Chapter 11

A Dynamic Reformulation

So far our analysis was confined to a two-period world; after the second period the world ceases to exist or just repeats itself. In this framework there is no trade-off between consumption and portfolio investment. Although the choice of portfolio is dictated by preferences over consumption, it is not possible to increase consumption at the expense of security purchase. This means that savings, as defined in the usual sense, are absent. Hence, this model has some static features.

In this chapter we suggest a dynamic extension in order to remedy these shortcomings. Our extended version contains an infinite horizon and introduces explicitly the trade-off between present period consumption and savings. We shall show that the dynamic version preserves some important characteristics of the two-period model. We shall also present an example which illustrates an equilibrium structure of the dynamic model.

11.1 THE MODEL

Let the intertemporal von Neumann–Morgenstern utility function of the home country be

$$U = \sum_{t=0}^{\infty} (\delta)^t u(c_1{}^t, c_2{}^t), \qquad 0 < \delta < 1 \tag{11.1}$$

where δ is a discount factor (which is equal to one over one plus the rate of time preference) and superscript t denotes a variable in time t. For example, $c_1{}^t$ is consumption of the first good in period t. This convention is also used below.

We denote now by α^t a state in period t. A state of the world is described by the infinite sequence $(\alpha^1, \alpha^2, \ldots, \alpha^t, \ldots)$. We assume that the set of states and the distribution over it are the same in each period. Put differently, α^t can be considered as an identical independently distributed random variable for all t.

We assume also that equities are one-period securities. This means that the purchase of an equity today entitles its holder only to the equity's share in profits tomorrow. After the distribution of profits, the equity has no more economic value. In every period, firms issue one-period equities in order to finance factor payments.

Given the portfolio composition from period $t-1$, (z_1^{t-1}, z_2^{t-1}), portfolio income in state α^t in period t is

$$z^t(\alpha^t) = p_1{}^t(\alpha^t)\theta_1(\alpha^t)z_1^{t-1} + p_2{}^t(\alpha^t)\theta_2(\alpha^t)z_2^{t-1} \tag{11.2}$$

The income of factors of production in period t, in state α^t, is equal to the value of newly issued equities by the home firms:

$$Z^t(\alpha^t) = q_1{}^t(\alpha^t)\{Z_1[q^t(\alpha^t)] + q^t(\alpha^t)Z_2[q^t(\alpha^t)]\} \tag{11.3}$$

(We omit factor endowments (L, K) in writing the supply functions $Z_i[\cdot]$.)

The home country can choose in period t–state α^t consumption and new equity purchase whose total value does not exceed portfolio plus factor incomes. Hence, the period t–state α^t budget constraint is

$$\begin{aligned} p_1{}^t(\alpha^t)c_1{}^t(\alpha^t) + p_2{}^t(\alpha^t)c_2{}^t(\alpha^t) + q_1{}^t(\alpha^t)[z_1{}^t(\alpha^t) + q^t(\alpha^t)z_2(\alpha^t)] \\ \leq p_1{}^t(\alpha^t)\theta_1(\alpha^t)z_1^{t-1} + p_2{}^t(\alpha^t)\theta_2(\alpha^t)z_2^{t-1} \\ + q_1{}^t(\alpha^t)\{Z_1[q^t(\alpha^t)] + q^t(\alpha^t)Z_2[q^t(\alpha^t)]\} \\ = z^t(\alpha^t) + Z^t(\alpha^t) \end{aligned} \tag{11.4}$$

In our two-period model, we had two numeraires; a numeraire for the first period and a numeraire for the second period. The only relative prices that mattered were relative commodity prices and relative security prices. There was no significance to a price of a good in terms of a security, or vice versa. This was of course a result of the structure in which securities were traded in one period and goods were traded in another period; goods and securities were never traded simultaneously.

Now the situation is different, as one can see from (11.4). Goods and securities are traded simultaneously in every period. Observe, however, that in the case of certainty it is still true that the relative price of goods is equal to the relative price of securities; the relative price of securities today has to be equal to the relative price of goods tomorrow (from arbitrage considerations). Suppose also that in the case of certainty the relative price of goods is the same in every period. Then the relative price of equities today equals the relative price of goods today. Does this imply that commodity prices equal equity prices? The answer is no, since the difference between the increase over time in the absolute price levels of goods and securities reflects the interest rate, which is positive due to time preference.

