Let \((X,d)\) be a metric space. Let \(P\) be a countable collection of closed subsets of \(X\) of finite diameter. Assume: (few large sets nearby)

There exist \(\zeta_1, \zeta_2 \geq 0\) such that for all \(S \in P\), \(\rho, t > 0\),

\[\{ S \in P : d(S, S') \leq \rho, \text{diam}(S') \geq t \} \leq e^{\zeta_1 + \zeta_2 + \text{diam}(S')}\]

Example: packing of shapes in \(\mathbb{R}^n\) whose volume is proportional to the \(n\)'th power of their diameter with a uniform proportionality constant.

E.g.: Circle packings, cube packings.

Theorem: There exists \(p_0 = p_0(\zeta_1, \zeta_2)\) such that

\[P_p(\exists \text{ connected comp. of open sets}) = 0, \forall p < p_0.\]

\(p\)-site percolation on the sets in \(P\). Open set retained.

Main lemma: There exists \(p_0\) s.t. for all \(p < p_0\), \(S_0 \in P\), \(r \geq 0, k \in \mathbb{Z}\),

\[P_p(S_0 \xrightarrow{\leq 2^k} r) \leq e^{\zeta_1 + \frac{3r}{2^k + 2} + \frac{3 \text{diam}(S_0)}{2^{k+2}}} p^{-\frac{r}{2^{k+5}}}\]

if \(r \geq \text{diam}(S_0)\).

The event \(\{ S_0 \xrightarrow{\leq 2^k} r \}\) is

Both \(S_0\) and the blue sets are open. Blue sets have diameter \(\leq 2^k\).

Proof of main lemma:
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It suffices to prove when \( \inf_{S \in \mathcal{P}} \text{diam}(S) > 0 \).

Since every finite path has an \( S \) with minimal positive diameter.

Scaling the metric, we may and will assume that \( \inf_{S \in \mathcal{P}} \text{diam}(S) = 1 \).

We prove the lemma by induction on \( k \).

Base case \( k = 0 \): For \( \{S_0 \subseteq V^3 \} \), all used sets must have diameter exactly 1.

Number of paths of length \( L \) using sets of diam. 1, starting at \( S_0 \) and going to distance \( r \) is

\[
\leq e^{c_1 + c_2 \text{diam}(S_0)} \cdot (c_1 + c_2)^{L-2}
\]

Prob. to be open for each path = \( p^L \) necessarily have \( L > 1/r \).

Due \( c_1 + c_2 \text{diam}(S_0) + (L-2)c_2 \leq L = 1 + 1/r \)

\[
\leq e^{c_2 \text{diam}(S_0)} \cdot p^{\frac{r}{c_2}} \text{ as required.}
\]

Induction step \( k \geq 1 \):

Assume lemma holds up to \( k-1 \) and prove for \( k \).

Decompose the event \( \{S_0 \subseteq V^3 \} \) according to the sets with diam. in \( [2^{k-1}, 2^k) \) that are used on the open path.

For \( m \geq 0 \) integer and \( S_0, \ldots, S_m \in \mathcal{P} \), with \( d(S_i, S_j) < r \) and \( \text{diam}(S_i) \in [2^{k-1}, 2^k) \) for each.
For $m \geq 0$ integer and $n_0, \ldots, n_m \in \mathbb{N}$, with $d(S_{i}, S_{i+1}) < r$ and $\text{diam}(S_i) \leq 2^{k}r$, let $E_{n_0, n_1, \ldots, n_m} = \left\{ S_0 \xrightarrow{\text{neigh.}} S_1 \text{ or } \ldots \text{ or } S_{m-1} \xrightarrow{\text{neigh.}} S_m \rightarrow \text{ dist. } r \text{ from } S_0 \right\}$ be a family of disjoint events. 

**Claim:** $P(E_{n_0, n_1, \ldots, n_m}) \leq \prod_{i=0}^{m-1} \min \left\{ p, \frac{\text{diam}(S_i, S_{i+1})}{2^{k}r} \right\}$.  

**Proof:** By the Van den Berg-Kesten Incy. 

$$P(E_{n_0, n_1, \ldots, n_m}) \leq \prod_{i=0}^{m-1} \left( P(S_i \xrightarrow{\text{neigh.}} S_{i+1}) \cdot P(S_m \rightarrow \text{ dist. } r \text{ from } S_0) \right)$$

**What to do with** $P(S_0 \xrightarrow{\text{neigh.}} S_1)$? 

**Trick:** It equals $P(S_0 \xrightarrow{\text{neigh.}} S_1)$. 

By swapping the states of $S_0$ and $S_1$. 

By the induction hypothesis, 

$$P(S_1 \xrightarrow{\text{neigh.}} S_0) \leq \min \left\{ e^{-e^{c_2} \cdot \frac{3 \text{diam}(S_0, S_1)}{2^{k}r}}, e^{-c_2^2 \cdot \frac{3 \text{diam}(S_0, S_1)}{2^{k}r}} \right\}.$$  

Since $S_0$ needs to be open, 

$$P \leq e^{-c_1(c_2+c_2^2)} \leq \min \left\{ p, \frac{\text{diam}(S_0, S_1)}{2^{k}r} \right\}.$$  

Apply similar reasoning (without swapping trick) to other terms. 

Main Lemma follows from Claim by summing over all $m$ and $S_0, \ldots, S_m$. 

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