Aggregating Non-Additive Beliefs

*** Extremely preliminary ***

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1 Preliminaries

A single cooperative game, $G = (N; v)$ is defined by a set of events N and a single characteristic function v which assigns a real number to every non empty set of events $S \in P(N)$ ($P(N) \equiv \{S \neq \phi | S \subseteq N\}$) and zero to the empty set. Typically $v(S)$ is interpreted as the prior the agent hold for the subset of events S. We extend this definition to our setting of multiple priors by defining a multi-game as a set of events and a set of characteristic functions.

Definition 1. *An m-prior Multi-Game $\tilde{G}$ is a pair $\tilde{G} = (N; V)$ where V is a set of characteristic functions $V = \{v_1, v_2, \ldots, v_m\}$ such that for every $v_j \in V$, $v_j : P(N) \to \mathbb{R}$, $v_j(\emptyset) = 0$ and $v_j(N) = 1$.*

It will be convenient to denote the single cooperative game that is defined by the j^{th} characteristic function of the multi-game $\tilde{G}$ by $\tilde{G}_j = (N; v_j)$.

An aggregate additive probability measure of the multi-game $\tilde{G}$ describes the probability assigned to each event taking all the multiple priors into account.

Definition 2. *$x \in \mathbb{R}^n_+$ is an aggregate additive probability measure of the multi-game $\tilde{G} = (N; \{v_1, v_2, \ldots, v_m\})$ if $\sum_{i=1}^{n} x_i = 1$.*

Definition 3. *Let $G = (N; v)$ be a single cooperative game where v is monotonic and non-negative. Let X be a vector of individual resources (random variable in Lehrer (2009)). Define $\int^{cav} X dv = \min\{f(X)\}$ where the minimum is taken over all concave and homogeneous functions $f : R^n \to R$ such that for every $S \subseteq N$, $f(\chi^S) \geq v(S)$.*

Lemma 1 (Lemma 1(i) in Lehrer (2009))**.** *For every X,*

$$\int^{cav} X dv = \max\left\{ \sum_{S \in P(N)} F(S)v(S) | \sum_{S \in P(N)} F(S)\chi^S = X, \forall S \in P(N) : F(S) \geq 0 \right\}$$

2 Decomposition Lemma

The lemma provides necessary and sufficient condition for the decomposition of a given efficient aggregate payoff vector x into m vectors such that the first belongs to the core of the first issue, the second to the core of second issue and so on. This will identify those elements in the multi-core that are trivial and those that are not.

The following generalizes the Shapley-Bondareva's system of balancing sets to account for systems of weights whose total weights may differ across players.

Definition 4. *Let $F : P(N) \to \mathbb{R}_+$ be a system of weights. The vector of weights induced by F is denoted by $W^F = \sum_{S \in P(N)} F(S)\chi^S$. We say that F_1 and F_2 are W-equivalent if $W^{F_1} = W^{F_2}$.*

The W-equivalence relation induces a partition on the set of all system of weights.[1] Let us denote the set of all W-equivalence classes by Γ. Further, for every class $\gamma \in \Gamma$, we denote the players weights by W^γ and for every characteristic function v, $T_v^\gamma \equiv \max_{F \in \gamma} \sum_{S \in P(N)} F(S)v(S)$.

[1] The set of all systems of Shapley-Bondareva's balancing weights is identical to the class of functions F such that W^F is the vector of ones.

Lemma 2. *Let $x \in \mathbb{R}^n$ be a an efficient payoff vector. There exist m vectors $x_1, ..., x_m$ such that $\forall v_j \in V : x_j \in C(\tilde{G}_j)$ and $\sum_{v_j \in V} x_j = x$ if and only if every class $\gamma \in \Gamma$ satisfies $\sum\limits_{v_j \in V} T^{\gamma}_{v_j} \leq \sum\limits_{i \in N} W^{\gamma}[i] x_i$.*

To prove this lemma we construct a set of linear inequalities that characterize the set of all the decompositions of a given payoff vector such that all vectors belong to the cores of the respective individual games. We use Farkas' Lemma, or alternatively, the hyperplane separation theorem, to show that this set of inequalities has a solution if and only if the above condition is satisfied. We conclude the proof by proving the following conclusion that establishes that the lemma holds for a single game as well as for multiple games. Trivially, if one of the games has an empty core this condition is violated.[2]

