The Survival of Social Security and Immigration
: A Political Economy Model
(Preliminary Draft, November 2006)

Edith Sand and Assaf Razin

December 6, 2006

Abstract
In the political debate people express the idea that immigrants are good because they can help pay for the old. The paper explores this idea in a dynamic political-economy setup. For this purpose we develop an OLG political economy model of social security and migration to explore how migration policy and a pay-as-you-go (PAYG) social security system are jointly determined. In the paper we characterize sub-game perfect Markov equilibria for different patterns of fertility rates among native born and migrants. We flash out the political economy mechanism whereby migration strengthens the social security system in the presence of ageing. In our analysis, migration policy is endogenously determined by its conventional negative effect of migration on wages, but also by strategic considerations. The latter imply that immigrants are desirable for the sustainability of the social security system because the political system is able to manipulate the ratio of old to young and thereby the coalition for future high social security benefits.

1 Introduction
All over the world, declining population growth rates and rising life expectancy have strong impact on the social security system, as we know it. Due to decreased fertility rates and longer life expectancy, the EU population, in particular, is undergoing a long term trend of ageing, leading to a likely fall in the working population in the 25 states from 303 million to 297 million by 2020. A smaller labour force means less economic growth and faltering social security
system. In this context, migration is regarded by many as one of the necessary factors for the sustainability of the this system. The analysis of the of the inter-generational and intra-generational aspects of the sustainability of social security has had a revival of sorts in recent time. The analytical underpinnings of a political-economy equilibrium model in which migration and taxes interact, are yet to be worked out.\footnote{Immigration does not provide in itself a full-fledged long-term solution to falling birth rates and an ageing population, but it is one of the available tools within a broader policy mix.}

Standard theory of the determinants of the size of the government in a direct democracy highlight the relationship between the scope of redistribution, i.e. the extent of the welfare state, and pre-tax income inequality. Two interpretations have been suggested to explain this dependence: Lovell (1975) emphasizes the size of the government as a provider of public goods, while others such as Meltzer and Richard (1981) emphasize the role of the government in redistributing income. See Persson and Tabellini (1999) for a survey. As Razin, Sadka and Swagel (2002a; 2002b) have recently pointed out, a third potential channel is the leakage of welfare benefits to groups such as immigrants or the elderly.

Razin, Sadka and Swagel (2001, 2002) develop an OLG model where the extent of taxation and redistribution policy is generally determined as a political-economy equilibrium by a balance between those who gain from higher taxes/transfers and those who lose. In a stylized model of migration and human capital formation, they show — somewhat against the conventional wisdom — that low-skill immigration may lead to a lower tax burden and less redistribution than would be the case with no immigration, even though migrants (naturally) join the pro-tax/transfer coalition. The model captures two conflicting effects of migration on taxation and redistribution. On the one hand, migrants who are net beneficiaries of the welfare state will join forces with the low income native-born voters in favor of higher taxes and transfers. On the other hand, redistribution becomes more costly to the native-born as the migrants share the redistribution benefits with them. But the governments transfers, in this model, accrued uniformly to young and old. Thus the intra-generational transfer is the trigger for the so-called fiscal leakage from the median voter to the net beneficiaries of the welfare system, but not the inter-generational transfer as in a pure social security model.

Instead of considering migration as determined at the source, and workers
entering the "open doors to heaven" (Borjas 1999), the question who is allowed into a country depends on active immigration policy on part of the receiving countries. They more often than not enact quotas, point systems, and the like, in order to select those immigrants whom they deem most desirable. This view presupposes that the country under consideration is attractive for potential immigrants. In the US immigration debate the vast majority of citizens favor much tougher immigration rules. This could be due to congestion in public goods and/or dislike of foreigners (mostly Mexicans). However, immigration survives unchecked because henceforth it has not been a salient issue in elections and there is a large business lobby keeping it in place.2

Pay-as-you-go pensions redistribute income not only between generations but also between skill groups. Cooley and Soares [1999] and Boldrin and Rustichini [2] study the interaction between capital accumulation and social security in general equilibrium models of a closed economy with constant population growth rates where levels of social security payments are determined periodically by majority voting. Cooley and Soares [1999] constructed a useful general equilibrium model in which a pay-as-you-go social security system can be adopted and sustained as a political and economic equilibrium. Prescott and Rios-Rull (2000) argue that a necessary feature for equilibrium is that beliefs about the behavior of other agents are rational. We argue that in stationary OLG environments this implies that any future generation in the same situation as the initial generation must do as well as the initial generation did in that situation. They conclude that the existing equilibrium concepts in the literature do not satisfy this condition. We then propose an alternative equilibrium concept, organizational equilibrium, that satisfies this condition. They

\[ Razin and Sadka (1999, 2004) make an argument that highlights the importance of migration in enlarging the labor force in OLG models with pay-as-you-go fiscal system, for current and future generations of native born. They consider an overlapping-generations model, where each generation lives for two periods. In each period a new generation with a continuum of individuals is born. Each individual possesses a one unit of labor-schooling time endowment in the first period, when young. There is a pay-as-you-go (PAYG) pension system, which employs payroll taxes (at a flat rate) on the working young in order to finance a uniform benefit (b) to the aged. In an infinite-horizon, overlapping generations economy, this net burden is perfectly consistent with a net gain to the native born population. The additional obligation of the fiscal system to pay pension benefits to the incoming migrants, when they retire, could be shifted forward, in effect, indefinitely. If, hypothetically, the world would come to a stop at a certain point of time in the future, the young generation at that point would bear the deferred cost of the present migration. But in an ever-lasting economy, the migrants, by supplying work and helping the financing the pension benefit of period zero to native born retirees, are a boon to the host country population: old, young, and future generations. \]

3
show that equilibrium exists, it is unique, and it improves over autarky without achieving optimality. Moreover, the equilibrium can be readily found by solving a maximization program. Michele Boldrin, and Aldo Rustichini (2000) model PAYG social security systems as the outcome of majority voting within a OLG model with production. When voting, individuals make two choices: pay the elderly their pensions or default, which amount to promise themselves next period. Under general circumstances, there exist equilibria where pensions are voted into existence and maintained. Our analysis uncovers two reason for this. The traditional one relies on intergenerational trade and occurs at inefficient equilibria. A second reason relies on the monopoly power of the median voter. It occurs when a reduction in current savings induces a large enough increase in future return on capital to compensate for the negative effect of the tax. On the sustainability of pay-as-you-go social security. [see also Hassler, Mora, Storesletten and Zilliboti (2003), and Bergstrom and Hartman (2005).] Yet, a fundamental question awaiting to be answered is how migration policy cum social security are determined a political-equilibrium setup, even in the absence of any intra-generational distributions motives.

The main purpose of the present paper is to highlight the intergenerational-transfer mechanism through which migration, determined endogenously, can work to strengthen the social security system, in a political economy setup. To focus on the pure intergenerational aspects of the social security sustainability issue we completely abstract from intra-generational income transfers. There is a pay-as-you-go (PAYG) social security system, which employs payroll taxes (at a flat rate) on a representative working young in order to finance a uniform benefit (b) to the aged population. We characterize subgame-perfect Markov equilibria for different patterns of population growth among the native born and the immigrants populations. The migration policy is endogenously determined by the conventional negative effect of migration on wages, savings, etc. But it is also driven by strategic considerations concerning the effect of the current tax-migration policy on the next period tax-migration policy. The current young would like to influence the old-young composition of next period voters through the current migration policy and taxes.

The paper is organized as follows. Section 2 provides analysis of a baseline model, where factor prices are exogenous and the the economy does not accumulate capital inputs. Section 3 extends the baseline model to include capital accumulation and endogenous factor prices. Section 4 considers the effect of ageing in the extended model. Section 5 concludes.
2 A Base Line Model

The economy is populated by overlapping generations of identical individuals. Individuals live for two periods. When young, the representative individual works and makes labor-leisure and consumption-savings decisions. When old, the individual retires, and receives social security benefits, and income from private savings. The tax-transfer system is "pay as you go" where in every period the government levies a flat tax on the young’s wage income, which fully finances the social security benefits paid to the old. Immigrants enter the economy when young, and gain the right to vote only in the next period, when old. They have the same preferences as those of the native born, except from having a higher population growth rate. Immigrants are fully integrated into the social security system upon arrival into the country. Offsprings of immigrants are like native born in all respects (in particular, they have the same rate of population growth).

We assume that the utility of the representative young individual is logarithmic \(3\), given by:

\[
U^y(w_t, \tau_t, b_{t+1}) = \log[w_t l_t (1 - \tau_t) - \Psi^{t+1} + \beta \log[b_{t+1}]]
\]

\[
U^o(b_t) = b_t
\]

where \(U^y\) and \(U^o\) are the utility functions of young and old individuals, \(\beta \in [0,1]\) is the discount factor, and \(\Psi > 0\) a disutility parameter (also equals to the labor supply elasticity with respect to the wage rate). The transfer payments to the old at period \(t\), \(b_t\), are financed by collecting a flat income tax rate, \(\tau_t \in [0,1]\), from the young individual’s wage income at the same period, \(w_t l_t\), where \(l_t\) denotes the hours worked.

Labor is a single input in the production of a homogenous final good. The production function is linear:

\[
Y_t = N_t
\]

where \(Y_t\) and \(N_t\) are period \(t\) output and labor supply, respectively. Competitive equilibrium wage rate, equals to the marginal productivity of labor, is constant and normalized to unity. A worker can be either native born or immigrant, perfectly substitutable, and with equal productivities. The migration

\[^3\text{Note that this type of utility function implies that there are no income effects on the demand for leisure (Greenwood, Hercowitz, and Huffman (1988)).}\]
quota is expressed as a certain percentage of the number of young individuals in the native born population, $\gamma \in [0, 1]$. Labor supply is given by:

$$N_t = L_t l_t (1 + \gamma_t)$$

(4)

where $L_t$ is the number of young individuals in the native born population (old people do not work).

