On the Parcel Method and the Baroclinic Wedge of Instability

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(Manuscript received 23 October 1996, in final form 2 July 1997)

ABSTRACT

The heuristic explanation, suggested by the parcel method, for the baroclinic instability mechanism is reexamined. The parcel method argues that an air parcel displaced within the wedge of instability, that is, between the horizontal and the isentropes, is vertically accelerated by the buoyancy force and hence becomes unstable. However, in the synoptic scale, the buoyancy is balanced by the vertical pressure gradient force perturbation, which is neglected by the parcel method, and thus the parcel acceleration is essentially horizontal. For the unstable Eady normal modes, the horizontally averaged buoyancy work is found to maximize at the steering level and to vanish at the boundaries, but the horizontally averaged parcel kinetic energy growth is minimized at the steering level and maximized at the boundaries. It is shown that the buoyancy work is vertically redistributed by the pressure gradient force perturbation throughout the secondary circulation.

The parcel method also assumes that a parcel displaced adiabatically within the wedge of instability finds itself warmer than its new surroundings and thus contributes toward both vertical and meridional positive heat fluxes. However, since the temperature difference between the parcel and the environment from which it departed cannot be neglected, the slope of the instantaneous displacement is not a sufficient criterion to determine the signs of the heat fluxes. It is shown here that for the Eady normal modes solution, the four combinations of ascending or descending of initially colder or warmer parcels make jointly the vertical heat flux maximize at the steering level and the meridional heat flux remain constant with height.

1. Introduction

Since Eady (1949), the parcel method is often used to provide a heuristic explanation for the basic mechanism of the baroclinic instability, for example, Green (1960), Pedlosky (1987), Thorpe et al. (1989), and Holton (1992). This explanation contains two basic arguments as follows.

When an air parcel is displaced within the *wedge of instability,* that is, when the displacement slope in the y–z plane is between the horizontal and the isentropic slope, then the buoyancy force b' performs positive work on the parcel (Fig. 1). Therefore, although the atmosphere is stably stratified to vertical displacements, the buoyancy force rather than being restoring will vertically accelerate the parcel from its initial position increasing the parcel kinetic energy. This argument is associated with the instability of the parcel itself and will be denoted hereafter as the *buoyancy argument.*

Also, a parcel displaced adiabatically within the wedge of instability, upward and poleward, becomes warmer than its new environment. This contributes toward both vertical and meridional positive heat flux perturbations. Positive vertical heat flux converts perturbation available potential energy to perturbation kinetic energy, while positive meridional heat flux extracts perturbation energy from the mean flow. This argument is associated with the instability of the flow as a whole and will be denoted hereafter as the *heat flux argument.*

The *buoyancy argument* is based on the parcel method assumption that an air parcel is moving sufficiently slowly to instantaneously adjust its pressure to its surroundings. Therefore, the perturbed pressure gradient force, $-\rho_s^{-1}\nabla_3 p'$ (hereafter PGF$'$, where ρ is the density, p the pressure, the subscript s indicates the rest state, and prime the perturbation), is neglected when compared to the buoyancy and the latter remains the single force to perform work on the parcel. But the adjusting pressure assumption seems to be not applicable to baroclinic instability since in large-scale dynamics the PGF$'$ is by no means negligible. This can be easily shown from the vertical momentum equation

$$\frac{dw'}{dt} = b' - \frac{1}{\rho_s}\frac{\partial p'}{\partial z},$$

where w' is the vertical velocity,

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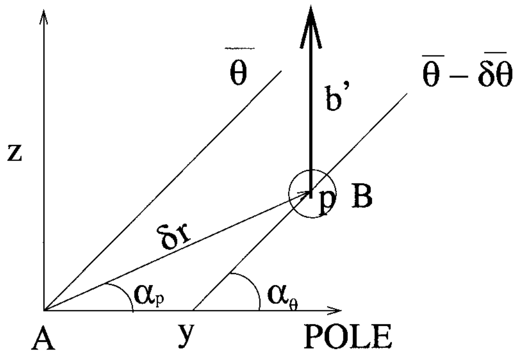


FIG. 1. A parcel displacement $\delta\mathbf{r}$ in the y–z plane, from initial position A to position B, with an angle slope α_p, in a stably stratified baroclinic atmosphere. Isentropic cross sections surfaces, inclined by the α_θ angle, are indicated by the solid lines. The buoyancy force is represented by $\mathbf{b}'$.

$$b' = -\frac{\rho'}{\rho_s}g = \frac{\theta'}{\theta_s}g$$

is the buoyancy, g the gravity, and θ is the potential temperature. The parcel method neglects the vertical component of the PGF′ (the latter term of the rhs) versus the buoyancy (the former), while under the quasi-hydrostatic balance, the buoyancy is canceled at leading order by the vertical component of the PGF′ and the acceleration of the parcel is essentially horizontal.