Conclusion 3. *Denote by v_0 the characteristic function that attaches 0 to every coalition. Let $G = (N; v)$ be a cooperative game and let $\tilde{G} = (N; \{v, v_0\})$. $x \in C(G)$ if and only if every class $\gamma \in \Gamma$ satisfies $T^{\gamma}_v \leq \sum_{i \in N} W^{\gamma}[i] x_i$.*

Conjecture 4. *Suppose τ is the Bloch and de Clippel (2010) partition on characteristic functions. v and v' are members of the same equivalence set in τ if and only if for every γ, $\arg\max_{F \in \gamma} \sum_{S \in P(N)} F(S) v(S) = \arg\max_{F \in \gamma} \sum_{S \in P(N)} F(S) v'(S)$*

[2]Suppose $C(G_1)$ is empty. Consider the W-equivalence class γ where $W^{\gamma} = 1$. Then, by the Bondareva-Shapley Theorem there exists one F_1 such that $\sum_{S \in P(N)} F_1(S) v_1(S) > v_1(N)$. For $\forall j = 2, ..., m$ let $F_j(S) = 0$ for every $S \subset N$ and $F_j(N) = 1$ which leads to $\sum_{S \in P(N)} F_j(S) v_j(S) = v_j(N)$. Then,

$$\sum_{v_j \in V} \sum_{S \in P(N)} F_j(S) v_j(S) = \sum_{S \in P(N)} F_1(S) v_1(S) + \sum_{v_j \in V \setminus \{v_1\}} \sum_{S \in P(N)} F_j(S) v_j(S) >$$
$$v_1(N) + \sum_{j=2}^{m} v_j(N) = \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} W^{\gamma}[i] x_i$$

Therefore, there exists a $\gamma \in \Gamma$ such that $\sum_{v_j \in V} T^{\gamma}_{v_j} > \sum_{i=1}^{n} W^{\gamma}[i] x_i$ which violates the condition in Lemma 2.

3 Concave Integrals

Conclusion 5. *For every* $\gamma \in \Gamma$, $\int^{cav} W^{\gamma} dv = T^{\gamma}_{v}$.

Lemma 6 (The Sum of Individual Cores is Closed and Convex)**.** $C(\tilde{G})$ *is a closed and convex set.*

Proof. The set of vectors that satisfy a weak linear inequality is a closed set. Therefore. the core of a single game is a closed set. Therefore, the set of vectors that can be represented as sum of core elements is closed.

Next, let $x, y \in C(\tilde{G})$. There exist $2m$ vectors $x_1, ..., x_m$ and $y_1, ..., y_m$ such that $\forall v_j \in V : x_j \in C(\tilde{G}_j), y_j \in C(\tilde{G}_j)$ and $\sum_{v_j \in V} x_j = x$ and $\sum_{v_j \in V} y_j = y$. The m vectors $\lambda x_1 + (1-\lambda) y_1, ..., \lambda x_m + (1-\lambda) y_m$ satisfy $\forall v_j \in V : \lambda x_1 + (1-\lambda) y_1 \in C(\tilde{G}_j)$ due to the convexity of the core of a single game. Since these vectors sum to $\lambda x + (1-\lambda) y$, $\lambda x + (1-\lambda) y \in C(\tilde{G})$. □

Lemma 7. *Let* $G = (N, v)$ *be a cooperative game. Let* H *be the set of extreme points of* $\hat{H}$ *where* $\hat{H} = \{h \in \mathbb{R}^n | \forall S \in P(N) : h(S) \geq v(S)\}$. *Then, for every vector* $Y \in \mathbb{R}^n$, $\int^{cav} Y dv = \min_{h \in H} Y \cdot h$.