Immigrants have the same preferences as the native born population, but different fertility rates. We assume that the native born population has a lower population growth rate, $n \in [-1, 1]$, than that of the immigrant population, $m \in [-1, 1]$, so that, $n < m$. We also assume that the immigrant’s descendants are completely integrated into the economy and therefore have the same population growth rate as the native born population does. The number of young native born individuals at period $t$ can be written as follows,

$$L_t = L_{t-1} (1 + n) + \gamma_{t-1} L_{t-1} (1 + m)$$

(5)

In addition, immigrants are also assumed to contribute to, or benefit from, the welfare state in the same way as native born. Because the social security system redistributes income from the young to the old, the balanced government budget constraint implies:

$$b_{t+1} L_t = \tau_{t+1} w_{t+1} L_{t+1}$$

(6)

Re-arranging the expression yields:

$$b_{t+1} = \frac{\tau_{t+1} w_{t+1} L_{t+1} [(1 + n) + \gamma_t (1 + m)] (1 + \gamma_{t+1})}{(1 + \gamma_t)}$$

(7)

Labor-leisure decisions of young individuals are derived, as usual, from utility maximization, taking the prices and policy choices as given:

$$l_t^\Psi = w_t (1 - \tau_t)$$

(8)

Substituting for $b_t$, $b_{t+1}$ and $l_t$ in equations (7) and (8) into equation (1), the indirect utility functions of the young individual can be written as:

$$V^\Psi(w_t, \tau_t, \tau_{t+1}, w_{t+1}) = \log \left[ \frac{\Psi w_t l_t (1 - \tau_t)}{\Psi + 1} \right] +$$

$$\beta \log \left[ \frac{\tau_{t+1} w_{t+1} l_{t+1} [(1 + n) + \gamma_t (1 + m)] (1 + \gamma_{t+1})}{(1 + \gamma_t)} \right]$$

(9)

\footnote{A ceiling for $\gamma$ is set equal to one, which means that the number of immigrants cannot surpass the number of native born.}
such that,

\begin{align*}
l_t^\Psi &= w_t(1 - \tau_t) \\
l_{t+1}^\Psi &= w_{t+1}(1 - \tau_{t+1})
\end{align*}

Substituting for \( b_t \) in equations (7) into equation (2), yields the the indirect utility functions of the old individual:

\[ V^o(b_t) = \frac{\tau_t w_t l_t [(1 + n) + \gamma_{t-1} (1 + m)] (1 + \gamma_t)}{(1 + \gamma_{t-1})} \]  

Note that the old individual prefers that the migration quota will be as large as possible, because the larger the current immigration is, the larger is the total amount of tax collected, and thus the social security benefits she receives. The old preferable tax rate is the "Laffer point" tax rate, where the tax revenues are maximized. The tax rate at that point is equal to \( \frac{\Psi}{\Psi + 1} \).

The young individual prefers naturally that the current tax rate is as low as possible, regarding migration quota. Other preferences are ambiguous. Remember, wage rate is not influenced by migration in the base line model because the marginal productivity of labor is an exogenously given constant. But, a larger quota increases next period social security benefits per old individual because migrants have population growth rate which is higher than that of the native born. Transfer payments. On the other hand larger migration quota typically influence the identity (old or young) of next period decisive voter. This strategic consideration results from the fact that a higher current migration quota not only increases the number of current young which will be the old voters in the next period but also increases the number of young voters (some of these are offspring of the current migrants) in the next period. Since the immigrants fertility rate is higher than that of the native born, \( m > n \), a higher migration quota will increase the number of young voters by more than the number of old voters in the next period. Thus the current young voter tend to favor the most liberal migration policy. When this consideration dominates, it may lead to an equilibrium where the decisive voter identity alternate over time from young to old, to young, to..., etc.

### 2.0.1 A Political-Economic Equilibrium

We employ a subgame-perfect Markov equilibrium of perfect foresight, as our equilibrium concept (see Krusell and Rios-Rull (1996)):
Definition 1 A subgame-perfect Markov equilibrium is defined as a vector of policy decision rules, $\Psi = (T, G)$, where $T : [0, 1] \rightarrow [0, 1]$, is the taxation policy rule, $T(\gamma_{t-1})$, and $G : [0, 1] \rightarrow [0, 1]$, is the immigration quota policy rule, $G(\gamma_{t-1})$, such that the following functional equation holds:

1. $\Psi(\gamma_{t-1}) = \arg \max_{\pi} V^i(\gamma_{t-1}, \pi_t, \pi_{t+1})$ subject to $\pi_{t+1} = \Psi(\gamma_t)$, where $\pi_t = (\tau_t, \gamma_t)$ is defined as the vector of policy platform, and $V^i$ is the indirect utility of the current decisive voter.

2. The fixed-point condition requires that if next period policy outcome is derived by the vector of policy decision rules- $\Psi$, the maximization of the indirect utility of the current decisive voter will reproduce the same law of motion, $\Psi(\gamma_{t-1}) = \Psi(\gamma_t)$.

The Markov perfect political equilibrium notion states that the expected policy function, which depends on the minimal present state variables, must be self-fulfilling. The policy variables which are the tax rate, $\tau_t$, and the openness rate, $\gamma_t$, have to maximize the decisive voter’s indirect utility function, while taking into account that next period decision rules depend on the state variable i.e. the current openness rate of the economy towards immigration. A subgame-perfect Markov equilibrium has a "switching" strategy, in the sense that under the assumption that immigrants enter the country while young and gain the right to vote only in the next period when they are old, voters take into account the effect of admitting a certain number of immigrants on the composition of voters and their voting preferences in the next period. Moreover, when the number of young exceed the number of old in the population, the young decisive voter admits a limited number of immigrants, in order to change the decisive voter’s identity from young to old in the next period.

The equilibrium path depends on the native born and immigrant’s fertility rates. If the fertility rates of the native born and immigrants are both positive, there is a steady state with no taxation/social security benefits. If alternatively, the sum of the fertility rates is negative, there is also a steady state, but with a certain level of taxation/social security benefits (the "Laffer point" tax rate) with maximal migration quota. Otherwise, the sum of the fertility rates can be positive and the native born population’s fertility rate negative. In this case, some degree of openness to immigration always prevails while there is an alternate period by period taxation/social security policy, depending on the identity of the decisive voter. In a given period there is a certain amount of taxation/social security benefits (the "Laffer point" tax rate) with maximum migration quota, while in the next period there is no social security and a more
restrictive policy towards immigration.

As already mentioned, the state variable does not only influence the decisive voter’s indirect utility but also the distribution of next period voters. Since immigrants gain the right to vote only in the second period of their life in the host economy, the next period ratio of old to young voters in the native born population, denoted by $u_{t+1}$, is given by:

$$u_{t+1} = \frac{(1 + \gamma_t)}{(1 + n) + \gamma_t(1 + m)} \quad (13)$$

Assuming that in case of a tie the old will be the decisive, the condition, $u_{t+1} < 1$, assures a majority of young individuals in the next period, while the condition, $u_{t+1} \geq 1$, assures a majority of old individuals. Therefore, the state variable of the economy, affects the next period ratio of young to old voters, $u_{t+1}$, which sets the profile of the next period decisive voter.

The Markov Perfect political equilibrium of the baseline model and its possible equilibrium paths, which depend on the fertility rates of the native born and immigrant populations, can be formalized as follows:

**Proposition 2** There exist an equilibrium with the following feature:

$$T(\gamma_{t-1}) = \begin{cases} 
\tau_t = 0 & \text{if } u_t(\gamma_{t-1}) < 1 \\
\tau_t = \frac{\Psi_t}{\Psi_{t+1}} & \text{otherwise}
\end{cases} \quad (14)$$

$$G(\gamma_{t-1}) = \begin{cases} 
\gamma_t = -\frac{n}{m} & \text{if } u_t(\gamma_{t-1}) < 1 \\
\gamma_t = 1 & \text{otherwise}
\end{cases} \quad (15)$$

where $\gamma_t$ is restricted to be between zero and one. Under the assumption that the native born population growth rate is lower than that of the immigrant’s, there are three possible equilibrium paths, depending on the population growth rates of the native born and immigrant population, as follows: 1. if $n > 0$, there is no taxation/social security benefits; 2. if $m + n < 0$, there is full openness to immigration and there is a positive level of taxation/social security benefits (the "Laffer point" tax rate). 3. if $n < 0$ and $m + n > 0$, there is an alternate taxation/social security policy where some positive level of immigration always prevails; in periods where the current decisive voter is old, there is full openness to immigration and there is also a positive level of taxation/social security benefits (the "Laffer point" tax rate), whereas in periods where the current decisive voter is young, there is no taxation/social security benefits and a more restrictive policy towards immigration.
The proposition is proved in the appendix. The interpretation of the proposition is as follows.

The young decisive has two conflicting considerations regarding the desired immigration restrictions. One one hand, if there is full openness towards immigration, there will be more young working people in period $t+1$, and therefore, the tax revenue that which will be collected from a larger work force, needed to support retirement benefits, will increase. Therefore, the young decisive voter in period $t$, who will be old in period $t+1$, would benefit from the more generous social security benefits. On the other hand, if the immigration policy is excessively large, the decisive voter in period $t+1$ will be a young voter, who will want the tax rate reduced to zero; hence no social security benefits whatsoever. The threshold level of immigration restrictions, $\gamma_t = -n/m$, is exactly the level of the immigration policy that would equate the number of old and the number of young in period $t+1$. Thus, by choosing the immigration restrictions at that level, the decisive voter in period $t$ would finely balance the two conflicting forces on her period $t+1$ social security benefits, so as to maximize these benefits. This young voter’s preferable openness rate is therefore chosen strategically, in order to change the identity of the decisive voter in the next period from young voter to old.

If the current old-young ratio is smaller one, the decisive voter in the next period is an old voter. This voter will naturally vote for the most liberal immigration policy possible because only the current social security benefits matter to her. In this case the immigration quota is equal to one. The maximum tax revenue is achieved at the "Laffer point", where the tax rate at that point is equal to $\Psi/\Psi_{t+1}$.