The problem of the economy now is to maximize the expected value of (11.1) subject to the infinite sequence of budget constraints (11.4). This is a dynamic programming problem which can be solved by means of Bellman's principle of optimality.

Let $V^t(z_1^{t-1}, z_2^{t-1}, \alpha^t)$ be the maximum of

$$E \sum_{\tau=t}^{\infty} (\delta)^{\tau-t} u[c_1(\alpha^\tau), c_2(\alpha^\tau)]$$

subject to (11.4) with τ substituted for t. Then, we have by the principle of optimality (omitting α^t from $c_1{}^t, c_2{}^t, z_1{}^t, z_2{}^t$)

$$(11.5) \quad V^t(z_1^{t-1}, z_2^{t-1}, \alpha^t) = \max[u(c_1, c_2) + \delta \mathop{E}_{\alpha^{t+1}} V^{t+1}(z_1{}^t, z_2{}^t, \alpha^{t+1})], \qquad t = 0, 1, \ldots$$

subject to the single-period budget constraint (11.4) and $(c_1{}^t, c_2{}^t, z_1{}^t, z_2{}^t) \geq 0$. An α^{t+1} below E indicates that the expectation is over α^{t+1}.

This implies that, given the functions $V^t(\cdot), t = 1, 2, \ldots$, the economy chooses consumption, savings, and its portfolio composition so as to solve in period t and state α^t the single-period problem

(11.6) choose $c_1{}^t(\alpha^t), c_2{}^t(\alpha^t), z_1{}^t(\alpha^t), z_2{}^t(\alpha^t) \geq 0$

to maximize

$$U^t[c_1{}^t(\alpha^t), c_2{}^t(\alpha^t), z_1{}^t(\alpha^t), z_2{}^t(\alpha^t)] = u[c_1{}^t(\alpha^t), c_2{}^t(\alpha^t)] + \delta \mathop{E}_{\alpha^{t+1}} V^{t+1}[z_1{}^t(\alpha^t), z_2{}^t(\alpha^t), \alpha^{t+1}]$$

subject to (11.4)

$U^t(\cdot)$ represents the preferences of period t over consumption and equities. We now present a diagrammatic exposition of the solution to (11.6). However, before we do so it may prove useful to consider the accounts of the balance of payments in this setup.

Let $X^t(\alpha^t)$ be the value of commodity sales in period t and state α^t of the home country. $X^t(\alpha^t)$ depends on the allocation of factors of production in period $t-1$ and the realization of commodity prices and the technological parameters in period t:

$$X^t(\alpha^t) = p_1{}^t(\alpha^t)\theta_1(\alpha^t)Z_1^{t-1} + p_2{}^t(\alpha^t)\theta_2(\alpha^t)Z_2^{t-1}$$

$X^t(\alpha^t)$ is also equal to the dividend payments of the industries of the home country in period t and state α^t.

Let

$$C^t(\alpha^t) = p_1{}^t(\alpha^t)c_1{}^t(\alpha^t) + p_2{}^t(\alpha^t)c_2{}^t(\alpha^t)$$

be the value of consumption and

$$K^t(\alpha^t) = q_1{}^t(\alpha^t)z_1{}^t(\alpha^t) + q_2{}^t(\alpha^t)z_2{}^t(\alpha^t)$$

the value of security purchase in period t and state α^t. Then, using the definitions of $z^t(\alpha^t)$ and $Z^t(\alpha^t)$, dividend income and factor income [see (11.2) and (11.3)], (11.4) can be rewritten as

$$C^t(\alpha^t) - z^t(\alpha^t) + K^t(\alpha^t) - Z^t(\alpha^t) = 0 \tag{11.7}$$

Adding and subtracting $X^t(\alpha^t)$ from the left-hand side of (11.7), we get

$$[C^t(\alpha^t) - X^t(\alpha^t)] + [X^t(\alpha^t) - z^t(\alpha^t)] + [K^t(\alpha^t) - Z^t(\alpha^t)] = 0 \tag{11.8}$$

Equation (11.8) represents the balance of payments accounts. The term in the first set of brackets represents the trade account deficit, the term in the second set represents the service account deficit (net dividend outflows), and the last term represents the capital account deficit (net equity purchase). The sum of the deficits in the trade and service accounts is $C^t(\alpha^t) - z^t(\alpha^t)$ and it is equal to the deficit in the current account. If there is no international trade in securities, then in equilibrium, $Z_i{}^t = z_i{}^t$, $i = 1, 2$, and the deficits in the capital and service accounts are zero.