Proof. By Lemma 1,

$$\int^{cav} Y dv = \max_{F \in \mathbb{R}^{2^n}} \left\{ \sum_{S \in P(N)} F(S) v(S) | \sum_{S \in P(N)} F(S) \chi^S = Y, \forall S \in P(N) : F(S) \geq 0 \right\}$$

By the strong duality theorem,

$$\int^{cav} Y dv = \min_{h \in \mathbb{R}^n} \left\{ Y \cdot h | \forall S \in P(N) : h(S) \geq v(S) \right\}$$

Then, $\displaystyle\int^{cav} Y dv = \min_{h \in \hat{H}} Y \cdot h$. Moreover, for every $Y \in U$, $Y \cdot h$ is a linear function of h. Since $\hat{H}$ is convex, the minimum is achieved on (at least) one of the extreme points. Since $H = \{h \in \hat{H} | h \text{ is an extreme point}\}$, $\displaystyle\int^{cav} Y dv = \min_{h \in H} Y \cdot h$. □

Lemma 8. *For every $G = (N, v)$ with non-empty core there is a neighborhood U of 1_N such that every $Y \in U$ satisfies $\int^{cav} Y dv = \min_{x \in C(v)} Y \cdot x$.*

Proof. Denote $\hat{H} = \{h \in \mathbb{R}^n | \forall S \in P(N) : h(S) \geq v(S)\}$. By Lemma 7, there is a finite set, H (the set of all extreme points of $\hat{H}$) such that $\int^{cav} Y dv = \min_{h \in H} Y \cdot h$. Also, note that $C(v) = \{h \in \hat{H} | h(N) = v(N)\}$ and therefore $\min_{x \in C(v)} Y \cdot x \geq \min_{h \in \hat{H}} Y \cdot h$. Moreover, by Lemma 7, $\min_{h \in \hat{H}} Y \cdot h = \min_{h \in H} Y \cdot h$ and therefore, $\min_{x \in C(v)} Y \cdot x \geq \min_{h \in H} Y \cdot h$.

Suppose, to the contrary, that there is a sequence Y_n that converges to 1_N and satisfies $\int^{cav} Y_n dv \neq \min_{x \in C(v)} Y_n \cdot x$. Since $\int^{cav} Y_n dv = \min_{h_n \in H} Y_n \cdot h_n$, we get $\min_{x \in C(v)} Y_n \cdot x > \min_{h_n \in H} Y_n \cdot h_n$. As a result, $h_n(N) > v(N)$ for all n.

However, since H is finite, there is a subsequence of Y_n that converges to 1_n and its corresponding h_n subsequence is constant at h. Let us consider $\lim_{n \to \infty} \int^{cav} Y_n dv$. On the one hand, $\lim_{n \to \infty} \int^{cav} Y_n dv = \int^{cav} \lim_{n \to \infty} Y_n dv = \int^{cav} 1_n dv = v(N)$, the last equality is due to Shapley-Bondareva Theorem ($G = (N, v)$ has a non-empty core). On the other hand, $\lim_{n \to \infty} \int^{cav} Y_n dv = \lim_{n \to \infty} Y_n \cdot h_n = \{\lim_{n \to \infty} Y_n\} \cdot h = 1_n \cdot h = h(N)$. Hence, $h(N) = v(N)$. Contradiction. □

Lemma 9 (Alternative Statement)**.** *Let $x \in \mathbb{R}^n$ be an efficient payoff vector. $x \in C(\tilde{G})$ if and only if every $Y \in R^n_+$ satisfies* $\sum_{v_j \in V} \int^{cav} Y dv_j \leq Y \cdot x$.

Proof. Suppose that $x \in C(\tilde{G})$. Then, there exist m vectors $x_1, ..., x_m$ such that $\forall v_j \in V : x_j \in C(\tilde{G}_j)$ and $\sum_{v_j \in V} x_j = x$. Thus, $\forall S \subseteq N : x_j(S) \geq v_j(S)$. Moreover, for every $Y \in R^n_+$, for every function $F_Y(S) : P(N) \to [0, \infty)$ such that $\sum_{S \in P(N)} F_Y(S) \chi^S = Y$, we get $\sum_{S \subseteq N} F_Y(S) \times x_j(S) \geq \sum_{S \subseteq N} F_Y(S) \times v_j(S)$. Note that $\sum_{S \subseteq N} F_Y(S) \times x_j(S) = Y \cdot x_j$, meaning that, for every $Y \in R^n_+$, for every function $F_Y(S)$, $Y \cdot x_j \geq \sum_{S \subseteq N} F_Y(S) \times v_j(S)$. In particular, for every $Y \in R^n_+$, $Y \cdot x_j \geq \max_{F_Y(s)} \sum_{S \subseteq N} F_Y(S) \times v_j(S)$. By Lemma 1, for every $Y \in R^n_+$, $Y \cdot x_j \geq \int^{cav} Y dv_j$. Summing over all the issues, for every $Y \in R^n_+$, $\sum_{v_j \in V} Y \cdot x_j \geq$