There are three possible equilibrium paths depending on the population growth rates of the native born and immigrant populations. The first equilibrium path is the one where the population growth rates of the native born and immigrant populations are positive, $n, m > 0$. In this case, there is a corner solution in the political economy game. There is no taxation/social security benefits. This is due to the fact that for any positive values for any immigration restrictions the number of next period young voters exceeds the number of next period old voters. Because the decisive voter in the current and next periods is young, and her preferences are for zero labor tax, no social security benefits will be paid to the old. The young is indifferent to immigration because it does not influences her current income, or the next period decisive voter’s identity. As a result, the equilibrium path is one where in every period there is a majority of young voters, who therefore destroys the social security system.
for ever. The second and the third equilibrium paths are characterized by a "switching" strategy of the current and next period young voters. The current decisive voter is using the immigration restrictions as a means to affect the identity of the next period decisive voter through the mix of old and young voters in the next period. In the second equilibrium type, where the sum of the native born and immigrant population growth rates is negative, \( m + n < 0 \), there is a majority of old who set the migration quota at its maximum level. Therefore, the number of next period old voters exceeds the number of next period young voters. Thus, along the equilibrium path a majority of old will always prevail, which validates a permanent existence for the social security system and a the maximum flow of immigrants.

In the third equilibrium type, where the native born and immigrant populations population growth rates are \( n < 0 \), and \( m + n > 0 \), the equilibrium path is characterized by an alternate taxation/social security policy where some level of immigration always prevails. The reason is that when there is a majority of old their preferable openness rate is maximal and their tax rate is at the "Laffer point". But since the old decisive voter set a maximal rate of openness and the sum of the fertility rate are positive, the number of next period old voters does not exceed the number of next period young voters. When the decisive voter is young, he votes for a minimal tax rate, and votes strategically for a certain level of openness rate which changes the identity of the next period decisive voter to an old voter (there exist such an openness rate since the native born fertility rate is negative). This creates a cycling effect of an alternate taxation/social security policy, with a certain level of immigration, depending on the identity of the decisive voter.

3 The Extended Model: Capital Accumulation and Endogenous Factor Prices

In this section, the same basic assumptions regarding the model feature are presented. The aggregate savings of the young which generates next period aggregate capital are being used as a factor of production, in a constant return to scale production function. The new features are that young individual is able to save, and in the presence of capital input, the wage rate and the rate of interest are endogenously determined. Social security benefits are financed, as before, by a payroll tax in a Pay-As-You-Go system.

The utility of the representative young individual, as before, is logarithmic.
\[ U^y(w_t, \tau_t, s_t, r_{t+1}, b_{t+1}) = \log(w_t l_t (1-\tau_t) - s_t - \frac{r_{t+1}^\psi + 1}{\Psi + 1}) + \beta \log(b_{t+1} + (1+r_{t+1}) s_t) \] (16)

\[ U^o(b_t) = b_t + (1 + r_t) s_{t-1} \] (17)

where \( r_t \) is the interest rate, and \( s_t \) is the savings of the young at period \( t \).

The production function is a Cobb-Douglas production function which is assumed to use both labor and capital as its factors of production:

\[ Y_t = N_t^{1-a} K_t^\alpha \] (18)

where \( K_t \) is the aggregate amount of capital and \( N_t \) is defined as in the previous section. The wage rate and interest rate are determined by the marginal productivity conditions (capital is assumed to depreciate completely at the end of the period):

\[ w_t = (1-a)(1+\gamma_t)^{-1} l_t^{1-a} k_t^\alpha \] (19)

\[ r_t = \alpha(1+\gamma_t)^{-1} l_t^{1-a} k_t^\alpha - 1 \] (20)

where \( k_t \) is capital per native-born work force. The balanced government budget constraint is derived as in the previous section:

\[ b_{t+1} = \tau_{t+1} w_{t+1} l_{t+1} [(1+n) + \gamma_t (1+m)] (1+\gamma_{t+1}) \] (21)

The saving-consumption decision of young individuals are made by maximizing their utility while taking the prices and policy choices as given, and the labor-leisure decision is given as in the previous section:

\[ s_t = \frac{1}{1+\beta} \left( \frac{\beta \Psi}{\Psi + 1} w_t l_t (1-\tau_t) - \frac{b_{t+1}}{1 + r_{t+1}} \right) \] (22)

\[ l_t^\psi = w_t (1-\tau_t) \] (23)

The market clearing condition requires that the net domestic saving generates net domestic investment:

\[ s_t = k_{t+1} \left( \frac{1+n + \gamma_t (1+m)}{(1+\gamma_t)} \right) \] (24)
Solving for $b_{t+1}$ from equations (21) and (22), and substituting $b_{t+1}$ in equations (14), the utility function of the young can be written as follows:

$$V^y(w_t, \tau_t, r_{t+1}, \tau_{t+1}) = \log \left( \frac{1}{1 + \Psi_{t+1}} w_t l_t (1 - \tau_t) (1 + \beta f(\tau_{t+1})) \right)$$

$$+ \beta \log \left( \frac{1}{1 + \Psi_{t+1}} w_t l_t (1 - \tau_t) (1 + \beta f(\tau_{t+1})) (1 + r_{t+1}) \right)$$

(25)

where $f(\tau_{t+1}) = \frac{1 - \alpha}{1 + \beta \tau_{t+1}}$, such that,

$$k_{t+1} = \frac{\beta}{1 + \beta \Psi_{t+1}} \left( 1 + \gamma_t l_t (1 - \tau_t) (1 - \gamma_t) (1 + m) \right)$$

(26)

$$l^\Psi_t = w_t (1 - \tau_t)$$

(27)

$$l^\Psi_{t+1} = w_{t+1} (1 - \tau_{t+1})$$

(28)

and substituting $b_t$ from equation (21) and $k_t$ from equation (24), in equations (17), the utility function of the old can be written as follows:

$$V^o(\gamma_{t-1}, k_t, w_t, r_t, \tau_t) = \tau_t w_t l_t (1 + \gamma_{t-1}) (1 + \gamma_t) (1 + m) +$$

$$\left( 1 + r_t \right) k_t \left( \frac{1 + \gamma_{t-1} (1 + m)}{1 + \gamma_t} \right)$$

(29)

As in the previous analysis, the old individual favors a positive level of tax rate at a "Laffer Point" ($\tau^* = \frac{\Psi}{\Psi_{t+1}}$), and a maximal openness rate of the economy towards immigration.

With endogenous factor prices and capital accumulation we now have typically an interior solution for the desired migration quota. We denote the desired quota by $\gamma^*$.  

### 3.0.2 Political-Economics Equilibrium

The Markov sub-game Perfect equilibrium definition for the extended model is as follows:

**Definition 3** A Markov perfect political equilibrium is defined as a vector of policy decision rules, $\Psi = (T, G)$, and private decision rule, $S$, where $T : [0, 1] \rightarrow [0, 1]$, is the taxation policy rule, $\tau_t = T(\gamma_{t-1}, k_t)$, and $G : [0, 1] \rightarrow [0, 1]$, is the immigration quota policy rule, $\gamma_t = G(\gamma_{t-1}, k_t)$, and $S : [0, \infty) \rightarrow [0, \infty)$, is the saving decision rule, $k_{t+1} = S(\pi_t, k_t)$, such that the following functional equation holds:

1. $\hat{\Psi}(\gamma_{t-1}, k_t) = \arg \max_{\pi_t} V^i(\gamma_{t-1}, \pi_t, \pi_{t+1})$ subject to $\pi_{t+1} = \Psi(\gamma_t, S(\pi_t, k_t))$. 

The explicit value of $\gamma^*$ is given in the appendix.

14
2. \( S(\pi_t, k_t) = \frac{\beta \Psi}{1 + \Psi} \frac{(1+\gamma_t)w_t(1-f(\tau_{t+1}))}{1+\Psi(1+n)} \), with \( \tau_{t+1} = T(\gamma_t, S(\pi_t, k_t)) \).

3. The fixed-point condition requires that if next period policy outcome is derived by the vector of policy decision rules- \( \Psi \), the maximization of the indirect utility of the current decisive voter subject to the law of motion of capital, will reproduce the same law of motion, \( \Psi(\gamma_{t-1}, k_t) = \Psi(\gamma_{t-1}, k_t) \).

Similar to the definition of equilibrium in the base line model, policy variables have to maximize the decisive voter’s indirect utility function, while taking into account the law of motion of capital and the fact that next period decision rules depend on the state variables i.e. the current period openness rate and next period capital per native-born work force. Equilibrium paths depend in general on the native born and immigrant’s population growth rates (as in the baseline model) and on the initial stock of capital per native-born in the work force. There are three types of equilibria.

In the first type of Markov sub game Perfect equilibrium policy rules do not depend on the capital per native born work force state variable.\(^6\)

**Proposition 4** There exists an equilibrium with the following feature:

\[
T(\gamma_{t-1}) = \begin{cases} 
\tau_t = 0 & \text{if } u_t(\gamma_{t-1}) < 1 \\
\tau_t = \frac{\Phi}{1+\Psi} & \text{otherwise} 
\end{cases} \tag{30}
\]

\[
G(\gamma_{t-1}) = \begin{cases} 
\gamma_t = \text{Min}[\gamma^*, -\frac{n}{m}] & \text{if } u_t(\gamma_{t-1}) < 1 \\
\gamma_t = 1 & \text{otherwise} 
\end{cases} \tag{31}
\]

\[
S(\pi_t, k_t) = \begin{cases} 
S(\pi_t, k_t, \tau_{t+1} = 0) & \text{if } u_t(\gamma_{t-1}) < 1, \quad n < 0 \cap m > 0 \\
S(\pi_t, k_t, \tau_{t+1} = \frac{\Phi}{1+\Psi}) & \text{if } m > 0 \cap n < 0 \\
S(\pi_t, k_t, \tau_{t+1} = 0) & \text{otherwise } n > 0 
\end{cases} \tag{32}
\]

where \( \gamma_t \) is restricted to be between zero and one, and \( \gamma^* \) is given explicitly in the appendix. The equilibrium paths depend on the fertility rates and on the initial amount of capital per native-born work force the economy is endowed with. There are three main types of equilibrium paths which are similar to the previous section: 1. if \( n > 0 \), there is no taxation and social security benefits and some restrictions on immigration. 2. if \( m + n < 0 \), there is full openness to immigration and a positive level of taxation (the "Laffer point" tax rate). 3. if \( n < 0 \) and \( m + n > 0 \), there is an alternate taxation policy where some level...

