Q. J. R. Meteorol. Soc. (1994), **121**, pp. 227–228 551.511.32

A comment on the geostrophic wind divergence

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(Received 17 January 1994; revised 18 February 1994)

The geostrophic wind divergence was given by Houghton (1989) as

$$\nabla_{\mathrm{H}} \cdot \mathbf{V}_{\mathrm{g}} = -\frac{v_{\mathrm{g}}}{r}(\tan\phi + \cot\phi), \tag{1}$$

where ∇_{H} is the horizontal divergence, $\mathbf{V}_{\mathrm{g}} = (u_{\mathrm{g}}, v_{\mathrm{g}})$ the geostrophic wind vector, r the distance from the earth's centre and ϕ the latitude. Although the two terms on the right-hand side of (1) appear as a result of the earth's sphericity, it is suggested from what follows that the $\tan\phi$ term is superfluous and should be omitted.

The horizontal wind divergence on a spherical surface was given by Pedlosky (1979) as

$$\nabla_{\mathrm{H}} \cdot \mathbf{V} = \frac{1}{r\cos\phi}\left\{\frac{\partial u}{\partial\lambda} + \frac{\partial}{\partial\phi}(v\cos\phi)\right\} \tag{2}$$

where λ is the longitude.

In the common coordinate system in use in meteorology for which x and y are directed eastwards and northwards, respectively, one can substitute dx for $r\cos\phi\,\mathrm{d}\lambda$ and dy for $r\,\mathrm{d}\phi$. Equation (2) then yields (e.g. Haltiner and Williams 1980)

$$\nabla_{\mathrm{H}} \cdot \mathbf{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - \frac{v\tan\phi}{r}. \tag{3}$$

The last term in (3) results directly from the eastwards variation of the unit vector $\mathbf{j}$ in this 'quasi-Cartesian' coordinate system (Holton 1979), given by

$$\frac{\partial\mathbf{j}}{\partial x} = -\frac{\tan\phi}{r}\mathbf{i},$$

and can be derived from

$$\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}\right) \cdot (\mathbf{i}u + \mathbf{j}v).$$

This coordinate system is not strictly Cartesian because the directions of the unit vectors vary over the sphere.

Substituting

$$\mathbf{V} = \mathbf{V}_{\mathrm{g}} = \left(-\frac{1}{\rho f}\frac{\partial p}{\partial y}, \frac{1}{\rho f}\frac{\partial p}{\partial x}\right)$$

into (3) and assuming that ρ is constant (for the corresponding expression with horizontal density variations, see e.g. Dutton 1976) yields

$$\nabla_{\mathrm{H}} \cdot \mathbf{V}_{\mathrm{g}} = \underbrace{-\frac{1}{\rho f}\frac{\partial^2 p}{\partial x\,\partial y}}_{A} + \underbrace{\frac{1}{\rho f}\frac{\partial^2 p}{\partial y\,\partial x}}_{B} - \frac{v_{\mathrm{g}}}{r}\cot\phi - \frac{v_{\mathrm{g}}}{r}\tan\phi. \tag{4}$$

The $\cot\phi$ term is the $\beta \equiv \partial f/\partial y$ contribution since

$$-\frac{v_{\mathrm{g}}}{r}\cot\phi \equiv -v_{\mathrm{g}}\frac{\beta}{f}.$$

If the first two terms A and B did indeed cancel each other, then (4) would have simplified

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down to (1). But this seems not to be the case here because sphericity is involved, and in spherical coordinates the dx, dy differentiations are not interchangeable. In fact, substituting $dx = r\cos\phi\, d\lambda$, $dy = r\, d\phi$ in the terms A, B in (4) yields

$$A + B = -\frac{1}{\rho f}\frac{\partial}{r\cos\phi\,\partial\lambda}\left(\frac{\partial p}{r\,\partial\phi}\right) + \frac{1}{\rho f}\frac{\partial}{r\,\partial\phi}\left(\frac{\partial p}{r\cos\phi\,\partial\lambda}\right) = +\frac{v_g}{r}\tan\phi. \tag{5}$$

Now, (5) and (4) together give the geostrophic wind divergence equation, viz.

$$\nabla_H \cdot \mathbf{V}_g = -\frac{v_g}{r}\cot\phi, \tag{6}$$

Notice that the $\tan\phi$ term from (1) has cancelled out. Equation (6) would seem, therefore, to be the proper representation for the geostrophic wind divergence (see also e.g. Pedlosky 1979; Cammas and Ramond 1989).

The $\cot\phi$ term is quite small and is generally neglected. It is interesting to note that it was suggested as an explanation of the asymmetry observed in the Intertropical Convergence Zone (ITCZ) in which the vertical velocities and precipitation south of the convergence line are greater than those to the north of it. Falkovich (1979) argues that, when integrating the boundary-layer divergence, the $-(\beta/f)v_g$ or $\cot\phi$ term is the principal one because its contribution increases with height whereas the other terms are limited. To the south of the ITCZ axis $v_g > 0$. Thus the beta effect should lead to an increase of the boundary-layer convergence to the south of the ITCZ centre and to a decrease to the north of it. At 5 to 10 degrees latitude the value of $\cot\phi$ is in the range 5 to 11, which is one to two orders of magnitude larger than $\tan\phi$. This scaling, however, is not the reason for omitting the $\tan\phi$ term, as has been discussed above.

Acknowledgement

We thank A. Khain for the reference to Falkovich's work.

References

Cammas, J. P. and Ramond, D. 1989 Analysis and diagnosis of the composition of ageostrophic circulations in jet-front systems. *Mon. Weather Rev.*, **117**, 2447–2462

Dutton, J. A. 1976 *The ceaseless wind, An introduction to the theory of atmospheric motion.* McGraw-Hill, New York

Falkovich, A. I. 1979 *Dynamics and energetics of the intertropical convergence zone.* Gidrometizdat, Leningrad. (in Russian)

Haltiner, G. J. and Williams, R. T. 1980 *Numerical prediction and dynamic meteorology*, 2nd ed. John Wiley, New York

Holton, J. R. 1979 *An introduction to dynamic meteorology*, 2nd ed., Academic Press, New York

Houghton, J. T. 1989 *The physics of atmospheres*, 2nd ed., reprinted 1991. Cambridge University Press, Cambridge

Pedlosky, J. 1979 *Geophysical fluid dynamics.* Springer-Verlag, New York