$\sum_{v_j \in V} \int^{cav} Y dv_j$ or $Y \cdot x \geq \sum_{v_j \in V} \int^{cav} Y dv_j$. We have shown that $x \in C(\tilde{G})$ implies that for every $Y \in R^n_+$, $Y \cdot x \geq \sum_{v_j \in V} \int^{cav} Y dv_j$.

Next, suppose $x \notin C(\tilde{G})$. By Lemma 6 $C(\tilde{G})$ is closed and convex. Thus, by a separating hyperplane theorem there is a separating $Z = (Z_1, ..., Z_n)$ between x and $C(\tilde{G})$. That is, for every $w \in C(\tilde{G})$, $x \cdot Z < w \cdot Z$. Thus, $x \cdot Z < \min_{w \in C(\tilde{G})}\{w \cdot Z\}$. For a positive constant c denote $Z_c = \frac{Z+c}{c}$. Note that $Z_c \to_{c \to \infty} 1_N$. Since $x(N) = w(N)$, $x \cdot (Z + c) < \min_{w \in C(\tilde{G})}\{w \cdot (Z + c)\}$. Since $c > 0$, $x \cdot Z_c < \min_{w \in C(\tilde{G})}\{w \cdot Z_c\}$. For every $w \in C(\tilde{G})$ there exist $w_1, ..., w_m$ such that $\forall v_j \in V : w_j \in C(\tilde{G}_j)$ and $\sum_{v_j \in V} w_j = w$. Therefore, for every positive constant c, $x \cdot Z_c < \sum_{v_j \in V} \min_{w_j \in C(\tilde{G}_j)}\{w_j \cdot Z_c\}$.

For every issue $v_j \in V$, let U_j be the neighborhood of 1_N that satisfies Lemma 8. That is, $\int^{cav} Y dv_j = \min_{x_j \in C(\tilde{G}_j)}\{x_j \cdot Y\}$ for every $Y \in U_j$. Let $U = \cap_j U_j$. Therefore, $\int^{cav} Y \cdot dv_j = \min_{x_j \in C(\tilde{G}_j)}\{x_j \cdot Y\}$ for every $v_j \in V$ and $Y \in U$. As a consequence, for every $Y \in U$, $\sum_{v_j \in V} \int^{cav} Y \cdot dv_j = \sum_{v_j \in V} \min_{x_j \in C(\tilde{G}_j)}\{x_j \cdot Y\}$. Note that there is a c large enough such that $Z_c \in U$ and Z_c is non-negative. This implies that $\sum_{v_j \in V} \int^{cav} Z_c \cdot dv_j = \sum_{v_j \in V} \min_{x_j \in C(\tilde{G}_j)}\{x_j \cdot Z_c\}$. Therefore, $x \cdot Z_c < \sum_{v_j \in V} \int^{cav} Z_c \cdot dv_j$. Hence, $x \notin C(\tilde{G})$ implies that there exists $Y \in R^n_+$ that does not satisfy $\displaystyle\sum_{v_j \in V} \int^{cav} Y dv_j \leq Y \cdot x$. Alternatively, if every $Y \in R^n_+$ satisfies $\displaystyle\sum_{v_j \in V} \int^{cav} Y dv_j \leq Y \cdot x$ then $x \in C(\tilde{G})$. □

References

Bloch, Francis and Geoffroy de Clippel (2010) "Cores of Combined Games," *Journal of Economic Theory*, Vol. 145, pp. 2424–2434.

Lehrer, Ehud (2009) "A new integral for capacities," *Economic Theory*, pp. 157–176.