\(^6\)In essence this equilibrium is a reduced form of the Markov sub game Perfect equilibrium of the base line model.
of immigration always prevails: in periods where the decisive voter is old, the economy is fully opened to immigration and there is a positive level of taxation (the "Laffer point" tax rate); and in periods where the decisive voter is young, there is no taxation and there are some restrictions on immigration.

The intuition is the same as in section 2. Because the decision rules in type one equilibrium do not depend on the capital per native-born work force, the extended model is essentially a reduced form of the Markov sub game Perfect equilibrium in the base line model.

The second and third types of Markov Perfect equilibrium of the extended model are specified as follows:

**Proposition 5** Under several conditions on the parameters of the model, which are specified in the appendix, there exist two other equilibria types, i = 1, 2, with the following features:

\[
T^i(\gamma_{t-1}, k_t) = \begin{cases} 
\tau_i(k_t) & \text{if } k_t \in [F(\tau_i), F(\tau_i)] \\
0 & \text{if } u_t(\gamma_{t-1}) < 1 \\
\frac{\Psi}{\lambda + \beta} & \text{otherwise}
\end{cases} \quad (33)
\]

\[
G^i(\gamma_{t-1}, k_t) = \begin{cases} 
1 & \text{if } k_t \in [F(\tau_i), F(\tau_i)] \\
\min[\gamma^*, -\frac{n}{m}] & \text{otherwise} \\
1 & \text{otherwise}
\end{cases} \quad (34)
\]

\[
S^i(\pi_t, k_t) = \begin{cases} 
S^i(\pi_t, k_t, \tau_{t+1} = \tau_i(k_{t+1})) & \text{if } k_t \in [F(\tau_i), F(\tau_i)] \\
S^i(\pi_t, k_t, \tau_{t+1} = \frac{\Psi}{\lambda + \beta}) & \text{otherwise} \\
S^i(\pi_t, k_t, \tau_{t+1} = g_i(F(\tau_i))) & \text{if } k_t \in [g_i(F(\tau_i)), g_i(F(\tau_i))] \\
S^i(\pi_t, k_t, \tau_{t+1} = 0) & \text{otherwise} \\
S^i(\pi_t, k_t, \tau_{t+1} = \frac{\Psi}{\lambda + \beta}) & \text{if } m + n < 0 \\
S^i(\pi_t, k_t, \tau_{t+1} = \tau_i(k_{t+1})) & \text{if } k_t \in [F(\tau_i), F(\tau_i)] \\
S^i(\pi_t, k_t, \tau_{t+1} = 0) & \text{otherwise}
\end{cases} \quad (35)
\]

where \( x = 1 + \frac{(1+\beta)\alpha}{\Psi + \alpha} \), \( \Psi = \frac{\beta(1+\beta) + \alpha}{\Psi(1+\beta) + \alpha + \beta} \) and \( g_i(F) \) and \( F(\tau) \) are functions given in the appendix. The two equilibrium, \( i = 1, 2 \), differ essentially in the implicit functions of the tax rate denoted by \( \tau_i(k_t) \), which are defined in the appendix. In the first type of equilibrium, \( \tau_1(k_t) \) is a decreasing function in \( k_t \), while in the second type \( \tau_2(k_t) \) is an increasing function in \( k_t \). The equilibrium
paths, depends on the fertility rates of the native born and immigrant populations and on the initial capital the economy is endowed with: 1. if $n > 0$ and $k_t \in [F(\tau), F(\tau_i)]$, there is a positive tax rate which depends on the capital per native-born work force state variable and full openness to immigration. If $k_t \notin [F(\tau), F(\tau_i)]$, there are at least few periods in which there is no taxation and some restriction on immigration. 2. if $m + n < 0$, there is fixed and positive tax rate (the "Laffer point" tax rate), and a full openness to immigration. 3. if $n < 0$ and $m + n > 0$, there is a range of $k_t$, for which there is a positive tax rate which depends on the capital per native-born work force state variable and full openness to immigration. If $k_t$ is not in this range, there are at least few periods in which there is an alternate taxation policy where some level of immigration always prevails: in periods where the decisive voter is old, the economy is fully opened to immigration and there is a positive level of taxation (the "Laffer point" tax rate); and in periods where the decisive voter is young, there is no taxation and there are some restrictions on immigration.

The proposition is proved in the appendix.

The tax rate is positive, and typically smaller than the "laffer point" rate. It depends on the amount of capital, per native-born work force. The migration quota is set at the ceiling. A non "switching" strategy is optimal due the more complicated interactions in the extended model. Current policy variables influence the amount of capital per native-born work force; which in turn influences the next period policy variables. This influence works not only through the current migration quota, but also indirectly through the adjustment in the amount of capital per native-born work force. In one of the (non switching) equilibrium, the tax rate is decreasing in the amount of capital per native-born work force. In the other "non switching" equilibrium, the tax rate is increasing in the amount of capital per native-born work force.

The difference in the two "non switching" equilibrium tax rate, as a function of capital per native-born work force, is due to conflicting forces of the effect of the next period tax rate on next period capital, per native-born work force. On one hand, a higher tax rate in the next period raises future social security benefits. Larger benefits, tend to reduce current savings. This would cause a reduction in next period capital per native-born work force ("Effect One"). On the other hand, a higher next period tax rate tend to decrease the amount of hours worked next period and lower the next period interest rate. The consequent fall in the current young future income induces more savings. This tends to increase next period capital per native-born work force ("Effect Two").
In the "non switching" equilibrium, where the tax rate is decreasing in the amount of capital per native-born work force, "Effect One" is stronger than "Effect Two". In the "non switching" equilibrium, where the tax rate is increasing in the amount of capital per native-born work force, "Effect One" is weaker than "Effect Two".

Equilibrium paths depend on the fertility rates and the amount of capital per native-born work force:

1. The fertility rates of the native born and immigrant populations are positive, \( n, m > 0 \). In this case, the number of next period young voters exceeds the number of next period old voters, which means that the decisive voter is always young. Therefore, if the capital per native-born work force is in the range \([F(\tau), F(\tau_i)]\), than the optimal strategy of the young is always to vote for a positive tax rate. The rate depends on the capital per native-born work, and on the exogenously given ceiling on migration quotas. If the initial capital is not in the range \([F(\tau), F(\tau_i)]\), zero tax rate and a positive migration quota, below the ceiling, are chosen by the young. Capital evolves in a way that it is possible to have a period where the amount of capital per native-born work force enters the range \([F(\tau), F(\tau_i)]\); if it does, from then on the current young again vote for a positive tax rate, and sets the migration quota at the exogenously given ceiling.

2. If the sum of the fertility rates is negative, then the number old voters always exceeds the number of young voters. This means that the decisive voter is always young. In that case the old sets the tax rate at the "Laffer point", and the migration quota at the ceiling.

3. If the sum of the fertility rate is positive, but the native born fertility rate is negative, there are two possible equilibrium paths. If the capital per native-born work force is in the range \([F(\tau), F(\tau_i)]\), the optimal strategy of the young is to set a tax rate that depends on the capital per native-born work force, and to set the migration quota at the ceiling level. If the capital per native-born work force is outside the range \([F(\tau), F(\tau_i)]\), tax policy alternate period by period, and migration quota is below the ceiling level but above zero. Capital evolves over time in a way that there could be a period where the capital per

\footnotesize
\begin{enumerate}
\item The equilibrium property of the tax function in the first equilibrium, is already noted by Forni (2005).
\item The relevant range for \( k_t \) is as follows: If the decisive voter is young, and \( k_t \in [F(\tau), F(\tau_i)] \), the optimal decision rules are \( \pi_t = (\tau_i(k_t), 1) \), from that period and on. If the decisive voter is old and \( k_t \in [g_i(F(\tau_i)), g_i(F(\tau_i))] \), than next period decisive voter is young and \( k_{t+1} \in [F(\tau), F(\tau_i)] \). Thus, in that case also, the optimal decision rules are \( \pi_t = (\tau(k), 1) \) from that period and on.
\end{enumerate}
native-born work force enters the range \([F(\tau), F(\tau_i)]\); Once in this range, the optimal strategy of the young is to set a tax rate that depends on the capital per native-born work force, and to set the migration quota at the ceiling level.

4 The Effect of Aging

We are now in position to conduct a comparative statics. We analyze the effect of aging of the native and the immigrant populations on the size of the social security system and on immigration restrictions. Aging of the population is specified by a reduction in the fertility rate of the population (life expectancy is assumed to be exogenously fixed). The effect of aging on the immigration policy results from the fact that aging changes the ratio of young to old individuals in the next period\(^9\). Aging of the native born decreases the number of young to old in the next period. This allows the old to lift the restrictions on immigration. By admitting more immigrants the ratio of young to old in the next period rises. Aging of the immigrants’ population leads to opposite results. A higher openness rate increases more the number of next period young to old since the immigrants fertility rate is higher. Thus, with a lower immigrants fertility rate leads to more tight migration restrictions.

Proposition 6 1. Aging of both populations can move the system to an equilibrium with a certain level of taxation/social security benefits (the "Laffer point" tax rate) and to the ceiling level of the migration quota.

2. The aging of the native born (immigrants) population increases (decreases) there will be some restrictions on immigration. The migration quota decreases (increases) in the stock of capital.

3. In the "non switching" equilibrium types, aging can move the system to an equilibrium path where the tax rate is positive and depends on the capital per native born work force, \(\tau_t = \tau(k_t)\), whereas the migration quota is at the ceiling level, \(\gamma_t = 1\).