11.2 A DIAGRAMMATIC EXPOSITION

For the purpose of the diagrammatic exposition, let us aggregate consumption into the Hicks composite good $C^t(\alpha^t)$. Then, we can

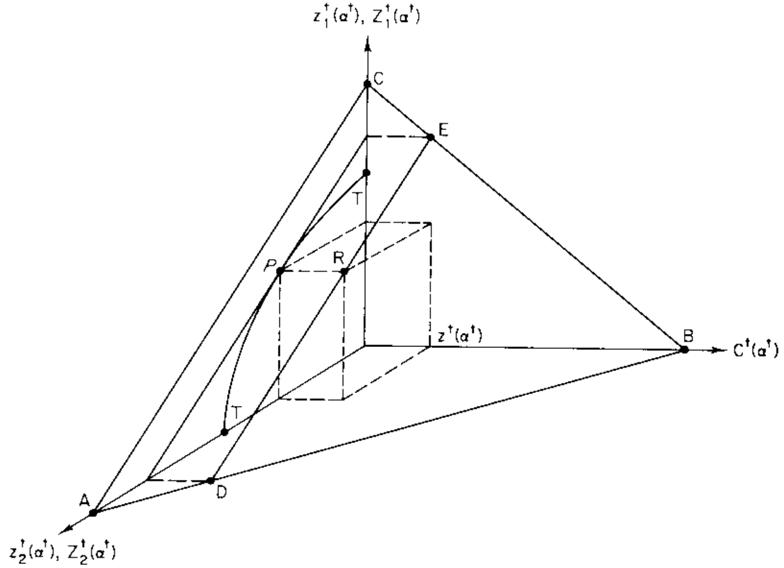

FIGURE 11.1

represent the budget constraint of the economy (11.4) by the plane ABC in Figure 11.1. The construction of this plane is as follows. Its slopes have to reflect relative prices. Hence, because of our aggregation of consumption, the slope of CB toward the $C^t(\alpha^t)$ axis is $1/q_1{}^t(\alpha^t)$, and the slope of AB toward the same axis is $1/[q_1{}^t(\alpha^t)q^t(\alpha^t)] = 1/q_2{}^t(\alpha^t)$. The slope of AC toward the $z_2{}^t(\alpha^t)$ axis is $q^t(\alpha^t)$. It remains, therefore, to determine only the height of the plane. This can be done by choosing a feasible point at which (11.4) is satisfied with equality.

We have drawn in Figure 11.1 the transformation curve between $Z_1{}^t$ and $Z_2{}^t$. Given $q^t(\alpha^t)$, production is chosen at point P, at which a line parallel to AC is tangent to the transformation curve. Clearly, since $z^t(\alpha^t) > 0$, the economy can choose a portfolio which consists of $(Z_1{}^t, Z_2{}^t)$ and consume the income from dividends; that is, choose $C^t(\alpha^t) = z^t(\alpha^t)$. This choice is represented by point R, where R lies on a horizontal line through P and the distance between R and P equals $z^t(\alpha^t)$.

We can now impose the preference map generated by $U^t(\cdot)$ on Figure 11.1 in order to find the optimal portfolio and consumption point [$U^t(\cdot)$ has to be modified in the usual way in order to represent aggregate consumption].

If there is no international trade in equities, then equilibrium security prices are such that the optimal point is R. In this case investors hold the domestically issued equities and the value of consumption equals portfolio income, just as in the two-period model. The deficits in each of the three balance of payments accounts are zero.

If there is international trade in securities, then any point on ABC can be an equilibrium point. If the equilibrium point is to the right of DE, that is, $C^t(\alpha^t) > z^t(\alpha^t)$, then there is a deficit in the current account and a surplus in the capital account. If the equilibrium point lies to the left of DE, then there is a surplus in the current account and a deficit in the capital account.