Appendix

Lemma 2

Proof. Suppose $m > 1$. There exist m vectors $x_1, ..., x_m$ such that $\forall j \in \{1, ..., m\}$: $x_j \in C(\tilde{G}_j)$ and $\sum_{j=1}^{m} x_j = x$ if and only if the following conditions are satisfied,

$$\begin{cases} \forall j \in \{1,\ldots,m\}, \forall S \in P(N)\backslash\{N\} : {\chi_S}' x_j \geq v_j(S) \\ \forall j \in \{1,\ldots,m\} : {\chi_N}' x_j = v_j(N) \\ \sum_{j=1}^{m} x_j = x \end{cases}$$

Or,

$$\begin{cases} \forall j \in \{1,\ldots,m\}, \forall S \in P(N) : {\chi_S}' x_j \geq v_j(S) \\ \forall j \in \{1,\ldots,m\} : {\chi_N}' x_j \leq v_j(N) \\ \sum_{j=1}^{m} x_j = x \end{cases}$$

Since $m > 1$ then

$$\begin{cases} \forall j \in \{1,\ldots,m-1\}, \forall S \in P(N) : {\chi_S}' x_j \geq v_j(S) \\ \forall S \in P(N) : {\chi_S}'[x - \sum_{j=1}^{m-1} x_j] \geq v_m(S) \\ \forall j \in \{1,\ldots,m-1\} : {\chi_N}' x_j \leq v_j(N) \\ {\chi_N}'[x - \sum_{j=1}^{m-1} x_j] \leq v_m(N) \end{cases}$$

Or,

$$\begin{cases} \forall j \in \{1,\ldots,m-1\}, \forall S \in P(N) : -{\chi_S}' x_j \leq -v_j(S) \\ \forall S \in P(N) : {\chi_S}'[\sum_{j=1}^{m-1} x_j] \leq {\chi_S}' x - v_m(S) \\ \forall j \in \{1,\ldots,m-1\} : {\chi_N}' x_j \leq v_j(N) \\ -{\chi_N}'[\sum_{j=1}^{m-1} x_j] \leq -{\chi_N}' x + v_m(N) \end{cases}$$

The next step is to write these systems of inequalities in a compact matrix notation. Let $\mu : P(N)\backslash\emptyset \to \{1, 2, \ldots, 2^n - 1\}$ be an ordering on the set of non-empty coalitions (non-zero characteristic vectors).[3] For clarity we assume, with no loss of generality, that $\mu(N) = 2^n - 1$. Denote the inverse function that attaches a unique coalition to every number in $\{1, 2, \ldots, 2^n - 1\}$ by μ^{-1}.

Denote by R_j^S ($j \in \{1, \ldots, m-1\}$ and $S \in P(N)$) the row vector of length $(m-1)n$ where the characteristic vector of S lies starting at element $(j-1)n+1$ while all other elements are zeros. In addition, denote by Q^S ($S \in P(N)$) the row vector of length $(m-1)n$ where characteristic vector of S is replicated $m-1$ times. First, let A be an $m2^n \times (m-1)n$ matrix constructed in the following way. For the first $m(2^n - 1)$ rows, if $k \mod m \neq 0$ the k^{th} row corresponds to $-R_{k \mod m}^{\mu^{-1}(\lceil \frac{k}{m} \rceil)}$, otherwise it corresponds to $Q^{\mu^{-1}(\frac{k}{m})}$. Each row j of the next $m-1$ rows corresponds to R_j^N while the last row is $-Q^N$. Next, let b be a vector of length $m2^n$ constructed in the following way. For the first $m(2^n - 1)$ rows, if $k \mod m \neq 0$ the k^{th} element is $-v_{k \mod m}(\mu^{-1}(\lceil \frac{k}{m} \rceil))$, otherwise it corresponds to $\sum_{i \in \mu^{-1}(\frac{k}{m})} x[i] - v_m(\mu^{-1}(\frac{k}{m}))$. Each element j of the next $m-1$ elements are $v_j(N)$ while the last element is $-\big(\sum_{i\in N} x[i] - v_m(N)\big)$. Finally, let w be a vertical concatenation of $x_1, \ldots, x_{m-1}$. Then there exist $x_1, \ldots, x_m$ such that $\forall j \in \{1, \ldots, m\} : x_j \in C(\tilde{G}_j)$ and $\sum_{j=1}^m x_j = x$

[3] The choice of the specific ordering is inconsequential to the rest of the proof.

if and only if there is a vector $w \in \mathbb{R}^{(m-1)n}$ that satisfies $Aw \leq b$.[4]

By Farkas' Lemma, or alternatively, by applying the separating hyperplane theorem to a set of linear inequalities, given a matrix A and a vector b, the system $Aw \leq b$ has a solution w, if and only if, for every non-negative vector z, $A'z = 0$ implies $b'z \geq 0$.