The intuition of the result is as follows. Both the aging of native-born and immigrants populations, can move the system to an equilibrium where the sum of the fertility rates are negative, \(n + m < 0\). In this case, the old are in the

\(^9\)In the case where \(\gamma_t = -\frac{n}{m}\), the ratio affect the openness rate through the number of young to old voters next period, and in the case where \(\gamma_t = \gamma^*\), the ratio affect the openness rate through its capital per native born in the next period (more old individual in the next period, increases the total amount of capital, but more young in the next period, decreases the per capital per native born work force).
majority every period. The old set the tax rate at the "Laffer point", and liberalize migration as much as possible.

Aging of the native-born (immigrants) population increases (decreases) the migration quota when there is a majority of the young. They set some restrictions on immigration, \( \gamma_t = \min(\gamma^*, -\frac{\tau_t}{\gamma^*}) \). A higher (lower) migration quota increases (decreases) the number of workers, which by raising (lowering) the wage rate and the savings of the young, generate a higher (lower) amount of capital in the next period.

Aging affect the capital per native born work force, and thus can move the system from and to the range of capital per native born work force state variable:

\[ k_t \in [F(\tau_t), F(\tau_i)] \]

where the tax rate is positive and depends on the capital per native born work force, \( \tau_t = \tau(k_t) \), and the openness rate is maximal \( \gamma_t = 1 \).

5 Conclusion

We demonstrate a political economy mechanism whereby migration strengthens the social security system in the presence of aging. That is, the older are the native born and migrant populations the more likely is that the migration policy is liberalized and that the social security system will survive. But when there exists some restriction on migration, aging of the native born population decreases migration restrictions, while the aging of migrants increases them.

The framework of the model is an OLG model of identical individuals who live for two periods. The social security system is a pay-as-you-go, which employs payroll taxes on the working young in order to finance a social-security benefit to the aged. Immigrants enter the economy when young, and gain the right to vote only in the next period, when old. Except from having a higher population growth rate, they have the same preferences and contribute to and benefit from the welfare state in the same way as the native born. Their offspring are assumed to be completely integrated into the country and have the same population growth as the native born.

The model is a political economy model where the political decisions regarding labor taxation and immigration quotas are taken simultaneously, through majority voting. Markov sub-game perfect political equilibria of the game feature a dynamic of repeated voting where individuals have a forward looking property, in the sense that they take into account the effect of their current voting on the next period voting decisions.

There are three Markov sub-game perfect types of equilibria. The first equilibrium type, is characterized by a "switching" strategy, in the sense that under
the assumption that immigrants gain the right to vote only in the next period when they are old, voters take into account the effect of admitting a certain number of immigrants on the composition of voters and their voting preferences in the next period. Moreover, when the number of young exceed the number of old in the population, the young decisive voter admits a limited number of immigrants, in order to change the decisive voter’s identity from young to old in the next period.

The two other equilibria include, in addition to the migration policy considerations of the first equilibrium, also the effect of current policy variables on next period policy variables through their effect on savings. Since there exists another channel of influence on next policy variable (in addition to migration policy) through savings, the young decisive voter does not engage in a "switching" strategy, but admit the maximum amount of immigrant, and in so doing she renders a majority of young every period. These equilibria are characterized by a range of values for the state variables, for which the tax rate policy depends on the amount of capital per capita and the migration policy is liberalized.

An interesting extension is to introduce heterogeneity within the native born and the immigrants population in terms of labor productivity. This would introduce intra-generational distribution aspects to the current model.

6 Appendix

6.1 Proposition I:

Proof. We must show that the vector of policy decision rules, \( \Psi = (T, G) \), as defined in the proposition, satisfies the equilibrium conditions:

1. \( \hat{\Psi}(\gamma_{t-1}) = \arg \max_{\pi_t} V^i(\gamma_{t-1}, \pi_t, \pi_{t+1}) \) subject to \( \pi_{t+1} = \Psi(\gamma_t) \).
2. \( \hat{\Psi}(\gamma_{t-1}) = \Psi(\gamma_{t-1}) \).

If \( u_t \geq 1 \) than the decisive voter is old. Substituting for \( l_t \) from equation (8) into (12), the utility of the old can be rewritten as:

\[
V^o(\gamma_{t-1}) = \frac{\tau_t (1 - \tau_t) \Psi_{l_t}}{1 + \gamma_{t-1}} (1 + n) + \gamma_{t-1} (1 + m) (1 + \gamma_t)
\]

It is straightforward to show that \( V^o(\gamma_{t-1}) \) is maximized by setting \( \gamma_t = 1 \) and \( \tau_t = \frac{\Psi}{1 + \gamma_{t-1}} \).

If \( u_t < 1 \) than the decisive voter is young. From equation (9), the utility of
the young voter subject to $\pi_{t+1} = \Psi(\gamma_t)$ is given by:

$$V^y(\gamma_{t-1}) = \begin{cases} 
\log\left[\frac{\Psi}{\Psi_t^T}(1 - \tau_t)\right]^{\frac{\Psi}{\Psi_t}} & \text{if } u_{t+1} < 1 \\
\log\left[\frac{\Psi}{\Psi_t^T}(1 - \tau_t)\right]^{\frac{\Psi}{\Psi_t}} + \beta \log\left[\frac{\Psi}{\Psi_t^T}(1 - \tau_t)\right]^{\frac{\Psi}{\Psi_t}} & \text{otherwise}
\end{cases}
$$

In that case $V^y(\gamma_{t-1})$ is maximized by setting $\gamma_t = -\frac{\Psi}{\Psi_t}$ and $\tau_t = 0$. It should be noted that in the case where the fertility rates are positive, than for every openness rate there is a majority of young in every period, and thus the young decisive voter in every period will indifferent between all possible openness rates.

6.2 Proposition II:

Proof. As in the previous proposition, we must show that the vector of policy decision rules, $\Psi = (T, G)$, as defined in the proposition, satisfies the equilibrium conditions:

1. $\check{\Psi}(\gamma_{t-1}) = \arg \max_{\pi_t} V^t(\gamma_{t-1}, \pi_t, \pi_{t+1})$ subject to $\pi_{t+1} = P(\gamma_t)$.
2. $\check{\Psi}(\gamma_{t-1}) = \Psi(\gamma_{t-1})$.
3. $S(\pi_t, k_t) = \frac{\beta}{1 + \beta} \Psi(1 + \gamma_t)^{u_t} (1 - \tau_t) (1 - f(\tau_{t+1}))$, with $\tau_{t+1} = T(\gamma_t)$.

Consider first the case where there is a majority of old in period $t$, i.e. $u_t \geq 1$.

Using the fact that,

$$w_t l_t = \left(1 - \alpha\right) k_t^{\alpha} (1 + \gamma_t)^{-\alpha} \left(1 - \tau_t\right)^{\frac{1 - \alpha}{1 + \gamma_t}}$$

and

$$1 + \tau_t = \alpha \left(1 - \alpha\right) k_t^{\alpha} (1 + \gamma_t)^{\alpha} \left(1 - \tau_t\right)^{\frac{1 - \alpha}{1 + \gamma_t}}$$

the utility of the old voter can be rewritten as:

$$V^o(\gamma_{t-1}, k_t) = \frac{\tau_t (1 - \alpha)(1 + \gamma_t)^{-\alpha} k_t^{\alpha}}{(1 - \tau_t) \left(1 + \gamma_t\right)^{\alpha}} - \alpha \left(1 - \alpha\right) k_t^{\alpha} (1 + \gamma_t)^{\alpha} \left(1 - \tau_t\right)^{\frac{1 - \alpha}{1 + \gamma_t}}$$

It can be proved that $V^o(\gamma_{t-1}, k_t)$ is maximized by setting $\gamma_t = 1$ and $\tau_t = \Psi(\gamma_t)$.

Consider next the case where there is a majority of young in period $t$, i.e. $u_t < 1$. Substituting for $w_t l_t (1 - \tau_t)$ and $1 + \tau_{t+1}$ from equations (38) and (39), the utility of the young voter subject to $\pi_{t+1} = \Psi(\gamma_t)$, can be written in the Lagrangian form, in the

22
is higher by setting \( \gamma \) is maximized by \( \gamma \) setting \( L \) Lagrangian multiplier, and \( L(k_t) \) is defined as follows:

\[
L(k_t) = \begin{cases} 
L(k_t) \text{ with } \tau_{t+1} = 0, \text{ and } \gamma_{t+1} = \text{Min}[\gamma^*, \frac{m}{m}] & \text{if } u_{t+1} < 1 \\
L(k_t) \text{ with } \tau_{t+1} = \frac{\Psi}{\Psi + 1}, \text{ and } \gamma_{t+1} = 1 & \text{otherwise}
\end{cases}
\]

where \( A = (1 + \beta) \log \left( \frac{\beta}{\Psi + 1} (1 - \alpha) \right) + \beta \log \left( \alpha ((1 - \alpha))^{\frac{1 - \alpha}{\Psi + 1}} \right) \), \( \lambda_1 \) is the Lagrangian multiplier, and \( L(k_t) \) is defined as follows:

\[
L(k_t) = \begin{cases} 
A + (1 + \beta) \log \left( k_t^\alpha (1 + \gamma_t)^{-\alpha} (1 - \tau_t) \right)^{\frac{\beta}{\Psi + 1}} (1 + \beta f(\tau_{t+1})) & \\
+ \beta \log \left( k_t^\alpha (1 + \gamma_t)^{-\alpha} (1 - \tau_t) \right)^{\frac{\beta}{\Psi + 1}} (1 + \beta f(\tau_{t+1})) & \\
- \lambda_1 (k_{t+1}^{\alpha} - \beta \Psi (1 + \gamma_t)^{-\alpha} (1 - \tau_t) \right)^{\frac{\beta}{\Psi + 1}} \left( 1 + \beta f(\tau_{t+1}) \right)
\end{cases}
\]

and the indirect utility of the young subject to constant next period policy variables, is maximized by setting \( \gamma_t = \gamma^* \) and \( \tau_t = 0 \), where \( \gamma^* \in [0, 1] \) and is defined as follows:

\[
\gamma^* = \frac{\beta(1 - \alpha) \Psi (n - m) + \alpha(1 + \Psi)(1 + n)x}{-\alpha(1 + \Psi)(1 + n)x}
\]