The *heat flux argument* refers to the *instantaneous* displacement and tacitly assumes that the displaced parcel was in a perfect balance with the environment from which it departed, position A in Fig. 1. Hence $\theta'(A) \equiv \theta_p(A) - \theta_e(A) = 0$, where θ is the potential temperature, and the subscripts p and e indicate the parcel and the environment, respectively. But $\theta'(A)$ does not vanish in general.[1] Thus, if a parcel locally colder than its environment, that is, $\theta'(A) < 0$, is lifted adiabatically poleward to B within the wedge of instability and remains colder than its new environment at position B, it certainly contributes toward *negative* heat fluxes.

In this paper we use the Eady normal modes solution to reexamine these two arguments. In section 2, we compare the role of the PGF′ to the role of the buoyancy in determining the parcel and flow instabilities. The contribution of the initial temperature perturbation to the heat fluxes is demonstrated in section 3. Concluding remarks appear in section 4.

[1] Here we implicitly relate Lagrangian properties (parcel, environment) to Eulerian ones (perturbation, mean flow). Under the linearity assumption, the mean flow remains constant with time and hence can be identified with the parcel's undisturbed environment. Thus if $\mathcal{F}(\mathbf{r}, t)$ represents some property of the fluid, at location $\mathbf{r}$ and time t, it can be associated with the parcel there so that $\mathcal{F}_p(\mathbf{r}, t) \equiv \mathcal{F}(\mathbf{r}, t) \equiv \overline{\mathcal{F}}(\mathbf{r}) + \mathcal{F}'(\mathbf{r}, t) \equiv \mathcal{F}_e(\mathbf{r}) + \mathcal{F}'(\mathbf{r}, t)$, where the bar denotes the mean flow.

2. PGF′ versus buoyancy

The volume average of the perturbed kinetic energy growth of a Boussinesq flow in the quasigeostrophic framework with no horizontal mean shear, can be written, for example, Pedlosky (1987, p 522), as

$$\frac{\partial\langle K'\rangle}{\partial t} = \langle w'b'\rangle, \tag{1}$$

where $K' = (u_g'^2 + v_g'^2)/2$ is the geostrophic perturbation kinetic energy and $\langle\ \rangle$ indicate a 3D channel volume average; $\langle w'b'\rangle$ can be interpreted either as the volume-averaged vertical heat flux or as the averaged buoyancy work. Choosing the latter interpretation, we may say that Eq. (1) manifests the dependency of the flow instability on the averaged buoyancy work.

On the other hand, the quasigeostrophic Lagrangian kinetic energy growth of a single parcel is equal to the work that is performed by the the total *horizontal* PGF; that is,

$$\frac{d_g}{dt}K_p = \frac{d_g}{dt}\left[\frac{(\overline{U} + v_g')^2 + v_g'^2}{2}\right]$$
$$= -\frac{1}{\rho_s}(\overline{\mathbf{v}} + \mathbf{v}')\cdot\boldsymbol{\nabla}_2(\overline{p} + p') = \frac{d_gK'}{dt} + \overline{U}fv_a', \tag{2}$$

where

$$\frac{d_gK'}{dt} = -\frac{1}{\rho_s}\mathbf{v}'\cdot\boldsymbol{\nabla}_2 p' = -\frac{1}{\rho_s}\mathbf{v}_a'\cdot\boldsymbol{\nabla}_2 p',$$
$$\frac{dg}{dt} = \frac{\partial}{\partial t} + u_g\frac{\partial}{\partial x} + v_g\frac{\partial}{\partial y} \tag{3}$$

is the quasigeostrophic Lagrangian time derivative, the upper bar indicates the temporal zonal mean; $\rho_s^{-1}\cdot\boldsymbol{\nabla}_2(\overline{p} + p')$ is the total horizontal PGF; $\mathbf{v}$ is the horizontal wind vector (where $\overline{\mathbf{v}} = (\overline{U}, 0)$); $\mathbf{v}_a'$ the horizontal ageostrophic wind vector where v_a' is its meridional component, and f the Coriolis parameter. However, when the quasi-hydrostatic assumption,

$$\frac{1}{\rho_s}\frac{\partial(\overline{p}+p')}{\partial z} = (\overline{b} + b'),$$

where

$$\overline{b} = \frac{\overline{\theta}}{\theta_s}g,$$

is applied, Eqs. (2), (3) become

$$\frac{d_gK_p}{dt} = -\frac{1}{\rho_s}(\overline{\mathbf{u}} + u')\cdot\boldsymbol{\nabla}_3(\overline{p} + p') + w'\overline{b} + w'b', \tag{4}$$

where $\mathbf{u}$ is the three-dimensional wind vector and $\rho_s^{-1}\boldsymbol{\nabla}_3(\overline{p} + p')$ is the total 3D PGF′ (excluding the rest state vertical component $-\rho_s^{-1}(\partial p_s/\partial z)$). If we average Eq. (4) on a 3D channel domain, that is, the flow is confined vertically and meridionally by four rigid walls, we will end up, for a zonally periodic perturbation, un-