Denote $Z_0 = \{z \in \mathbb{R}_+^{m2^n} | A'z = 0\}$. Thus, there exist $x_1, ..., x_m$ such that $\forall v_j \in V : x_j \in C(\tilde{G}_j)$ and $\sum_{j=1}^m x_j = x$ if and only if every vector $z \in Z_0$ satisfies $b'z \geq 0$.[5]

Denote, $Z_1 = \{z \in Z_0 | \forall v_j \in V : z[m2^n - 2m + j] \geq z[m2^n - m + j]\}$. Obviously, $Z_1 \subseteq Z_0$. By Lemma 10 (below), for every z such that $A'z = 0$ there exists $\hat{z}$ such that $A'\hat{z} = 0$, for all $v_j \in V : \hat{z}[m2^n - 2m + j] \geq \hat{z}[m2^n - m + j]$ and $b'z = b'\hat{z}$.

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[4] For $N = \{1, 2, 3\}$, $V = \{v_1, v_2, v_3\}$ and μ such that $\mu(\{1\}) = 1, \mu(\{2\}) = 2, \mu(\{3\}) = 3, \mu(\{1, 2\}) = 4, \mu(\{1, 3\}) = 5, \mu(\{2, 3\}) = 6, \mu(\{1, 2, 3\}) = 7$:

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix} ; \quad w = \begin{pmatrix} x_1[1] \\ x_1[2] \\ x_1[3] \\ x_2[1] \\ x_2[2] \\ x_2[3] \end{pmatrix} ; \quad b = \begin{pmatrix} -v_1(\{1\}) \\ -v_2(\{1\}) \\ x[1] - v_3(\{1\}) \\ -v_1(\{2\}) \\ -v_2(\{2\}) \\ x[2] - v_3(\{2\}) \\ -v_1(\{3\}) \\ -v_2(\{3\}) \\ x[1] - v_3(\{3\}) \\ -v_1(\{1,2\}) \\ -v_2(\{1,2\}) \\ x[1] + x[2] - v_3(\{1,2\}) \\ -v_1(\{1,3\}) \\ -v_2(\{1,3\}) \\ x[1] + x[3] - v_3(\{1,3\}) \\ -v_1(\{2,3\}) \\ -v_2(\{2,3\}) \\ x[2] + x[3] - v_3(\{2,3\}) \\ -v_1(\{1,2,3\}) \\ -v_2(\{1,2,3\}) \\ x[1] + x[2] + x[3] - v_3(\{1,2,3\}) \\ v_1(\{1,2,3\}) \\ v_2(\{1,2,3\}) \\ -(x[1] + x[2] + x[3] - v_3(\{1,2,3\})) \end{pmatrix} .$$

[5] The matrix A' has $(m-1)n$ rows, one for each player-issue pair (excluding the last issue), and $m2^n$ columns, one for each non-empty coalition-issue pair (the grand coalition pairs appear twice). In the first $m(2^n - 1)$ columns, $A'[k, l] = -1$ if the player corresponding to the row is a member of the coalition corresponding to the column ($k \mod n \in \mu^{-1}(\lceil \frac{l}{m} \rceil)$, where $k \mod n = 0$ refers to Player n) and both the row and column correspond to the same issue ($\lceil \frac{k}{n} \rceil = l \mod m$) while $A'[k, l] = 1$ if the player corresponding to the row is a member of the coalition corresponding to the column ($k \mod n \in \mu^{-1}(\lceil \frac{l}{m} \rceil)$) and the column corresponds to the m^{th} issue ($l \mod m = 0$), otherwise $A'[k, l] = 0$. In the last m columns, $A'[k, l] = -1$ if the column corresponds to the m^{th} issue ($l \mod m = 0$) while $A'[k, l] = 1$ if both the row and column correspond to the same issue ($\lceil \frac{k}{n} \rceil = l \mod m$), otherwise $A'[k, l] = 0$.