We will prove that \( V^\gamma(\gamma_{t-1}, k_t) \) is maximized by \( \gamma_t = \text{Min}[\gamma^*, \frac{m}{m}] \). If \( \gamma^* \leq \frac{m}{m} \) than it is sufficient to prove that the indirect utility of the young is higher by setting \( \gamma_{t+1} = 1, \tau_{t+1} = 1 \) than by setting \( \gamma_{t+1} = \gamma^*, \tau_{t+1} = 0 \). It is easy to see that the higher is next period openness rate the higher is the indirect utility of the young as it increases next period interest rate. Regarding next period tax rate, it is sufficient to prove that:

\[
0 = \log[(1 + \beta f(0))^{1 + \beta}] \leq \log \left( 1 + \beta f \left( \frac{\Psi}{\Psi + 1} \right)^{1 + \beta} \right)^{\frac{\beta}{\Psi + 1}} (1 - \frac{\Psi}{\Psi + 1})^{\frac{\beta}{\Psi + 1}} \]

since,

\[
0 = \log[(1 - f(0))^{-\Psi \beta^{\frac{1 - \alpha}{\Psi + 1}}}] \leq \log \left( 1 - f \left( \frac{\Psi}{\Psi + 1} \right)^{-\Psi \beta^{\frac{1 - \alpha}{\Psi + 1}}} \right)^{\frac{\beta}{\Psi + 1}} \]

Define the function: \( D(\Psi) = \log \left( 1 + \beta f \left( \frac{\Psi}{\Psi + 1} \right)^{1 + \beta} \right)^{\frac{\beta}{\Psi + 1}} (1 - \frac{\Psi}{\Psi + 1})^{\frac{\beta}{\Psi + 1}} \). The

---

10If the fertility rates are both positive, than it is straightforward to see that \( V^\gamma(\gamma_{t-1}, k_t) \) is maximized by \( \gamma_t = \text{Max}[\gamma^*, \frac{m}{m}], \tau_t = 0 \).
function, under the assumption that next period there is a majority of young voters the policy decisions for Proof. The proof will consist of two parts. The

6.3 Proposition III:

Since the derivative of \( D(\Psi) \) is positive for every \( \Psi > 0 \), and \( D(\Psi = 0) = 0 \), than \( D(\Psi) \) is positive for every \( \Psi > 0 \).

Otherwise, if \( \gamma^* > -\frac{n}{m} \), we must prove that the following holds,

\[
Log \left( (1 + \gamma^*)^{-n} \right)^{\frac{1+\beta}{1+\gamma^*}} + Log \left( \frac{(1+\gamma^*)(1+\gamma^*)^{-n}}{1+n+\gamma^*(1+m)} \right)^{-\Psi} \leq 0
\]

As \( D(\Psi) \) is positive for every \( \Psi > 0 \), and for \( \gamma^* > -\frac{n}{m} \) the following holds,

\[
(1 + \beta)Log \left( \frac{(1 + \gamma^*)^{-n}}{1 - \frac{n}{m} (1 + m)} \right) + \beta Log \left( \frac{1 + \gamma^*}{1 - \frac{n}{m}} \right)^{-\Psi} \leq 0
\]

it is sufficient to prove that,

\[
\beta Log \left( 1 + n - \frac{n}{m} (1 + m) \right)^{-\Psi} \leq 0
\]

Substituting \( \gamma^* \) from equation (43) into equation (49), we can rewrite the inequality in the following way,

\[
\alpha(1 + \Psi)x \frac{1}{m} \left( 1 + \frac{1 - \alpha}{\alpha} + \frac{1}{\Psi + 1} \right) - \beta(1 - \alpha) \Psi \geq 0
\]

Since this expression is positive, it completes the proof that \( V^u(\gamma_{t-1}, k_t) \) is maximized by setting \( \gamma_t = Min[\gamma^*, -\frac{n}{m}] \) and \( \tau_t = 0 \).}

6.3 Proposition III:

Proof. The proof will consist of two parts. The first part will prove that when there is a majority of young voters the policy decisions for the tax rate and openness rate stated maximizes the young indirect utility function, under the assumption that next period
decisive voter is young. The second part will complete the proof and show that under certain conditions on the models parameters, the vector of policy decision rules as defined in the proposition, satisfies the two equilibria conditions.

The first part of the proof:
We follow the proof of Forni (2004) to obtain the policy variables. The policy variables are obtained by using as a constrain the first derivative with respect to the policy variables of the logarithm of the capital accumulation equation. The policy variables are the following:

\[
\left( 1 + \frac{1 - \alpha}{\alpha} \tau_t(k_t) \right)^{1+\beta} \left( 1 - \tau_t(k_t) \right)^{\frac{\beta(1-\alpha)}{\psi}} = k_t^{-x} c \tag{51}
\]

\[
\gamma_t = 1 \tag{52}
\]

where \( x = 1 + \frac{(1+\Psi)\alpha_\beta}{\Psi+\alpha}, \) and \( c \) is a positive constant of integration. The policy decision rule of the openness rate is at its maximal value, and the policy decision rule of the tax rate is implicitly given in equation (51). Define this implicit function by

\[
F(\tau) = \left( 1 + \frac{1 - \alpha}{\alpha} \tau_t(k_t) \right)^{1+\beta} \left( 1 - \tau_t(k_t) \right)^{\frac{\beta(1-\alpha)}{\psi}} .
\]

The function \( F(\tau) \) is decreasing in \( \tau \) for \( \tau \in [0, \tau] \), where \( \tau = \frac{\Psi(1+\beta)+\alpha}{\Psi(1+\beta)+\alpha+\gamma} \), and increasing in \( \tau \) for \( \tau \in [\tau, 1] \). Therefore for every value of capital per native born work force, \( k_t = F(\tau_t(k_t)) \), there are two solutions \( \tau_t(k_t) \) in the range \([0, 1]\), referred to by \( \tau_1(k_t) \) and \( \tau_2(k_t) \). Define by \( \tau_1(k_t) \), to be the smaller solution from those two.

The first solution, \( \tau_1(k_t) \), which is the policy taxation rule in the first equilibrium of the proposition \( i = 1 \), is decreasing in \( k_t \),

where \( k_t \in [F(\tau), F(0)] \). The second solution, \( \tau_2(k_t) \), which is the policy taxation rule in the second equilibrium of the proposition \( i = 2 \), is increasing in \( k_t \), where \( k_t \in [F(\tau), F(1)] \).

The solution for the policy variables given in equations (51) and (52), will be proved to satisfy the first order conditions of the problem.

Substituting for \( x_t(1 - \tau_t) \) and \( 1 + \gamma_t+1 \) from equations (38) and (39), the young voter’s indirect utility function under the assumption that next period decisive voter is young which set the next period policy decision rules for the tax rate and openness rate to be \( \tau_{t+1} = \tau_t+1(k_{t+1}) \) and \( \gamma_{t+1} = 1 \) respectively, can be written in its Lagrangian form as follows:
\[ L = A + (1 + \beta) \log (k_t^\alpha (1 + \gamma_t)^{-\gamma}(1 - \tau_t)^{-\tau}) \] 
\[ + \beta \log (k_t^\alpha (1 + \gamma_t)^{-\gamma}(1 - \tau_t)^{-\tau}) \] 
\[ - \lambda_1 (k_{t+1} - \frac{\beta}{\tau_t} (1 + \gamma_t)^{-\gamma}(1 - \tau_t)^{-\tau}) \] 
\[ - \lambda_2 (\tau_t - 1) - \lambda_3 (\gamma_t - 1) - \lambda_5 (\gamma_t) \] 

(53)

The Kuhn-Tucker conditions are:

\[ \frac{\partial L}{\partial \tau_t} = -1 + \frac{\psi}{\psi + \alpha} 1 - \tau_t - \lambda_1 \frac{1 + \psi}{\psi + \alpha} k_{t+1} - \lambda_2 = 0 \] 

(54)

\[ \frac{\partial L}{\partial \gamma_t} = -\alpha \frac{1 + \psi}{\psi + \alpha} 1 + \lambda_1 \frac{k_{t+1}}{1 + \gamma_t} \left( \frac{n - m}{1 + n + \gamma_t(1 + m)} - \alpha \frac{1 + \psi}{\psi + \alpha} \right) - \lambda_4 + \lambda_5 = 0 \] 

(55)

\[ \frac{\partial L}{\partial k_{t+1}} = \left( \frac{\beta(1 + \alpha)}{1 + \beta(1 + \gamma_t)w_t(1 - \tau_t)(1 - f(\tau_{t+1}(k_{t+1})) \right) \frac{\partial f(\tau_{t+1}(k_{t+1}))}{\partial \tau_{t+1}} \frac{\partial \tau_{t+1}(k_{t+1})}{\partial k_{t+1}} \] 

\[ - \beta(1 - \alpha) \frac{1}{\psi + \alpha} 1 - \tau_{t+1}(k_{t+1}) + \frac{1}{k_{t+1}} \left( -\beta \frac{\psi(1 - \alpha)}{\psi + \alpha} \right) - \lambda_5 \] 

(56)

\[ k_{t+1} = \frac{\beta}{1 + \alpha} \frac{(1 + \gamma_t)w_t(1 - \tau_t)(1 - f(\tau_{t+1}(k_{t+1}))}{1 + n + \gamma_t(1 + m)} \] 

(57)

\[ \tau_t - 1 \leq 0, \lambda_2 \geq 0 \text{ and } \lambda_2 (\tau_t - 1) = 0 \] 

(58)

\[ -\tau_t \leq 0, \lambda_3 \geq 0 \text{ and } \lambda_3 (\tau_t - 1) = 0 \] 

(59)

\[ \gamma_t - 1 \leq 0, \lambda_4 \geq 0 \text{ and } \lambda_4 (\gamma_t - 1) = 0 \] 

(60)

\[ -\gamma_t \leq 0, \lambda_5 \geq 0 \text{ and } \lambda_5 (\gamma_t) = 0 \] 

(61)

Substituting for \( \lambda_1 \) in equation (55) in equations (53) and (54), they can be rewritten as:

\[ \frac{\partial L}{\partial \tau_t} = -\lambda_2 + \lambda_3 = 0 \] 

(62)

\[ \frac{\partial L}{\partial \gamma_t} = \frac{(1 + \beta)}{1 + \gamma_t} \left( -\frac{n + m}{1 + n + \gamma_t(1 + m)} \right) - \lambda_4 + \lambda_5 = 0 \] 

(63)

Since we have assumed that \( m > n \) from equation (63) we derive that \( \gamma_t \) has to be a corner solution where \( \gamma_t = 1 \). The other constraint regarding \( \tau_t \), may be bounding or not, meaning that \( \tau_t = \tau_t(k_t) \in [0, 1] \).