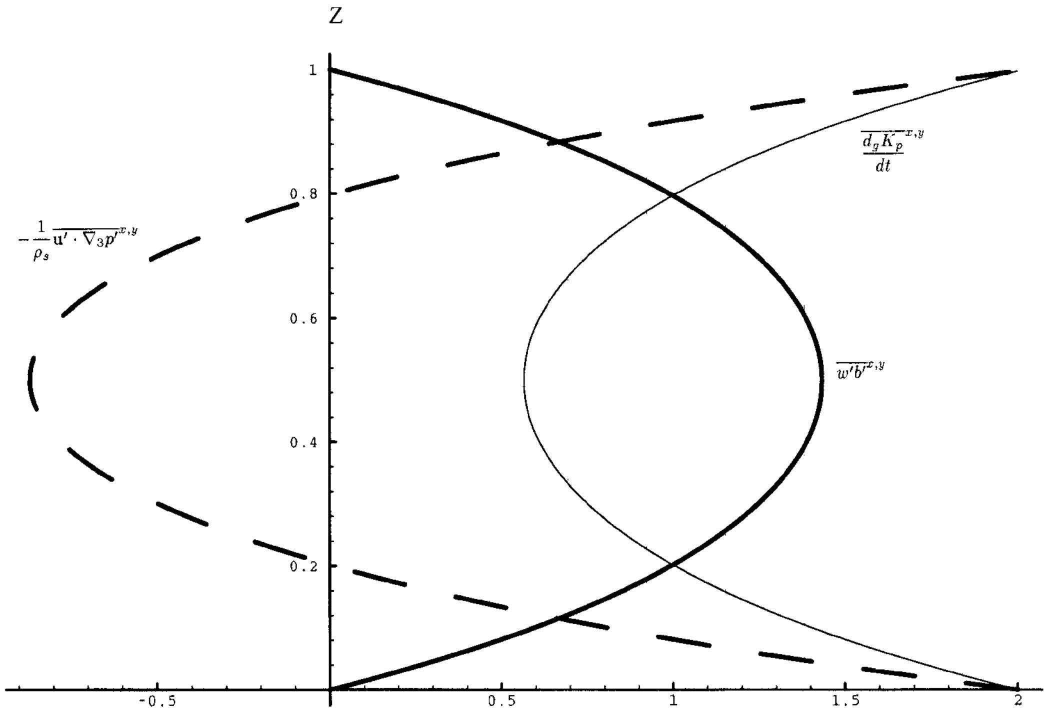


FIG. 2. Three horizontally averaged and normalized terms of Eq. (5). The parcel kinetic energy growth, the buoyancy work, and the work of the PGF′ are represented by the solid, heavy solid, and dashed lines, respectively.

der the Boussinesq approximation, with Eq. (1), that is, $d_g\langle K_p\rangle/dt = \partial\langle K'\rangle/\partial t = \langle w'b'\rangle$. In order to demonstrate the role of the PGF′ versus that of the buoyancy in determining the parcel kinetic energy growth, let us average Eq. (4) over the horizontal plane only,

$$\overline{\frac{d_g K_p}{dt}}^{x,y} = -\frac{1}{\rho_s}\overline{\mathbf{u}'\cdot\boldsymbol{\nabla}_3 p'}^{x,y} + \overline{w'b'}^{x,y}. \qquad (5)$$

Each term in this equation is normalized by the domain-averaged kinetic energy perturbation $\langle K'\rangle$ and by twice the growth rate, and then plotted in Fig. 2 for the most unstable mode of the Eady problem [i.e., zonal and meridional wave numbers $k = 3.1277$, $l_0 = \pi/2$, respectively, with the terminology as in Pedlosky (1987)]. As expected, the buoyancy work term, associated with the flow instability, is maximized right at the steering level and is zero at the upper and the lower boundaries. However, the parcel kinetic energy growth behaves in the opposite manner: it is minimized at the steering level and maximized at the upper and lower boundaries (note that this term is proportional to the square of the amplitude of the Eady unstable modes). In other words, on average, parcels displaced at the steering level gain minimal kinetic energy, although their contribution to the flow instability is maximal. By the same token, the parcels displaced horizontally, right at the boundaries, do not contribute at all to the flow instability but nevertheless gain maximal kinetic energy. This redistribution of the kinetic energy from the steering level throughout the fluid depth to the horizontal boundaries is due to the PGF′, see Fig. 2. Hence, the vertical structure of the perturbation kinetic energy growth in Fig. 2 (which is similar for all the unstable modes) cannot be explained by the buoyancy argument, which neglects the PGF′. The energy is redistributed by the secondary circulation energy flux, as can be shown from the relations

$$\begin{aligned}\overline{\frac{d_g K_p}{dt}}^{x,y} = \overline{\frac{d_g K'}{dt}}^{x,y} &= \overline{-\frac{1}{\rho_s}\boldsymbol{\nabla}\cdot(\mathbf{u}_a' p')}^{x,y} + \overline{w'b'}^{x,y}\\ &= \overline{-\frac{1}{\rho_s}\frac{\partial(w'p')}{\partial z}}^{x,y} + \overline{w'b'}^{x,y}\\ &= -\overline{\frac{p'}{\rho_s}\frac{\partial w'}{\partial z}}^{x,y} = \overline{\frac{p'}{\rho_s}\left(\frac{\partial u_a'}{\partial x} + \frac{\partial v_a'}{\partial y}\right)}^{x,y}. \qquad (6)\end{aligned}$$