Thus, there exist $x_1, ..., x_m$ such that $\forall v_j \in V : x_j \in C(\tilde{G}_j)$ and $\sum_{j=1}^{m} x_j = x$ if and only if every vector $z \in Z_1$ satisfies $b'z \geq 0$.

Every $z \in Z_1$ is a vector of length $m2^n$. For every $l \in \{1, \ldots, m2^n - 2m\}$, we refer to $z[l]$ as $z_{l \mod m}(\mu^{-1}(\lceil \frac{l}{m} \rceil))$ ($l \mod m = 0$ corresponds to the m^{th} issue). In addition, we denote $z_j(N) = z[m2^n - 2m + j] - z[m2^n - m + j] \geq 0$. Then, we can interpret every z_j as a coalitional weight function as defined in Definition 4.

$A'z = 0$ is a system of $(m-1) \times n$ equations, one for each player in each game (except the last). A pair of z_j and z_m satisfies the $((j-1) \times n + i)^{th}$ equation if and only if Player i's total weight in issue j equals her total weight in the last issue. Therefore, by Definition 4, $z \in Z_1$ if and only if all the z_js are non-negative and W-equivalent. Thus, there exist $x_1, ..., x_m$ such that $\forall v_j \in V : x_j \in C(\tilde{G}_j)$ and $\sum_{j=1}^{m} x_j = x$ if and only if every vector $z \in \mathbb{R}_+^{m2^n}$ that induces non-negative and W-equivalent z_js satisfies $b'z \geq 0$.

$$b'z = \sum_{S \in P(N)} z_m[S] \sum_{i \in S} x_i - \sum_{v_j \in V \setminus v_m} \sum_{S \in P(N)} z_j[S] v_j(S) - \sum_{S \in P(N)} z_m[S] v_m(S)$$

Denote the equivalence class of the z_js by γ

$$b'z = \sum_{i \in N} W^{\gamma}[i] x_i - \sum_{v_j \in V} \sum_{S \in P(N)} z_j[S] v_j(S)$$

Thus, there exist $x_1, ..., x_m$ such that $\forall v_j \in V : x_j \in C(\tilde{G}_j)$ and $\sum_{j=1}^{m} x_j = x$ if and only if for every m W-equivalent coalition weight functions $F_1, \ldots, F_m$,

$$\sum_{S \in P(N)} F_1(S) v_1(S) + \cdots + \sum_{S \in P(N)} F_m(S) v_m(S) \leq \sum_{i \in N} W^{\gamma}[i] x_i$$

Thus, there exist $x_1, ..., x_m$ such that $\forall v_j \in V : x_j \in C(\tilde{G}_j)$ and $\sum_{j=1}^{m} x_j = x$ if and only if for every $\gamma \in \Gamma$: $\sum_{v_j \in V} T_{v_j}^{\gamma} \leq \sum_{i \in N} W^{\gamma}[i] x_i$.

We complete the proof by showing that the lemma holds for a single game. Denote by v_0 the characteristic function that attaches 0 to every coalition. Let $G = (N; v)$ be a cooperative game and let $\tilde{G} = (N; \{v, v_0\})$. We show that $x \in C(G)$ if and only if every class $\gamma \in \Gamma$ satisfies $T_v^\gamma \leq \sum_{i \in N} W^\gamma[i] x_i$. First, $x \in C(G)$ if and only if x is an efficient payoff vector of $\tilde{G}$ since $C(G(N; v_0)) = \{0\}$. Also, $x \in C(G)$ if and only if there exist two vectors $x_1 = x$ and $x_2 = 0$ such that $\forall v_j \in V : x_j \in C(\tilde{G}_j)$ and $\sum_{v_j \in V} x_j = x$. Then, by Lemma 2, $x \in C(G)$ if and only if every class $\gamma \in \Gamma$ satisfies $\displaystyle\sum_{v_j \in V} T_{v_j}^\gamma \leq \sum_{i \in N} W^\gamma[i] x_i$. Since, for every class $\gamma \in \Gamma$, $T_{v_0} = 0$ then $x \in C(G)$ if and only if every class $\gamma \in \Gamma$ satisfies $\displaystyle T_v^\gamma \leq \sum_{i \in N} W^\gamma[i] x_i$. □