The optimal solutions should also satisfy the second order sufficient conditions, meaning that the bordered Hessian of the
Lagrangian should be negatively defined. Since the solution of the openness rate is a corner solution where the maximal openness rate maximizes the young voter’s indirect utility function, the bordered Hessian of the Lagrangian is equal to:

\[-g_r \left( g_r \frac{\partial^2 L}{\partial k_{t+1} \partial \tau_t} - g_k \frac{\partial^2 L}{\partial k_{t+1} \partial \tau_t} \right) + g_k \left( g_r \frac{\partial^2 L}{\partial \tau_t \partial k_{t+1}} - g_k \frac{\partial^2 L}{\partial \tau_t \partial k_{t+1}} \right) \]

(64)

where \( g_r \) and \( g_k \) are the derivatives of the capital per native born work force constraint, from equation (56), with respect to \( \tau_t \) and \( k_{t+1} \) respectively. The bordered Hessian can be rewritten in the following way:

\[-(1 + \beta)^2 + \frac{2x^2(1 + \frac{1-\alpha}{\alpha} \tau_t)^2(1 - \tau_t)^2 \left( \frac{1-\alpha}{\alpha} \right)^2}{(1 + \beta) \frac{1-\alpha}{\alpha}(1 - \tau_t) - \frac{\beta(1-\alpha)}{\Psi + \alpha} (1 + \frac{1-\alpha}{\alpha} \tau_t)} \]

(65)

This expression is positive for values of the tax rate, \( \tau_t \), which satisfies the following inequalities:

\[0 \leq -(1 + \beta) + \frac{\sqrt{2x}(1 + \frac{1-\alpha}{\alpha} \tau_t)(1 - \tau_t) \left( \frac{1-\alpha}{\alpha} \right)}{(1 + \beta) \frac{1-\alpha}{\alpha}(1 - \tau_t) - \frac{\beta(1-\alpha)}{\Psi + \alpha} (1 + \frac{1-\alpha}{\alpha} \tau_t)} \]

(66)

\[0 \geq (1 + \beta) + \frac{\sqrt{2x}(1 + \frac{1-\alpha}{\alpha} \tau_t)(1 - \tau_t) \left( \frac{1-\alpha}{\alpha} \right)}{(1 + \beta) \frac{1-\alpha}{\alpha}(1 - \tau_t) - \frac{\beta(1-\alpha)}{\Psi + \alpha} (1 + \frac{1-\alpha}{\alpha} \tau_t)} \]

(67)

Thus, \( \tau_t \in [\tau_1, \tau_2] \) is the range for which the bordered Hessian of the Lagrangian is negatively defined, where \( \tau_1 \) and \( \tau_2 \) denotes the solutions of these equations respectively\(^{11}\). The solution of the tax rate in the first equilibrium, \( \tau_1(k_t) \), is optimal in the range \( k_t \in [F(\Psi), F(\tau_1)] \), and the solution of the tax rate in the second equilibrium, \( \tau_2(k_t) \), is optimal in the range \( k_t \in [F(\Psi), F(\tau_2)] \).

**The second part of the proof:**

As in the previous proposition, we must show that the vector of policy decision rules, \( \Psi = (T, G) \), as defined in the proposition, satisfies the equilibrium conditions:

1. \( \hat{\Psi}(\gamma_{t-1}, k_t) = \arg \max_{\pi_t} V^1(\gamma_{t-1}, k_t, \pi_t, \pi_{t+1}) \) subject to \( \pi_{t+1} = \Psi(\gamma_t, k_t) \).
2. \( \hat{\Psi}(\gamma_{t-1}, k_t) = \Psi(\gamma_{t-1}, k_t) \).

\(^{11}\)Since these equations are second order polynomials, \([\tau^1, \tau^2]\) and \([\tau^1, \tau^2]\), denotes the ranges for which the tax rate is positive according to the first and second inequality respectively.
3. \( S(\pi_t, k_t) = \frac{\beta \Psi}{1 + \beta \Psi + 1} \cdot \frac{(1 + \gamma_t)\mu(1-\tau_t)(1-f(\tau_{t+1}))}{1+n+\gamma_t(1+m)} \), with \( \tau_{t+1} = T(\gamma_t, k_t) \).

Proposition III refers to the both equilibrium \( i = 1, 2 \). The first equilibrium

\[ \tau \in [\tau_1, \tau] \]

Thus, the solution in this case which is denoted by \( \tau_1(k_t) \), is decreasing in \( k_t \in [F(\tau), F(\tau_1)] \). The second equilibrium

\[ \tau = 2 \]

Thus, the solution in this case which is denoted by \( \tau_2(k_t) \), is decreasing in \( k_t \in [F(\tau_2), F(\tau)] \).

Consider first the case where there is a majority of old in period \( t \), i.e. \( u_t \geq 1 \).

The utility of the old voter is the same as in the previous proposition and thus \( V^o(\gamma_{t-1}, k_t) \) is maximized by setting \( \gamma_t = 1 \) and \( \tau_t = \frac{\Psi}{\Psi + 1} \). But unlike the previous proposition the saving of the young in period \( t \) also set next period policy variables.

In the case of a majority of old, the policy decision rules are set by \( \gamma_t = 1 \) and \( \tau_t = \frac{\Psi}{\Psi + 1} \). In order for the next period policy decision rules to be set according to the decision rules: \( \gamma_{t+1} = 1 \) and \( \tau_{t+1} = \tau_i(k_{t+1}) \), the next period capital per native born labor force, \( k_{t+1} \), should be in the range: \( [F(\tau), F(\tau_i)] \) where \( k_{t+1} \) is defined by the following equation:

\[ k_{t+1} = \frac{\beta \Psi}{1 + \beta \Psi + 1} \cdot \frac{2}{1 - \alpha} \cdot \left( 1 - \alpha \right)^{2 - \alpha} \cdot \left( 1 - \frac{\Psi}{\Psi + 1} \right)^{1 + \frac{\Psi}{\Psi + 1}} \cdot (1 - f(\tau_i(k_{t+1}))) \]  

The derivative of \( k_{t+1} \) by \( k_t \) when the policy decision rules are set by \( \gamma_t = 1 \) and \( \tau_t = \frac{\Psi}{\Psi + 1} \) and next period tax rate is set according to \( \tau_i(k_{t+1}) \), is negative in the first case for \( \tau \in [\tau_1, \tau] \) and positive in the second for \( \tau \in [\tau, \tau_2] \). Thus, the range of \( k_t \) for which \( k_{t+1} \in [F(\tau), F(\tau_i)] \) is \( k_t \in [g_1(F(\tau_1)), g_1(F(\tau))] \) in the first case and \( k_t \in [g_2(F(\tau)), g_2(F(\tau_2))] \) in the second, where
\( g_i(y) \) is defined by\(^{12}\):
\[
g_i(y) = \left( \frac{\beta}{1 + \beta} \frac{\psi}{\psi + 1} 2^{\alpha} (1 - f(\tau_i(y))) \left( (1 - \alpha) k^\alpha 2^{-\alpha} (1 - \frac{\psi}{\psi + 1}) \right)^{\frac{1+\psi}{\psi}} \right)^{-1} y
\]

(70)

Otherwise, if \( k_t \notin [g_1(F(\tau_1)), g_2(F(\tau))] \) in the first case or \( k_t \notin [g_2(F(\tau)), g_2(F(\tau_2))] \) in the second, than \( k_{t+1} \notin [F(\tau), F(\tau_1)] \), and next period policy variable are set by the young to be equal to \( \gamma_t = \gamma^* \) and \( \tau_t = 0 \). Thus, next period capital per native born labor force is equal to:
\[
k_{t+1} = \frac{\beta}{1 + \beta} \frac{\psi}{\psi + 1} 2^{\alpha} (1 - f(0)) \]

(71)

Consider next the case where there is a majority of young in period \( t \), i.e. \( u_t < 1 \).

If \( k_t \in [F(\tau), F(\tau_1)] \), we must prove that the indirect utility of the young voter subject to \( \pi_{t+1} = \Psi(\gamma_t, k_t) \), is maximized by \( \gamma_t = 1 \) and \( \tau_t = \tau_t(k_t) \). Substituting for \( w_t u_t (1 - \tau_t) \) and \( 1 + \tau_t \) from equations (38) and (39), the young voter’s indirect utility function subject to \( \pi_{t+1} = \Psi(\gamma_t, k_t) \), can be written in its Lagrangian form as follows:

\[
L^i(\gamma_{t-1}, k_t) =
\begin{cases}
L^i(k_t) \text{ with } \pi_{t+1} = \Psi(\tau_t(k_{t+1}), 1) & \text{if } k_{t+1} \in [F(\tau), F(\tau_1)] \\
L^i(k_t) \text{ with } \pi_{t+1} = \Psi(0, \text{Min}[\gamma^*, \frac{\alpha}{M}]) & \text{otherwise if } u_{t+1} < 1 \\
L^i(k_t) \text{ with } \pi_{t+1} = \Psi(\frac{\psi}{\psi + 1}, 1) & \text{otherwise}
\end{cases}
\]

(72)

The first part of the proposition, proves that if next period decision rules are set by \( \tau_{t+1} = \tau_t(k_{t+1}), \gamma_{t+1} = 1 \), the optimal solution for the young is to set \( \tau_t = \tau_t(k_t) \), and \( \gamma_t = 1 \). In addition, we have shown that under the assumption that next period policy decision rule are given according to equations (30) and (31), the young voter’s indirect utility function is maximized by

\[
k_{t+1} = \frac{\beta}{1 + \beta} \frac{\psi}{\psi + 1} 2^{\alpha} (1 - f(0)) \left( (1 - \alpha) k^\alpha 2^{-\alpha} (1 - \frac{\psi}{\psi + 1}) \right)^{\frac{1+\psi}{\psi}} (1 - f(0))
\]