In the same manner, the PGF′ redistributes the horizontally averaged meridional heat flux,

$$\overline{\frac{g}{\theta_s}\frac{f}{N^2}\frac{\partial\overline{U}}{\partial z}v'\theta'}^{x,y}$$

which is constant with height for the Eady normal modes, so that the total energy change would be minimized at the steering level and maximized at the horizontal boundaries. Similar vertical redistribution of the turbulent kinetic energy, by the PGF′, is also found in the planetary boundary layer; see Stull (1988).

In the next section we use the Eady growing normal modes solution to reexamine the heat flux argument.

3. The heat fluxes and the wedge of instability

The heat flux argument claims that a parcel displaced within the wedge of instability contributes toward positive heat fluxes and hence toward the instability of the flow. Thus, in order to validate this argument we plotted the parcel's instantaneous displacements in the y–z plane for three normal modes: $k = 0.5, 3.1277, 4.5$; Figs. 3a,b,c. These modes represent the long waves, the most unstable mode, and the almost neutral short waves, respectively. (Notice that in Fig. 3 the parcel's displacements, as well as the angle of the basic-state isentropes, indicated by the dashed lines, are in the y–z plane while variations are along the x axis.) These figures seem to fit nicely to the description of the baroclinic instability as a sort of a slantwise convection and support the heat flux argument. For the unstable modes, almost all the displacements lie within the wedge of instability. Near the horizontal boundaries the displacement slopes are forced to be horizontal, but the trajectories at the steering level vary from being almost horizontal for small k to almost parallel to the isentropes for the almost neutral critical k_c. For the most unstable mode $k_{\max} = 3.1277$, the trajectory slope, at the steering level, is equal to 0.61 (and 0.53 for the 2D, $l_0 = 0$, most unstable mode) of the isentropic slope. Recall that half of the isentropic slope is the optimal slope to generate kinetic energy perturbation[2]; Eady (1949). Trying to deduce from Figs. 3a,b what would be the vertical structure of the horizontally averaged vertical heat fluxes, $\overline{w'b'}^{x,y}$, we would expect it to maximize at the vicinity of the steering level and vanish at the boundaries. But, for the almost neutral (but still unstable) mode, Fig. 3c, we would expect to find a bimodal vertical structure for $\overline{w'b'}^{x,y}$ that is minimized at the steering level as well as at the boundaries and maximized somewhere in between them. However, when $\overline{w'b'}^{x,y}$ is plotted, it is found to maximize exactly at the steering level and to vanish at the boundaries for all the unstable modes. Thus, how can this paradox be explained?

The heat flux argument refers implicitly to the *instantaneous* displacement rather than the *total* displacement and neglects the initial temperature perturbation. The temperature difference θ', at some position $\mathbf{r}$ and time t is $\theta'(\mathbf{r}, t) = \theta_p(\mathbf{r}, t) - \theta_e(\mathbf{r}) = \theta_p(\mathbf{r}_0, t = 0) - \theta_e(\mathbf{r})$, where $\mathbf{r}_0$ indicates the initial position of the parcel. Thus, for adiabatic motion, $\theta'(\mathbf{r}, t)$ is determined by the initial value of the parcel's potential temperature at its initial position. Hence, the signs of the heat fluxes are affected by both the instantaneous parcel's displacement (according to v' and w') and by the total parcel's displacement (determined by θ') as demonstrated in the following.[3] Figure 4a (4b) combines the instantaneous displacement arrows diagram for $k = 4.5$, with the initial potential temperature (vertical heat flux) perturbation contours. For instance, the parcels indicated by A and B in Figs. 4a,b seem to experience similar instantaneous displacements, and both are within the wedge of instability. However, in Fig. 4a we can see that parcel A is initially warmer than its environment (solid contours) and parcel B is colder than its environment (dashed contours). Figure 4b shows that parcel B, although displaced upward and poleward, remains colder than its environment and contributes toward *negative* vertical heat flux. Actually, at each level besides the steering level, we can find all four combinations of ascending/descending of warmer/colder parcels. This leads to a partial cancellation in the heat flux when it is averaged horizontally. Since at the steering level the correlation between the parcels' vertical displacements and the temperature perturbations is always positive, the horizontal average of the vertical heat flux is maximized there. From similar reasons, it can be shown that the horizontally averaged meridional heat flux remains constant with height.