11.3 THE BASIC PROPOSITIONS

It is clear from the discussion in the previous section that our conclusions from Chapter 7, concerning the basic propositions of international trade theory, are valid in the present framework. If there is no international trade in securities, countries need not specialize according to comparative advantage in the Ricardian model, and in the Heckscher Ohlin model factor prices need not be equalized, and the Stolper–Samuelson and Rybczynski theorems need not hold. If, however, there is international trade in equities, these propositions are restored. One has to be careful only to formulate the Stolper–Samuelson and Rybczynski theorems in terms of equity prices instead of commodity prices. Observe that an important feature of the two-period model is preserved here; that is, the allocation of production resources depends on relative equity prices (which in turn determine factor prices) and not on commodity prices. This happens despite the fact that consumption does take place simultaneously with production.

An additional remark on the Stolper–Samuelson theorem is in order. In the two-period model with trade in equities, an increase in the price of an equity, other things being equal, increases the real reward of one factor and reduces the real reward of the other factor, where real rewards are measured in terms of equities. However, given a constant distribution of commodity prices, the gaining factor can increase proportionately its equity holdings and thus assure itself of more consumption in the second period in every state of the world. In the present context, the gaining factor can buy in the present period both more equities and more consumption.

11.4 AN EXAMPLE

We present now an example of a two-country world for which we are able to work out a stationary equilibrium. In this equilibrium the commodity prices have a genuine distribution which is the same in every period, while equity prices are constant over time and across states. The allocation of factors of production and the composition of portfolios are also constant over time and across states, but consumption and commodity trade levels are random variables with an identical independent distribution in every period.

Let the temporal utility functions of the home and foreign countries be

$$u(c_1, c_2) = -(1-\beta)\log(1-\beta) - \beta \log \beta + (1-\beta)\log c_1 + \beta \log c_2, \qquad 0 < \beta < 1 \tag{11.9}$$

$$u^*(c_1^*, c_2^*) = -(1-\beta^*)\log(1-\beta^*) - \beta^* \log \beta^* + (1-\beta^*)\log c_1^* + \beta^* \log c_2^*, \qquad 0 < \beta^* < 1 \tag{11.10}$$

and let $\theta_i(\alpha)$, $i = 1, 2$, be (genuine) independent identically distributed random variables which are the same for both countries. Every country has its own transformation curve between real equities. These transformation curves are summarized by the supply functions $Z_i(q)$, $Z_i^*(q^*)$, $i = 1, 2$ (see Chapter 5).

Let us begin with the case in which there is no international trade in securities. In this case $z_i = Z_i(q)$, $z_i^* = Z_i^*(q^*)$, $i = 1, 2$, is satisfied in every equilibrium. This implies that the value of consumption of each country is equal to its portfolio income. But, under these circumstances, commodity market clearing conditions imply the same structure of returns on both types of equities, as we have shown in Chapter 7 [see (7.20) and (7.21)]. In particular, it is shown there that in this case

$$\frac{\theta_2(\alpha)p_2(\alpha)}{\theta_1(\alpha)p_1(\alpha)} = \frac{\beta z_1 + \beta^* z_1^*}{Z_2 + Z_2^* - \beta z_2 - \beta^* z_2} = q = q^* \tag{11.11}$$

is a necessary condition for equilibrium [see (7.17) and (7.21)].

We want to show that there is a stationary equilibrium in which the portfolio composition of each country is constant; state and time independent. In this case, since there is no international trade in equities, (11.11) implies

$$q = q^* = \frac{\beta Z_1(q) + \beta^* Z_1^*(q)}{(1-\beta)Z_2(q) + (1-\beta^*)Z_2^*(q)} \tag{11.12}$$

Now, since $Z_1(q)$ and $Z_1^*(q)$ are declining in q and both reach zero for sufficiently high values of q, and since $Z_2(q)$ and $Z_2^*(q)$, are increasing in q and both reach zero at a sufficiently low value of q, there is a unique value $\bar{q}$ for which (11.12) is satisfied. It is therefore clear that if there is an equilibrium of the type for which we are looking, it has to satisfy

$$q = q^* = \bar{q} \tag{11.13}$$

$$Z_i^t = \bar{Z}_i \equiv Z_i(\bar{q}), \qquad i = 1, 2 \tag{11.14}$$

$$Z_i^{*t} = \bar{Z}_i^* \equiv Z_i^*(\bar{q}), \qquad i = 1, 2 \tag{11.15}$$

$$\theta_2(\alpha^t)p_2{}^t(\alpha^t) = p_1{}^t(\alpha^t)\theta_1(\alpha^t)\bar{q}, \qquad \alpha^t = 1, 2, \ldots, S, \quad t = 0, 1, \ldots \tag{11.16}$$

We are free to choose a numeraire in every state–period combination. The normalization we choose is

$$p_1{}^t(\alpha^t) = \frac{1}{\theta_1(\alpha^t)}, \qquad \alpha^t = 1, 2, \ldots, S, \quad t = 0, 1, \ldots \tag{11.17}$$

This together with (11.16) implies

$$p_2{}^t(\alpha^t) = \frac{\bar{q}}{\theta_2(\alpha^t)}, \qquad \alpha^t = 1, 2, \ldots, S, \quad t = 0, 1, \ldots \tag{11.18}$$

Hence, the distribution of our prices is stationary; it is the same in every period.