Lemma 10. *For every vector x of length $m2^n$ define*

$$\kappa(x) \equiv \min_{v_j \in V} \{x[m2^n - 2m + j] - x[m2^n - m + j]\}$$

Let z be a non-negative vector of length $m2^n$ such that $A'z = 0$ and $\kappa(z) < 0$. Then, there exists $\hat{z}$, a non-negative vector of length $m2^n$, such that $A'\hat{z} = 0$, $\kappa(\hat{z}) = 0$ and $b'z = b'\hat{z}$.

Proof. For every $j \in \{1, \ldots, m2^n - 2m\} \cup \{m2^n - m + 1, \ldots, m2^n\}$ define $\hat{z}[j] \equiv z[j]$ and for every $j \in \{m2^n - 2m + 1, \ldots, m2^n - m\}$ define $\hat{z}[j] \equiv z[j] - \kappa(z)$.

First, since z is a non-negative vector of length $m2^n$ and since $\kappa(z) < 0$, it must be that $\hat{z}$ is a non-negative vector of length $m2^n$.

Second, let us show that $A'\hat{z} = 0$. For every $k \in \{1, \ldots, (m-1)n\}$ denote the corresponding issue by $I_k = \lceil \frac{k}{n} \rceil$. Then, for every $k \in \{1, \ldots, (m-1)n\}$

$$\sum_{l=1}^{m2^n} A'[k,l]\hat{z}[l] - \sum_{l=1}^{m2^n} A'[k,l]z[l] = \sum_{l=1}^{m2^n} A'[k,l](\hat{z}[l] - z[l]) =$$
$$= \sum_{l=m2^n-2m+1}^{m2^n-m} A'[k,l](\hat{z}[l] - z[l]) =$$
$$= -(\hat{z}[m2^n - 2m + I_k] - z[m2^n - 2m + I_k]) + (\hat{z}[m2^n - m]) - z[m2^n] - m) =$$
$$= \kappa(z) - \kappa(z) = 0$$

Thus, for every $k \in \{1, \ldots, (m-1)n\}$, $\sum_{l=1}^{m2^n} A'[k,l]\hat{z}[l] = \sum_{l=1}^{m2^n} A'[k,l]z[l]$. Therefore, $A'\hat{z} = A'z$ and we conclude that $A'\hat{z} = 0$.

Next, let us show that $\kappa(\hat{z}) = 0$. Recall that,

$$\begin{aligned}\kappa(\hat{z}) \equiv & \min_{v_j \in V}\{\hat{z}[m2^n - 2m + j] - \hat{z}[m2^n - m + j]\} = \\ & \min_{v_j \in V}\{z[m2^n - 2m + j] - \kappa(z) - z[m2^n - m + j]\} = \\ & \min_{v_j \in V}\{z[m2^n - 2m + j] - z[m2^n - m + j]\} - \kappa(z) = \\ & \kappa(z) - \kappa(z) = 0\end{aligned}$$

Last, let us show that $b'z = b'\hat{z}$,

$$b'\hat{z} - b'z = \sum_{l=1}^{m2^n} b'[l](\hat{z}[l] - z[l]) = \sum_{l=m2^n-2m+1}^{m2^n-m} b'[l](\hat{z}[l] - z[l]) =$$
$$= -\kappa(z) \sum_{l=m2^n-2m+1}^{m2^n-m} b'[l] = -\kappa(z)\big(\sum_{j=1}^{m-1} -v_j(\{N\}) + (\sum_{i=1}^{n} x[i] - v_m(\{N\}))\big) =$$
$$= \kappa(z)\big(\sum_{j=1}^{m} v_j(\{N\}) - \sum_{i=1}^{n} x[i]\big)$$

Since x is an efficient payoff vector, $\sum_{j=1}^{m} v_j(\{N\}) = \sum_{i=1}^{n} x[i]$. Hence, $b'\hat{z} - b'z = 0$, meaning that $b'\hat{z} = b'z$. $\square$