Another counterexample to the heat flux argument is the neutral modes of the semi-infinite Eady model, for example, Gill (1982, §13.2, p. 550–555). For these modes, the parcels' displacements are within the wedge of instability beneath the steering level and outside the wedge of instability above it. Nevertheless, the horizontally averaged heat fluxes, as well as the horizontal average of the parcel kinetic energy growth, are all zero everywhere (independent of height).

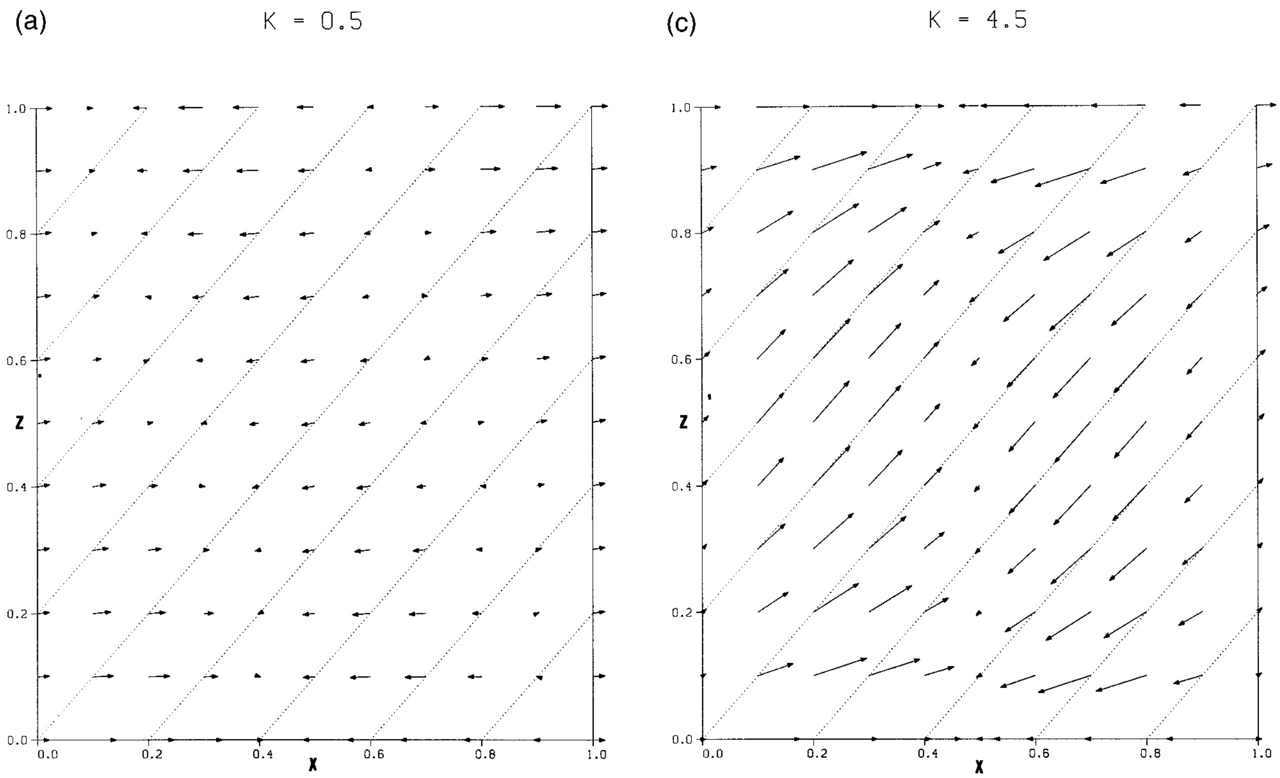
4. Concluding remarks

The parcel theory suggests a simple and intuitive explanation for the baroclinic instability mechanism and is presented in most textbooks. Therefore, we found it important to reexamine this theory and to understand its limitations.

The main argument of the theory, that parcels of an unstable baroclinic flow are displaced within the wedge of instability, was shown here to be correct for the Eady unstable normal modes. Also, the parcels' displacements, at the steering level, are indeed changing from being close to the horizontal for large wavelengths to almost being parallel to the isentropes near the shortwave cutoff. For the most unstable mode, the displacement slope is close to half of the isentropic slope, as already noticed by Eady (1949). The parcel method interprets these results by relating the parcel kinetic energy growth to the work performed by the buoyancy force. This is true only in the volume-averaged sense. Locally, the parcel is accelerated horizontally by the

[2] A short discussion on the optimal slope to extract the *total* energy perturbation appears in appendix A.

[3] A more rigorous derivation appears in appendix B.