It remains to determine the absolute prices of equities. Let

$$q_1 = \delta \quad \Rightarrow \quad q_2 = \delta\bar{q} \tag{11.19}$$

$$q_1^* = \delta^* \quad \Rightarrow \quad q_2^* = \delta^*\bar{q} \tag{11.20}$$

It remains to show that there exist $V^t(\cdot)$, $t = 0, 1, \ldots$, such that $z_i^t(\alpha^t) = \bar{Z}_i$, $i = 1, 2$, and

$$(11.21) \qquad c_1^t(\alpha^t) = \frac{(1-\beta)[p_1^t(\alpha^t)\theta_1(\alpha^t)\bar{Z}_1 + p_2^t(\alpha^t)\theta_2(\alpha^t)\bar{Z}_2]}{p_1^t(\alpha^t)}$$
$$= (1-\beta)(\bar{Z}_1 + \bar{q}\bar{Z}_2)\theta_1(\alpha^t),$$
$$\alpha^t = 1, 2, \ldots, S, \quad t = 0, 1, \ldots$$

$$(11.22) \qquad c_2^t(\alpha^t) = \frac{\beta(\bar{Z}_1 + \bar{q}\bar{Z}_2)\theta_2(\alpha^t)}{\bar{q}},$$
$$\alpha^t = 1, 2, \ldots, S, \quad t = 0, 1, \ldots$$

solve (11.5) for the home country, and that there exist $V^{*t}(\cdot)$, $t = 0, 1, \ldots$, such that $z_i^{*t}(\alpha^t) = \bar{Z}_i^*$, $i = 1, 2$, and

$$(11.23) \qquad c_1^{*t}(\alpha^t) = (1-\beta^*)(\bar{Z}_1^* + \bar{q}\bar{Z}_2^*)\theta_1(\alpha^t),$$
$$\alpha^t = 1, 2, \ldots, S, \quad t = 0, 1, \ldots$$

$$(11.24) \qquad c_2^{*t}(\alpha^t) = \frac{\beta^*(\bar{Z}_1^* + \bar{q}\bar{Z}_2^*)\theta_2(\alpha^t)}{\bar{q}},$$
$$\alpha^t = 1, 2, \ldots, S, \quad t = 0, 1, \ldots$$

solve (11.5) for the foreign country. It is easy to see that the commodity market clearing conditions are satisfied and that consumption levels are identically distributed in every period.

We show the solution to (11.5) only for the home country. The proof for the foreign country is the same.

Let $V^t(\cdot)$ be

$$(11.25) \qquad V^t(z_1^{t-1}, z_2^{t-1}, \alpha^t) \equiv V(z_1^{t-1}, z_2^{t-1}, \alpha^t)$$
$$\equiv \log[\theta_1(\alpha^t)]^{1-\beta}[\theta_2(\alpha^t)]^{\beta}(\bar{q})^{-\beta}$$
$$+ \frac{\delta}{1-\delta} E \log[\theta_1(\alpha)]^{1-\beta}[\theta_2(\alpha)]^{\beta}(\bar{q})^{-\beta}$$
$$+ \frac{1}{1-\delta} \log[\delta(\bar{Z}_1 + \bar{q}\bar{Z}_2)$$
$$+ (1-\delta)(z_1^{t-1} + \bar{q}z_2^{t-1})],$$
$$\alpha^t = 1, 2, \ldots, S, \quad t = 0, 1, \ldots$$

Then, it is easy to see that

$$(11.26)\quad \underset{\alpha^t}{E}V(z_1^{t-1}, z_2^{t-1}, \alpha^t) = \frac{1}{1-\delta} E \log[\theta_1(\alpha)]^{1-\beta}[\theta_2(\alpha)]^{\beta}(\bar{q})^{-\beta} + \frac{1}{1-\delta} \log [\delta(\bar{Z}_1 + \bar{q}\bar{Z}_2) + (1-\delta)(z_1^{t-1} + \bar{q}z_2^{t-1})]$$

The reader can now verify that $E_{\alpha^0}V(\bar{Z}_1, \bar{Z}_2, \alpha^0)$ is the expected value of the discounted sum of utilities at our proposed allocation. The reader can also verify that, using (11.25), our proposed allocation solves (11.5) for our proposed prices. Hence, our prices and allocation constitute an equilibrium.

