Lectures on disordered models

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Notes under construction!

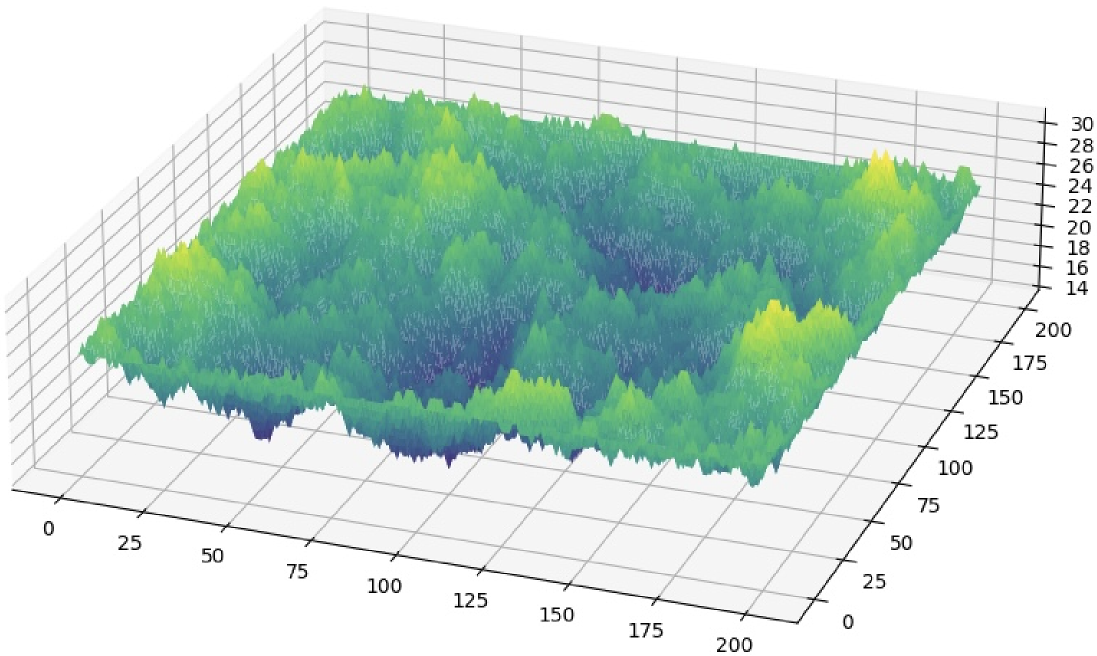

Figure 1: A minimal surface in independent disorder with $d = 2$, $n = 1$.

Consider putting some pictures on the title page, before the table of contents. Do we need an abstract?

Contents

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1 Introduction

These notes discuss some of the recent rigorous progress in the analysis of disordered models. Our focus is on disordered spin systems, first-passage percolation and minimal surfaces in random environments. Within these topics, we discuss the existence or absence of long-range order in disordered spin systems and questions of localization and delocalization of geodesics and minimal surfaces in random environment. Add more details, with pointers to the sections

These notes were initially written for lectures at the School on Disordered media, held in January 2025 at the Rényi Institute in Budapest, Hungary[1]. They were also used by the second author for lectures at the Seminar on Stochastic Processes, held in March 2025 at Indiana University, Bloomington[2], and at the UK Easter Probability meeting, held in March-April 2025[3]. We are grateful to the organizers for their kind invitation to deliver these lectures and for the impetus to write these notes.

2 Disordered spin systems

First give an introduction to pure (non-disordered) spin systems, explaining the phase transitions of the Ising, Potts and spin $O(n)$ models.

Mention that while we focus on the random-field examples, the results on the rounding of first-order phase transitions apply in wide generality to disordered spin systems, including even to the two-dimensional Edwards-Anderson model

2.1 Basic definitions

Lattice: We consider spin systems on the d-dimensional lattice $\mathbb{Z}^d$, regarded as a graph with nearest-neighbor edges $E(\mathbb{Z}^d) = \{\{u, v\} \colon \|u - v\|_1 = 1\}$ (where $\|x\|_p = (\sum_i |x_i|^p)^{1/p}$ is the standard ℓ^p norm). We sometimes write $u \sim v$ to indicate that $\{u, v\} \in E(\mathbb{Z}^d)$.

[1] Organized by Ágnes Backhausz, Gábor Pete, Balázs Ráth and Bálint Tóth.

[2] Organized by Wai-Tong (Louis) Fan, Nathan Glatt-Holtz, Elizabeth Housworth, Russell Lyons and Jing Wang.

[3] Organized by Tyler Helmuth, Ostap Hryniv, Ellen Powell, Kohei Suzuki, Andrew Wade and Mo Dick Wong.

