=1}^\infty \epsilon_i e_{g(i)}$ converges in the w^*-topology of $X^{**} \cong \ell_\infty$ and it follows that $|||\sum_{i=1}^\infty \epsilon_i e_{g(i)}||| \leq 2$. Moreover, setting $z^{**} = \sum_{i=1}^\infty \epsilon_i e_{g(i)}$ and $b^* = a\sum_{i=1}^\infty \epsilon_i \lambda_i e^*_{g(i)}$ we see that $b^* \in B_{X^*}$ and $z^{**}(b^*) = 2$. Therefore z^{**} attains its norm at the element $b^* \in B_{X^*}$ and $|||z^{**}||| = 2$. By a classical result [1], since X^* is separable, z^{**} attains its norm at an extreme point of B_{X^*} too. The latter set of points, in view of Milman's theorem, is contained in $\overline{B}^{w^*}$. It is easy to check that z^{**} does not attain its norm at a finitely supported (with respect to (e_i^*)) element of $\overline{B}^{w^*}$. Among the infinitely supported members of $\overline{B}^{w^*}$, it is clear that only b^* satisfies $z^{**}(b^*) = 2$, hence b^* is an extreme point of B_{X^*}. ∎

CLAIM 4. The set $\operatorname{ext} U^*$ has property (A).

Proof. Denote $E = \left\{a\sum_{i=1}^\infty \lambda_i e^*_{g(i)} : g \in \mathcal{G}_\infty\right\}$. By Claim 3, $E \subseteq \operatorname{ext} U^*$. So it is enough to prove that E has property (A). Our proof relies on the following easily verified fact.

FACT 1. For each two elements $u, v \in E$, if $u = a\sum_{i=1}^\infty \lambda_i e^*_{g_u(i)}$, $v = a\sum_{i=1}^\infty \lambda_i e^*_{g_v(i)}$ and $g_u(j) \neq g_v(j)$ then $\|u - v\| > \frac{1}{2^j}$.

Assume to the contrary that

$$E \subseteq \bigcup_{i=1}^\infty B_{X^*}(x_i, \epsilon_i), \quad \epsilon_i \to 0.$$

Since $B_{X^*} \subseteq 2B_{\ell_1}$ it follows that

$$E \subseteq \bigcup_{i=1}^\infty B_{\ell_1}(x_i, 2\epsilon_i).$$

Obviously, we can suppose that each $B_{\ell_1}(x_i, 2\epsilon_i)$ intersects E. For each i choose a representative $y_i \in B_{\ell_1}(x_i, 2\epsilon_i) \cap E$.

Choose m_0 sufficiently large so that for $m > m_0$ it holds that $2\epsilon_m < \frac{1}{4}$. Choose n_0 sufficiently large so that if $y \in E$ and $g_y(1) > n_0$ then

$$\max\{4\epsilon_1, \ldots, 4\epsilon_{m_0}\} < \|y - y_j\|,$$

for each $j \leq m_0$ (this is possible since $4\epsilon_i < 4$ and $E \subseteq 2S_{\ell_1}$). Denote by G_0 the set $\{1, 2, \ldots, n_0\}$. Choose $m_1 > m_0$ sufficiently large such that if $m > m_1$

then $2\epsilon_m < \frac{1}{8}$. Denote by G_1 the set $\{g_{y_{m_0+1}}(1), \dots, g_{y_{m_1}}(1)\}$. By Fact 1 if $x \in E$ and $g_x(1) \notin G_1$ then $\|x - y_j\| > \frac{1}{2}$ for $m_0 < j \le m_1$. Hence, $x \notin \cup_{i=m_0+1}^{m_1} B_{\ell_1}(x_i, 2\epsilon_i)$. Next we define inductively m_n and G_n such that

1) For every $m > m_n$, $2\epsilon_m < \frac{1}{2^{n+2}}$.
2) G_n is finite.
3) If $g_x(n) \notin G_n$ then $x \notin \cup_{i=m_{n-1}+1}^{m_n} B_{\ell_1}(x_i, 2\epsilon_i)$.

Choose m_{n+1} so that for $m > m_{n+1}$ it holds that $2\epsilon_m < \frac{1}{2^{n+3}}$. Denote by G_{n+1} the set $\{g_{y_{m_n+1}}(n+1), \dots, g_{y_{m_{n+1}}}(n+1)\}$. For every $x \in E$ and $m_n < j \le m_{n+1}$ if $g_x(n+1) \notin G_{n+1}$ then by Fact 1 $\|x - y_j\| > \frac{1}{2^{n+1}} > 4\epsilon_j$ and $x \notin \cup_{m_n+1}^{m_{n+1}} B_{\ell_1}(x_i, 2\epsilon_i)$. Define $b_1 = \max(G_0 \cup G_1) + 1$ and b_n to be $\max\big(\cup_{i=0}^{n} G_n \cup \{b_1, \dots, b_{n-1}\}\big) + 1$. Next define $g \in \mathcal{G}_\infty$ to be $g(n) = b_n$, $n = 1, 2, \dots$, and $x = \sum_{i=1}^{\infty} \lambda_i e^*_{g(i)}$. From our construction follows that $x \notin \bigcup_{i=1}^{\infty} B_{\ell_1}(x_i, 2\epsilon_i)$, a contradiction. ∎

The proof of Theorem 2 is complete. ∎

Proof of Theorem 1. By [3] Y contains c_0 (actually Y is c_0-saturated). Since Y is separable it follows [9] that c_0 is complemented in Y. Hence Y is isomorphic to the direct sum of Y_1 and c_0, where Y_1 is isometric to some subspace of Y and hence polyhedral. By Theorem 2, c_0 is isomorphic to a polyhedral Banach space X with the set $\operatorname{ext} B_{X^*}$ having property (A). Put $Z = (Y_1 \oplus_\infty X)$. Clearly, Z is polyhedral and $Y \cong Z$. Since $\operatorname{ext} B_{Z^*} = \operatorname{ext} B_{Y_1^*} \cup \operatorname{ext} B_{X^*}$ it follows that the set $\operatorname{ext} B_{Z^*}$ has property (A). The proof is complete. ∎

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