Our stationary solution can be represented in the framework of the two-period model. This is done in Figures 11.2 and 11.3. Each country has straight line assets–indifference curves with slope $\bar{q}$. These indifference curves are drawn in Figure 11.2a for the home country and in Figure 11.2b for the foreign country. Each country spends its income from dividends on consumption. Hence, it spends on securities its factor income. But given equity prices, producers choose the production point at the tangency of a line with slope $\bar{q}$ to the transformation curve. This line describes factor incomes and investors choose their portfolios on this line. Since assets–indifference curves are straight lines with slope $\bar{q}$, investors are indifferent between all points on the factor income line, and they may as well choose the points of tangency, E_s and $E_s{}^*$. This choice is repeated in every period independent of the state that realizes.

Figure 11.3 describes the state-α stationary consumption choice for the home country (the same applies to the foreign country).

Portfolio income in state α is given by the budget line BB whose slope is $p_2(\alpha)/p_1(\alpha) = \bar{q}\theta_1(\alpha)/\theta_2(\alpha)$. The consumer chooses his optimal consumption at the point of tangency of his temporal indifference curve and this budget line—point E_t. This choice depends only on the state that realizes; if the same state realizes in two different periods, then consumption and commodity trade will be the same in both periods.

It is now interesting to examine the structure of prices in our example. We have seen that relative equity prices are the same in both countries. It is therefore clear that factor prices in terms of equities

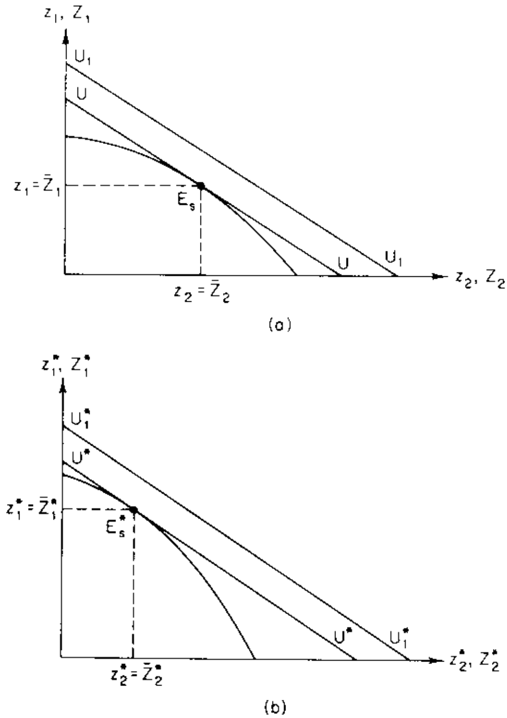

FIGURE 11.2

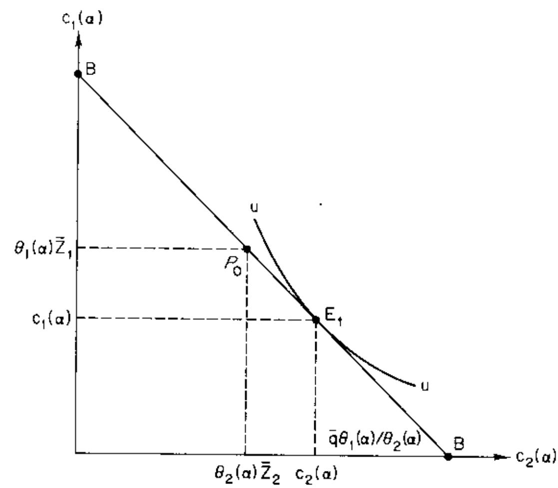

FIGURE 11.3

are also equal in both countries if they have the same production functions and no country specializes in production. Commodity prices are certainly equal in both countries because of the trade in commodities. But does this imply that factor prices in terms of commodities are equalized? The answer is not necessarily so; that is, although relative equity prices are the same, equity prices in terms of commodities need not be the same. It is clear from (11.19) and (11.20) that equity prices are equalized if and only if both countries have the same rates of time preference; that is, $\delta = \delta^*$. If the rates of time preference are the same, there will be both equity and factor price equalization, and if the rates of time preference differ, there will be neither equity nor factor price equalization [see Stiglitz (1970) for a discussion of factor price equalization in a deterministic dynamic model]. Observe, however, that in the formulation of the example we have not required the production functions to be the same in both countries; we only required the same risk structure in the same sector of each country.