(69)

\(^{12}\)If the sum of the fertility rates are negative, \( n + m < 0 \), than the identity of the decisive voter is always old, which means that next period capital per native born labor force always equal:
\(\gamma_t = \text{Min}[\gamma^*, -\frac{n}{m}]\) and \(\tau_t = 0\). Therefore we must show that if \(k_t \in [F(\tau), F(\tau)]\), the value of the young voter’s indirect utility function is higher under the following decision rules: \(\tau_t = \tau_t(k_t)\), and \(\gamma_t = 1\). Since the value of the young voter’s indirect utility function under the first decision rules is constant in \(k_t\), and the value of the young voter’s indirect utility function under the second decision rules is increasing in \(k_t\), the first condition on the parameters of the model is to require that both values of the indirect utility of the young should equate at \(k_t = F(\tau_t)\)\(^{13}\)

\[
\log \left(\left(\frac{2}{\tau_n + m} + \frac{\beta \Psi}{1 + \beta \Psi + 1}\right)^{-(1+\beta)} \left(2^{\Psi(1-\alpha)}\right)\right) = (1+\beta) \log \left(\left(1-\alpha\right)F(\tau_t)^\alpha \left(1+\gamma_t\right)^{-\alpha} \left(1+\beta \Psi + 1\right)^{\alpha(1+\gamma_t)} \right) + \log \left(\alpha(1-\alpha)(1-\frac{\Psi}{1+\alpha})2^{\Psi}\frac{\beta(1+\alpha)}{1+\beta \Psi + 1} (1+\gamma_t)^{-\alpha} \left(1-\frac{\Psi}{1+\beta \Psi + 1}\right)^{\alpha(1+\gamma_t)} \right)^\beta
\]

(73)

In addition, we must require that if \(k_t \in [F(\tau), F(\tau)]\), then also \(k_{t+1}\) which is equal to the following expression:

\[k_{t+1} = \frac{\beta}{1+\beta \Psi + 1} \left(1+\frac{2}{\tau_n + m} (1-\alpha)k_t^\alpha \left(1-\tau_t(k_t)\right) \left(1-f(\tau_t(k_{t+1}))\right)\right)^{\frac{\alpha}{1+\alpha}} \]

(74)

will be in the relevant range, i.e. \(k_{t+1} \in [F(\tau), F(\tau)]\)\(^{14}\). Since the derivative of \(k_{t+1}\) by \(k_t\) is negative in both equilibrium

\[i = 1, 2, \text{if } k_t \in [F(\tau), F(\tau)]\), then \(k_{t+1} \in [\underline{k}_t, \overline{k}_t]\), where \(\underline{k}_t\) and \(\overline{k}_t\) are defined respectively by the following equations:

\[
\underline{k}_t = \frac{\beta}{1+\beta \Psi + 1} \left(1+\frac{2}{\tau_n + m} (1-\alpha) (F(\tau))^\alpha \left(1-\tau_t(\overline{k}_t)\right) \left(1-f(\tau_t(\overline{k}_t))\right)\right)^{\frac{\alpha}{1+\alpha}} \]

(75)

\[
\overline{k}_t = \frac{\beta}{1+\beta \Psi + 1} \left(1+\frac{2}{\tau_n + m} (1-\alpha) (F(\tau))^\alpha \left(1-\tau_t(\underline{k}_t)\right) \left(1-f(\tau_t(\underline{k}_t))\right)\right)^{\frac{\alpha}{1+\alpha}} \]

(76)

\(^{13}\)If the fertility rates are both positive than next period policy variables are: \(\gamma_t = \text{Max}[\gamma^*, -\frac{n}{m}]\) and \(\tau_t = 0\), since next period decisive voter is young. Thus the condition in this case is simply: \(\log \left(\left(\frac{2}{\tau_n + m} + \frac{\beta \Psi}{1 + \beta \Psi + 1}\right)^{-(1+\beta)} \left(2^{\Psi(1-\alpha)}\right)\right) = (1+\beta) \log \left(\left(1-\alpha\right)F(\tau_t)^\alpha \left(1+\gamma_t\right)^{-\alpha} \left(1+\beta \Psi + 1\right)^{\alpha(1+\gamma_t)} \right) + \log \left(\alpha(1-\alpha)(1-\frac{\Psi}{1+\alpha})2^{\Psi}\frac{\beta(1+\alpha)}{1+\beta \Psi + 1} (1+\gamma_t)^{-\alpha} \left(1-\frac{\Psi}{1+\beta \Psi + 1}\right)^{\alpha(1+\gamma_t)} \right)^\beta \left(1+\gamma_t\right)^\Psi
\]

(73)

\(^{14}\)It should be noted that if \(k_{t+1} = \frac{\beta}{1+\beta \Psi + 1+\gamma_t(1+\beta \Psi + 1)} (1-\alpha) (1+\gamma_t)^{-\alpha} \left(1-f(\frac{\Psi}{1+\beta \Psi + 1})\right)\) is also in the relevant range, under the previous condition, the value of the young voter’s indirect utility function is higher under the first decision rules. Thus the optimal solution is \(\tau_t = \tau_t(k_t)\), and \(\gamma_t = 1\).
Therefore the required condition is that $[k_i, \bar{k}] \subseteq [F(\tau), F(\tau_i)]$.

Otherwise, if $k_t \notin [F(\tau), F(\tau_i)]$, we must prove that the indirect utility of the young voter subject to $\pi_{t+1} = \Psi(\tau_i, k_t)$, is

maximized by $\gamma_t = \text{Max}[\gamma^*, -\frac{m}{\tau}]$ and $\tau_t = 0$.

For $k_t > F(\tau_i)$ according to the first condition the value of the young voter’s indirect utility function is for $\gamma_t = \text{Max}[\gamma^*, -\frac{m}{\tau}]$ and $\tau_t = 0$. It should be noted that since this optimal solution changes next period decisive voter from young to old for all values of $k_{t+1}$, there are no additional conditions on $k_{t+1}$.

For $k_t < F(\tau)$ if $k_{t+1} > F(\tau_i)$, where $k_{t+1}$ is defined by:

$$k_{t+1} = \frac{\beta}{1 + \beta} \Psi + \frac{2}{1 + \beta} + \frac{m + n + m}{(1 - \alpha)k^0i2^{2-\alpha}(1 - \tau_i(k_t))} \frac{\Psi}{\Psi_i} (1 - f(\tau_i(k_{t+1})))$$

(77)

then as was proved the young voter’s indirect utility function is maximized by $\gamma_t = \text{Max}[\gamma^*, -\frac{m}{\tau}]$ and $\tau_t = 0$. In order to derive the condition that $k_{t+1} > F(\tau_i)$ for all $k_t < F(\tau)$, we must require that $F_i \geq F(\tau_i)$. Combining the previous condition with the latter, we derive the conditions: $k_t \geq F(\tau)$ and $F_i = F(\tau_i)$.

An additional condition is necessary in order to insures that $\tau_i(k_t)$ is the only solution in the range $[F(\tau), F(\tau_i)]$.

The condition requires that for $\tau_j(k_t)$ where $i \neq j$, if $k_t \in [F(\tau), F(\tau_i)]$ and $\tau_t = \tau_i(k_{t+1})$ then $k_{t+1} \notin [F(\tau), F(\tau_i)]$, where $k_{t+1}$ is defined by:

$$k_{t+1} = \frac{\beta}{1 + \beta} \Psi + \frac{2}{1 + \beta} + \frac{m + n + m}{(1 - \alpha)k^0i2^{2-\alpha}(1 - \tau_2(k_t))} \frac{\Psi}{\Psi_i} (1 - f(\tau_i(k_{t+1})))$$

(78)

Since $\tau_2(k_t)$ is increasing in $k_t$, $\tau_1(k_t)$ is decreasing in $k_t$, the derivative of $k_{t+1}$ by $k_t$ is positive in the first case, while negative in the other. Therefore the required conditions, should be that for $k_t \in [F(\tau), F(\tau_i)]$, either $k_{t+1} < F(\tau)$ or $k_{t+1} > F(\tau_i)$. Defined

15In the case where fertility rates are positive than we should also require that if $k_t > F(\tau) \Rightarrow k_{t+1} > F(\tau_i)$, where:

$\frac{\beta}{1 + \beta} \Psi + \frac{2}{1 + \beta} + \frac{m + n + m}{(1 - \alpha)F(\tau)^{\alpha}(1 + \gamma^*)^{-\alpha}(1 - 0)} \frac{\Psi}{\Psi_i} (1 - f(0))$

16This is due to the fact that as was proved in the first part of the proof, both solution for the tax rate $\tau_1(k_t)$ and $\tau_2(k_t)$ maximizes the young voter’s indirect utility function.
\( k_i \) and \( k \) by the following equations:

\[
\begin{align*}
\beta & \frac{\Psi}{1 + \beta \Psi + \frac{1}{2} n + m} \left((1 - \alpha)(F(\tau_i))^{\alpha}2^{-\alpha}(1 - \tau_j(F(\tau_i)))\right)^{\frac{\psi}{1+\psi}} (1 - f(\tau_i(k_i))) = k_i \\
(79) \\
\beta & \frac{\Psi}{1 + \beta \Psi + \frac{1}{2} n + m} \left((1 - \alpha)(F(\tau))^{\alpha}2^{-\alpha}(1 - \tau_j(F(\tau)))\right)^{\frac{\psi}{1+\psi}} (1 - f(\tau_i(k_i))) = k \\
(80)
\end{align*}
\]

Namely, the sufficient conditions are either \( k_i < F(\tau) \) or \( k > F(\tau_i) \).

7 References

Ted C. Bergstrom John Hartman, 2005, Demographics and the Political Sustainability of Pay-as-you-go Social Security, Department of Economics, UCSB.


References


References


References