Spin systems and Gibbs measures: The spin systems we consider are specified by:

1. A *state space* $(S, \mathcal{S}, \kappa)$, with $(S, \mathcal{S})$ a measurable space equipped with the (non-negative) measure κ. We write $\Omega := \{\sigma : \mathbb{Z}^d \to S\}$ for the set of *configurations* of the spin system.
2. A *formal Hamiltonian* H. For these notes, it suffices to consider H of the form

$$H(\sigma) = \sum_{u \sim v} h(\sigma_u, \sigma_v) + \sum_{v} h_v(\sigma_v) \tag{1}$$

   for some measurable $h : S \times S \to \mathbb{R}$ and $h_v : S \to \mathbb{R}$. In other words, we restrict to general single-site potentials and translation-invariant, isotropic nearest-neighbor pair interactions.

Given a temperature $T > 0$, a finite $\Lambda \subset \mathbb{Z}^d$ and a configuration $\tau \in \Omega$ (the boundary values), we obtain a probability measure $\mathbb{P}^T_{\Lambda,\tau}$, termed a *finite-volume Gibbs measure*, with the following standard prescription: Let

$$\Omega_{\Lambda,\tau} := \{\sigma \in \Omega \colon \sigma_v = \tau_v \text{ for } v \notin \Lambda\}. \tag{2}$$

be the configurations which agree with τ outside Λ. The *finite-volume Hamiltonian*

$$H_{\Lambda,\tau}(\sigma) := \sum_{\substack{u \sim v \\ \{u,v\} \cap \Lambda \neq \emptyset}} h(\sigma_u, \sigma_v) + \sum_{v \in \Lambda} h_v(\sigma_v) \tag{3}$$

consists of the terms in the formal Hamiltonian to which depend on the configuration in Λ. Then, $\mathbb{P}^T_{\Lambda,\tau}$ is the probability measure on $\Omega_{\Lambda,\tau}$ given by

$$d\mathbb{P}^T_{\Lambda,\tau}(\sigma) := \frac{1}{Z^T_{\Lambda,\tau}} e^{-\frac{1}{T} H_{\Lambda,\tau}(\sigma)} \prod_{v \in \Lambda} d\kappa(\sigma_v) \tag{4}$$

where the *partition function*

$$Z_{\Lambda,\tau} := \int_{\Omega_{\Lambda,\tau}} e^{-\frac{1}{T} H_{\Lambda,\tau}(\sigma)} \prod_{v \in \Lambda} d\kappa(\sigma_v) \tag{5}$$

normalizes $\mathbb{P}_{\Lambda,\tau}$ to be a probability measure, and it is tacitly assumed that $0 < Z_{\Lambda,\tau} < \infty$ so that this is possible.

*** add also zero temperature. To finite volume and perhaps also to infinite volume ****

We shall sometimes refer to (infinite-volume) Gibbs measures. These are the measures $\mathbb{P}^T$ on Ω which satisfy the following Dobrushin-Lanford-Ruelle condition: Suppose σ is sampled from $\mathbb{P}^T$. For any finite $\Lambda \subset \mathbb{Z}^d$, conditioned on the restriction of σ to Λ^c, the distribution of σ equals $\mathbb{P}^T_{\Lambda,\tau}$ where τ is any configuration which equals σ off Λ. *** mention that when we have a topology on the state space S then we can obtain such Gibbs measures by weak limits? should we then assume that the h and (h_v) are continuous? ***

We shall write $\langle \cdot \rangle^T_{\Lambda,\tau}$ for the *expectation operator* corresponding to the measure $\mathbb{P}^T_{\Lambda,\tau}$.

Sometimes, to distinguish different spin systems, we shall add the name of the system in a superscript for the above objects. We may also add parameters that the model depends on, including, particularly, the disorder variables for models with quenched disorder. For instance, for the (pure) Ising model, we may write H^{Ising} for its formal Hamiltonian, $\langle\cdot\rangle^{\text{Ising},T}_{\Lambda,\tau}$ for the expectation operator of its finite-volume Gibbs measures, etc. For the random-field Ising model, we may write $H^{\text{RF-Ising},\eta,\lambda}$, $\langle\cdot\rangle^{\text{RF-Ising},\lambda,\eta,T}_{\Lambda,\tau}$, etc., where η denotes the disorder variables and λ denotes the disorder strength *** refer to Section ***.

We refer to the books of Georgii [16] and of Friedli and Velenik [15] for an in-depth introduction to spin systems and their Gibbs measures.

2.2 Pure spin systems

In this section we introduce several classical spin systems that we will focus on. These are pure systems, in the sense that they are not placed in a random environment (disorder). In the next section, we shall discuss how the presence of disorder may alter the behavior of these models.