Consider the trade-off between present consumption and consumption in the next period. The consumer can buy a real equity of type 1 for δ dollars. This amounts to a sacrifice of the present consumption of $\delta/p_1(\alpha^t) = \delta\theta_1(\alpha^t)$ units of good 1, given state α^t. The real equity provides a sure return of one dollar in every state of the next period. The purchasing power of the dollar in terms of good 1 depends, however, on the state that will realize in the next period, and it is equal to $1/p_1(\alpha^{t+1}) = \theta_1(\alpha^{t+1})$ in state α^{t+1}. This means that the consumer can replace the sure consumption of $\delta\theta_1(\alpha^t)$ units of good 1 today with the random consumption tomorrow of $\theta_1(\alpha^{t+1})$ units of good 1. A similar calculation reveals that the sure consumption of $\delta\theta_2(\alpha^t)$ units of commodity 2 today can be traded for a random consumption of $\theta_2(\alpha^{t+1})$ units of commodity 2 tomorrow.

Consider the case in which there is international trade in securities. It is clear from the discussion in the last two paragraphs that if both countries have the same rate of time preference, then the no security–trade equilibrium is also an equilibrium in the presence of international financial markets. However, if the rates of time preference differ, our no security–trade equilibrium cannot be sustained in the presence of international financial markets, because with trade in securities the equity prices of the home country cannot differ from foreign equity prices. What is then the nature of a stationary equilibrium when countries have different rates of time preference? We shall show that in this case there is an equilibrium price structure which is similar but not identical to the price structure in the case of no trade in securities. But now the common price of type-1 real equities equals the largest discount factor; that is, if $\delta > \delta^*$, then $q_1 = \delta$, and the country with the lowest rate of time preference (largest discount factor) holds all financial assets. The other country holds no financial assets in this steady state and it consumes its factor income.

Assuming, without loss of generality, that the home country has a lower rate of time preference; that is, $\delta > \delta^*$, the equilibrium price structure is (the same prices prevail in both countries)

(11.27) $$q_1 = \delta > \delta^*$$

$$q_2 = \hat{q}\delta$$

(11.29) $$p_1{}^t(\alpha^t) = \frac{1}{\theta_1(\alpha^t)}, \qquad \alpha^t = 1, 2, \ldots, S, \quad t = 0, 1, \ldots$$

(11.30) $$p_2{}^t(\alpha^t) = \frac{\hat{q}}{\theta_2(\alpha^t)}, \qquad \alpha^t = 1, 2, \ldots, S, \quad t = 0, 1, \ldots$$

(11.31) $$\hat{Z}_i = Z_i(\hat{q}), \qquad i = 1, 2$$

(11.32) $$\hat{Z}_i{}^* = Z_i{}^*(\hat{q}), \qquad i = 1, 2$$

(11.33) $$\hat{z}_i = \hat{Z}_i + \hat{Z}_i{}^*, \qquad i = 1, 2$$

(11.34) $$\hat{z}_i{}^* = 0, \qquad i = 1, 2$$

(11.35) $$c_1{}^t(\alpha^t) = (1 - \beta)[\hat{Z}_1 + \hat{q}\hat{Z}_2 + (1 - \delta)(\hat{Z}_1{}^* + \hat{q}\hat{Z}_2{}^*)]\theta_1(\alpha^t),$$
$$\alpha^t = 1, 2, \ldots, S, \quad t = 0, 1, \ldots$$