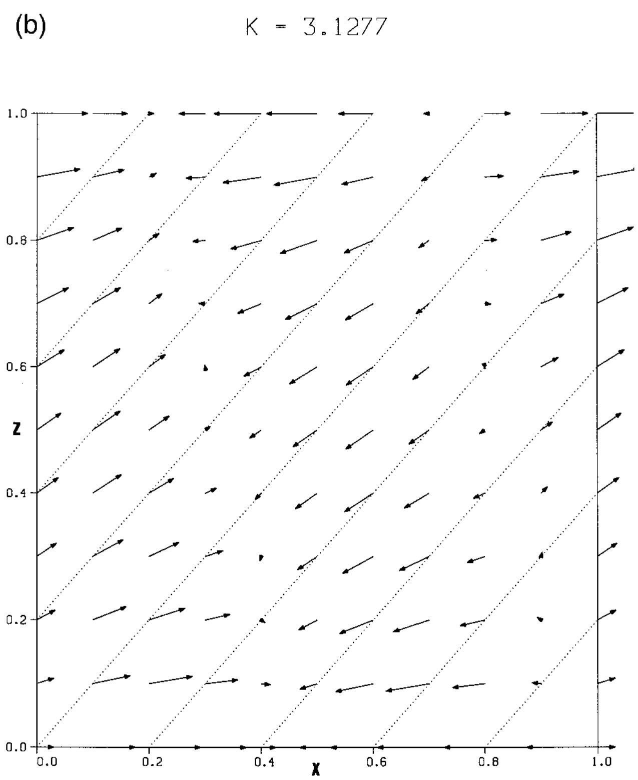


FIG. 3. Vector diagrams of the variation of the parcels displacements, in the y–z plane, along the x axis. Diagrams a, b, and c show the solution for three Eady normal modes: k = 0.5, 3.1277, 4.5, respectively, after Pedlosky (1987). Dashed lines indicate the angle of the basic-state isentropes. The x axis is distance normalized by the wavelength, while the z axis is depth normalized by the tropospheric scale height. Notice that the parcel displacements, as well as the angle of the isentropes, are in the y–z plane while variations are along the x axis.

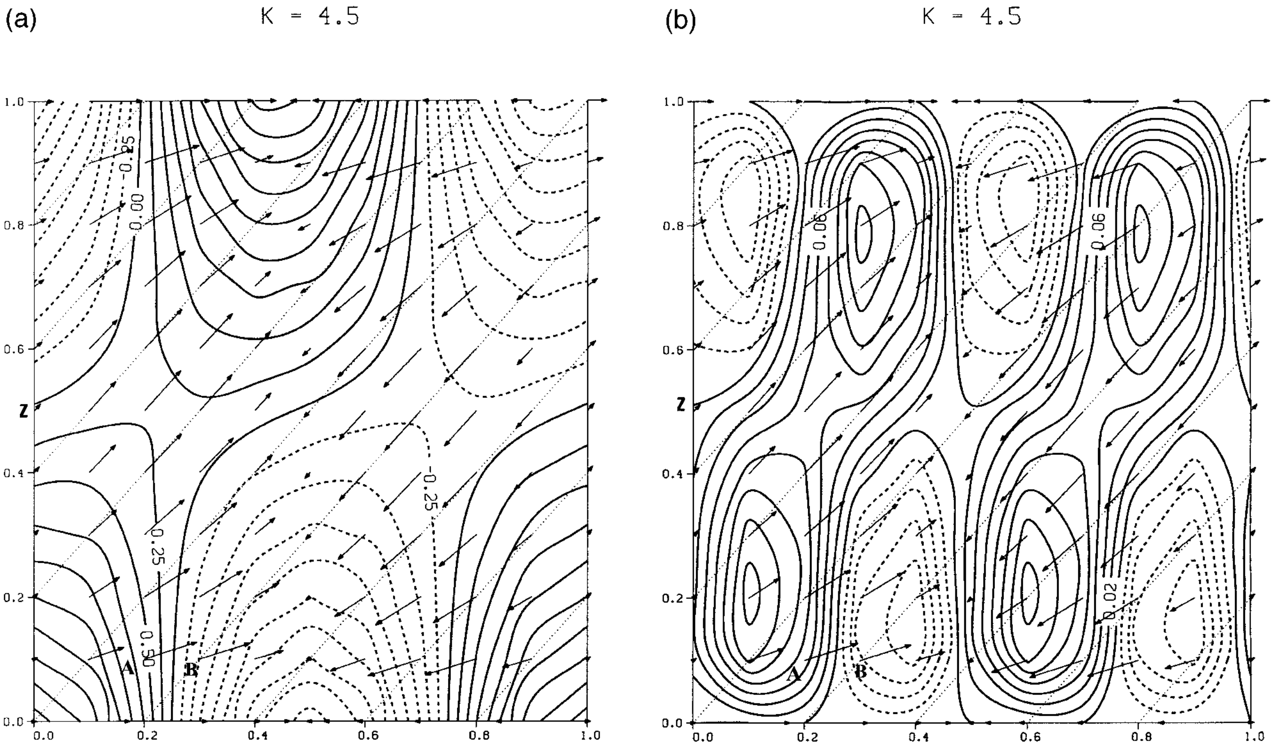


Fig. 4. Diagrams (a) and (b) combine the vector diagram from Fig. 3c (for $k = 4.5$) with the initial potential temperature θ' perturbation contours and the vertical heat flux contours $w'\theta'$, respectively. Positive (negative) contours are solid (dashed).

residual PGF′, that is, by the part of the PGF′ not balanced by the Coriolis force. The PGF′, which is intrinsically neglected by the parcel method, was shown to be responsible for the vertical redistribution of the parcel kinetic energy growth, from the steering level to the horizontal boundaries.

The parcel method suggests also that the slope of the parcel *instantaneous* displacement solely determines the signs of the heat fluxes. Here, we show that the temperature perturbation and therefore the heat fluxes are affected by the initial temperature perturbation and therefore by the *total* parcel's displacement. Thus, one cannot conclude from a snapshot of the parcel's displacement on the signs of the heat fluxes.

In summary, the parcel method, although appealing and elegant, should be considered with caution when applied to large-scale dynamics in general and to baroclinic instability in particular.

Acknowledgments. The authors wish to thank P. Bannon, C. Bishop, J. Clark, J. Evans, and S. Lee for their constructive remarks and warm encouragement.

APPENDIX A

Optimal Slope to Extract Total Energy Versus Kinetic Energy

If we refer, for a moment, only to the instantaneous displacement, then following Thorpe et al. (1989), the contribution of the parcel displacement to the vertical heat flux, that is, to the volume averaged kinetic energy growth is

$$\delta K \approx \frac{(\delta\mathbf{r})^2}{2\delta t} f \frac{\partial \overline{U}}{\partial z} \frac{\sin\alpha_p}{\sin\alpha_\theta} \sin(\alpha_\theta - \alpha_p) \qquad \text{(A1)}$$

(for more details, see appendix B). Similarly, it is easy to show that the contribution of the parcel displacement to the meridional heat flux, that is, to the averaged total energy growth E, is given by

$$\delta E \approx \frac{(\delta\mathbf{r})^2}{2\delta t} \left(\frac{f}{N} \frac{\partial \overline{U}}{\partial z}\right)^2 \frac{\cos\alpha_p}{\sin\alpha_\theta} \sin(\alpha_\theta - \alpha_p), \qquad \text{(A2)}$$

where f is the Coriolis parameter, N the Brunt–Väisälä frequency, $\overline{U}$ the mean wind, $\delta\mathbf{r}$ the parcel displacement, and α_θ, α_p are the slope angles of the isentropes and the parcel displacement, respectively (Fig. 1). Hence, the y–z plane may be divided into three regions, see Fig. A1. In the first region, which is the wedge of instability, the energy contributions to both δK and δE are positive (the growing modes). In the second region, the contributions to both δK and δE are negative (the decaying modes), while in the third region (which is not allowed for normal modes, but for transient growth solutions) the contribution is positive for the total energy δE, but negative for the kinetic energy δK; that is, the potential energy is restored but is not converted to kinetic energy.

Regions

1. $w'\theta' > 0$
   $v'\theta' > 0$
2. $w'\theta' < 0$
   $v'\theta' < 0$
3. $w'\theta' < 0$
   $v'\theta' > 0$

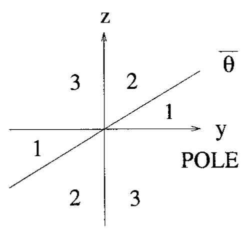


FIG. A1. The y–z plane is divided into three regions characterized by the different contributions of the parcel displacement to the averaged kinetic and total energies growth. In region 1, both of the contributions are positive. In region 2, both of the contributions are negative. In region 3 the contribution to the total energy growth is positive, while it is negative to the kinetic energy growth.

The maximal contribution to extract kinetic energy is given for the slopes,

$$\alpha_p = \frac{\alpha_\theta}{2} + n\pi \qquad \text{(region 1)}$$

and minimum (negative contribution) for the slopes,

$$\alpha_p = \frac{\alpha_\theta}{2} + (2n + 1)\frac{\pi}{2} \qquad \text{(region 3)}$$

(for $n = 0, 1, 2, \cdots$).

On the other hand, the maximal contribution to extract total energy is given for the slopes,

$$\alpha_p = \frac{\alpha_\theta}{2} + (4n + 3)\frac{\pi}{4} \qquad \text{(region 3)}$$

and minimal (negative contribution) for the slopes,

$$\alpha_p = \frac{\alpha_\theta}{2} + (4n + 1)\frac{\pi}{4}. \qquad \text{(region 2)}$$

Note that the optimal slope to extract kinetic energy lies within the wedge of instability, while the optimal slope to extract total energy lies outside of it, in region 3.

Hence, we might say in a very qualitative way that the baroclinic instability mechanism is more efficient in converting available potential energy perturbation to kinetic energy perturbation, rather than extracting the total energy perturbation from the mean flow.