(11.36) $$c_2{}^t(\alpha^t) = \frac{\beta[\hat{Z}_1 + \hat{q}\hat{Z}_2 + (1 - \delta)(\hat{Z}_1{}^* + \hat{q}\hat{Z}_2{}^*)]\theta_2(\alpha^t)}{\hat{q}},$$

$$\alpha^t = 1, 2, \ldots, S, \quad t = 0, 1, \ldots$$

(11.37) $$c_1^{*t}(\alpha^t) = (1 - \beta^*)\delta(\hat{Z}_1{}^* + \hat{q}\hat{Z}_2{}^*)\theta_1(\alpha^t),$$
$$\alpha^t = 1, 2, \ldots, S, \quad t = 0, 1, \ldots$$

(11.38) $$c_2^{*t}(\alpha^t) = \frac{\beta^*\delta(\hat{Z}_1{}^* + \hat{q}\hat{Z}_2{}^*)\theta_2(\alpha^t)}{\hat{q}},$$

$$\alpha^t = 1, 2, \ldots, S, \quad t = 0, 1, \ldots$$

Substituting the commodity demand functions into the commodity market clearing condition, we obtain, using (11.31)–(11.32),

$$\hat{q} = \frac{\beta Z_1(\hat{q}) + [(1-\delta)\beta + \delta\beta^*]Z_1^*(\hat{q})}{(1-\beta)Z_2(\hat{q}) + [1-(1-\delta)\beta - \delta\beta^*]Z_2^*(\hat{q})} \tag{11.39}$$

There is a unique $\hat{q}$ for which (11.39) is satisfied (the proof is the same as that used to prove uniqueness of $\bar{q}$), and this is the equilibrium price of type-2 real equities in terms of type-1 real equities. Observe also that a comparison of (11.39) with (11.12) implies

$$\hat{q} \gtreqless \bar{q} \qquad \text{as} \quad \beta \gtreqless \beta^* \tag{11.40}$$

Hence, if the home country has a stronger preference for the second commodity than the foreign country, the relative price of type-2 real equities will now be larger than in the case of no trade in securities and so will be the relative price of commodity 2, and if the foreign country has the stronger preference for the second commodity, q will now be lower and so will be $p_2(\alpha)/p_1(\alpha)$. If both countries have the same preference, then $\hat{q} = \bar{q}$ and there is no change in q as a result of the opening of international trade in equities. The reader can also see that in this case the price structure of the home country becomes the equilibrium price structure.

It remains to show that there exist functions $V^t(\cdot)$ and $V^{*t}(\cdot)$ for which the solution to (11.5), given the price structure, is (11.33)–(11.38). We present now the functions $V^{*t}(\cdot)$ (the functions $V^t(\cdot)$ are the same except that asterisks are omitted from the variables), and the reader can verify that the equilibrium allocation solves (11.5):

$$\begin{aligned} V^{*t}(z_1^{*t-1}, z_2^{*t-1}, \alpha^t) &\equiv V^*(z_1^{*t-1}, z_2^{*t-1}, \alpha^t) \\ &\equiv \log[\theta_1(\alpha^t)]^{1-\beta^*}[\theta_2(\alpha^t)]^{\beta^*}(\hat{q})^{-\beta^*} \\ &\quad + \frac{\delta^*}{1-\delta^*} \mathop{E}_{\alpha} \log[\theta_1(\alpha)]^{1-\beta^*}[\theta_2(\alpha)]^{\beta^*}(\hat{q})^{-\beta^*} \\ &\quad + \frac{1}{1-\delta^*} \log[\delta(\hat{Z}_1^* + \hat{q}\hat{Z}_2^*) \\ &\quad + (1-\delta)(z_1^{*t-1} + \hat{q}z_2^{*t-1})] \\ &\quad \alpha^t = 1, 2, \ldots, S, \quad t = 0, 1, \ldots \end{aligned} \tag{11.41}$$

Finally, let us comment on the likelihood of the existence of stationary solutions of the type presented in the example; that is, stationary solutions in which equity prices and factor allocations are state and time independent. We believe that this type of a stationary equilibrium will rarely exist, and that it may well be the case that the logarithmic utility function is the only one that yields it. The shape of the production functions is not important for this matter.

Our belief is based on the following reasoning. A necessary condition for (11.6) to have a solution with a state-independent portfolio choice is

(a) to have a utility function for which the marginal utility of income depends only on income, and
(b) to have state-independent income.

For income to be state independent, prices have to be proportional to the inverse of the technological coefficients, which seems to us can happen only with a Cobb–Douglas-type utility function. However, for a Cobb–Douglas-type utility function to have the marginal utility of income depend only on income, it has to be logarithmic.

REFERENCES

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