APPENDIX B

Instantaneous Displacement Versus the Prior Temperature Perturbation

For small displacements, the potential temperature perturbation can be expanded to

$$\theta'(\mathbf{r}, t) = \theta'(\mathbf{r} - \delta\mathbf{r}, t - \delta t) + \frac{d\theta'}{dt}\delta t, \qquad \text{(B1)}$$

where $\mathbf{r}$ is the position vector. The parcel conserves its potential temperature and therefore,

$$\frac{d\theta'}{dt} = \frac{d\theta_p}{dt} - \frac{d\overline{\theta}}{dt} = -\frac{d\overline{\theta}}{dt}, \qquad \text{(B2)}$$

where the bar indicates the mean flow. Hence, in a baroclinic zonal symmetric basic state,

$$\frac{d\theta'}{dt}\delta t = -\left(\frac{\partial\overline{\theta}}{\partial y}\delta y + \frac{\partial\overline{\theta}}{\partial z}\delta z\right) = -\frac{\partial\overline{\theta}}{\partial z}(\delta z - \delta y \tan\alpha_\theta), \qquad \text{(B3)}$$

where $\tan\alpha_\theta$ is the isentropes slope. The meridional and the vertical displacement projections are $\delta y = \delta\mathbf{r}\cos\alpha_p$; $\delta z = \delta\mathbf{r}\sin\alpha_p$ respectively (Fig. 1). Thus with the aid of the thermal wind relation,

$\frac{g}{\theta}\frac{\partial\theta}{\partial y} = -f\frac{\partial\overline{U}}{\partial z}$, Eq. (B1) becomes

$$\theta'(\mathbf{r}, t) = \theta'(\mathbf{r} - \delta\mathbf{r}, t - \delta t) + \frac{\overline{\theta}}{g}f\frac{\partial\overline{U}}{\partial z}\,\delta\mathbf{r}\,\frac{\sin(\alpha_\theta - \alpha_p)}{\sin\alpha_\theta}. \qquad \text{(B4)}$$

Therefore, the contribution of a parcel to the vertical and meridional heat flux is, respectively,

$$\frac{g}{\theta_s}w'\theta'|_{(\mathbf{r},t)} \approx \frac{g}{\theta}\frac{\delta\mathbf{r}\sin\alpha_p}{2\delta t}\theta'|_{(\mathbf{r}-\delta\mathbf{r},t-\delta t)} + \frac{(\delta\mathbf{r})^2}{2\delta t}f\frac{\partial\overline{U}}{\partial z}\frac{\sin\alpha_p}{\sin\alpha_\theta}\sin(\alpha_\theta - \alpha_p), \qquad \text{(B5a)}$$

$$\frac{g}{\theta_s}\frac{f}{N^2}\frac{\partial\overline{U}}{\partial z}v'\theta'|_{(\mathbf{r},t)} \approx \frac{g}{\theta}\frac{f}{N^2}\frac{\partial\overline{U}}{\partial z}\frac{\delta\mathbf{r}\cos\alpha_p}{2\delta t}\theta'|_{(\mathbf{r}-\delta\mathbf{r},t-\delta t)} + \frac{(\delta\mathbf{r})^2}{2\delta t}\times\left(\frac{f}{N}\frac{\partial\overline{U}}{\partial z}\right)^2\frac{\cos\alpha_p}{\sin\alpha_\theta}\sin(\alpha_\theta - \alpha_p). \qquad \text{(B5b)}$$

The first terms on the rhs of Eqs. (B5) represent the prior perturbation contribution, while the second terms represent the contribution associated with the instantaneous parcel displacement [Thorpe et al. (1989); they ignore the first terms]. The heat flux argument neglects the former terms when compared to the latter.

Actually, the condition on the parcel to be displaced within the wedge of instability,

$$\frac{w'}{v'} < \left(\frac{\delta z}{\delta y}\right)_{\overline{\theta}} = -\frac{\partial\overline{\theta}/\partial y}{\partial\overline{\theta}/\partial z}, \qquad \text{(B6)}$$

where $(\delta z/\delta y)_{\overline{\theta}}$ is the isentropic slope, is different from the conditions on the heat fluxes to be positive, as can be shown in the following. Substituting

$$w' = -\left[\frac{d\theta'}{dt} + v'\frac{\partial\overline{\theta}}{\partial y}\right]\Big/\frac{\partial\overline{\theta}}{\partial z}$$

into (B6) yields

$$\frac{d\theta'}{dt}\Bigg/\left(v'\frac{\partial\overline{\theta}}{\partial z}\right) > 0.$$

Since $\partial\overline{\theta}/\partial z$ is positive in a stratified atmosphere and also w'/v' is positive in the wedge of instability, then Eq. (B6) leads to

$$v'\frac{\mathrm{d}\theta'}{dt} > 0 \tag{B7a}$$

$$w'\frac{d\theta'}{dt} > 0, \tag{B7b}$$

which are different from the heat flux conditions for instabilities, $v'\theta' > 0$; $w'\theta' > 0$. In fact, for the wavenumber $k = 4.5$

$$\overline{w'\frac{d\theta}{dt}}^{x,y}$$

has a bimodal structure, while $\overline{w'\theta'}^{x,y}$ is maximized at the steering level, as discussed in section 3.

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