2.2.1 Models

We focus on the following models:

Ising model: The state space of the Ising model is $S = \{-1, 1\}$, endowed with the counting measure κ, and its formal Hamiltonian is

$$H^{\text{Ising}}(\sigma) := -\sum_{u\sim v}\sigma_u\sigma_v. \tag{6}$$

Potts model: Let $q \geq 2$ be an integer. The q-state Potts model has state space $S = \{1, 2\ldots, q\}$, equipped with the counting measure κ, and its formal Hamiltonian is

$$H^{\text{Potts}}(\sigma) := -\sum_{u\sim v}1_{\sigma_u=\sigma_v}. \tag{7}$$

The Ising model and the 2-state Potts model are equivalent (as their state spaces and formal Hamiltonians are related by affine transformations).

Spin $O(n)$ model: Let $n \geq 1$ be an integer (denoting the number of components). The state space of the spin $O(n)$ model is the $(n-1)$-dimensional sphere $S = \mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \colon \|x\| = 1\}$, viewed as a subset of $\mathbb{R}^n$ and equipped with the Borel sigma-algebra and rotationally-invariant probability measure κ. Its formal Hamiltonian is

$$H^{O(n)}(\sigma) := -\sum_{u\sim v}\sigma_u\cdot\sigma_v. \tag{8}$$

Here, we endow $\mathbb{R}^n$ with the Euclidean inner product $x\cdot y := \sum_{i=1}^n x_iy_n$ and norm $\|x\|^2 := x\cdot x$. The spin $O(1)$ model coincides with the Ising model.

2.2.2 Phase diagrams

*** Discuss here the phase diagrams for these pure models ***

*** From earlier writuep: ***

The pure (non-disordered) Ising, Potts and spin $O(n)$ models are well known to undergo a magnetization phase transition (see, e.g., [20]):

1. (Ising model). For $L \geq 0$ integer, let
$$\Lambda_L := \{-L, \dots, L\}^d \tag{9}$$
and consider the Ising model in Λ_L with $+$-boundary conditions, i.e., with $\tau \equiv +1$. Then in all dimensions $d \geq 2$ there exists a critical temperature $T_c^{\text{Ising}}(d)$ such that
$$\lim_{L\to\infty} \langle \sigma_0 \rangle_{\Lambda_L,+}^{\text{Ising},T} \begin{cases} = 0 & T > T_c^{\text{Ising}}(d) \\ > 0 & T < T_c^{\text{Ising}}(d) \end{cases}. \tag{10}$$
*** and it is further known that the limit is also zero at the critical temperature. Mention also the exponential rate of decay to zero at high temperatures? Divide into two parts, with the first part having supremum over boundary conditions? ***

2. (Potts model). When placing the Potts model under 1-boundary conditions (i.e., $\tau \equiv 1$) then in all dimensions $d \geq 2$ there exists a critical temperature $T_c^{\text{Potts}}(d)$ such that
$$\lim_{L\to\infty} \langle 1_{\sigma_0=1} \rangle_{\Lambda_L,1}^{\text{Potts},T} \begin{cases} = \frac{1}{q} & T > T_c^{\text{Potts}}(d) \\ > \frac{1}{q} & T < T_c^{\text{Potts}}(d) \end{cases}. \tag{11}$$

3. ($O(n)$ model with $n \geq 2$). The pure $O(n)$ models with $n \geq 2$ have a continuous symmetry - for all rotations R in $\mathbb{R}^n$, all domains Λ, boundary values τ and configurations σ, the Hamiltonians satisfy $H_{\Lambda,R\tau}^{O(n)}(R\sigma) = H_{\Lambda,\tau}^{O(n)}(\sigma)$ where $R\rho : \mathbb{Z}^d \to \mathbb{S}^{n-1}$ is the rotated configuration defined by $(R\rho)_v := R(\rho_v)$. The Mermin–Wagner theorem thus dictates the absence of a magnetization phase transition in dimension $d = 2$ at all positive temperatures $T > 0$:
$$\lim_{L\to\infty} \sup_{\tau:\mathbb{Z}^d\to\mathbb{S}^{n-1}} \|\langle \sigma_0 \rangle_{\Lambda_L,\tau}^{O(n),T}\| = 0. \tag{12}$$
An important fact, which will not be discussed here, is that a phase transition does occur in dimension $d = 2$: the famed Berezinskii–Kosterlitz–Thouless transition from a high-temperature regime with exponential decay of the above supremum to a low-temperature regime with power-law decay. In dimensions $d \geq 3$ a magnetization phase transition occurs: When placing the $O(n)$ model under $\rightarrow$-boundary conditions (i.e., $\tau \equiv (1, 0, \dots, 0)$) then in all dimensions $d \geq 3$ there exists a critical temperature $T_c^{O(n)}(d)$ such that is the existence of a single critical temperature known for $n \geq 3$?
$$\lim_{L\to\infty} \|\langle \sigma_0 \rangle_{\Lambda_L,\rightarrow}^{O(n),T}\| \begin{cases} = 0 & T > T_c^{O(n)}(d) \\ > 0 & T < T_c^{O(n)}(d) \end{cases}. \tag{13}$$

2.3 Random-field spin systems

*** Introduce the disordered models and discuss their phase diagram. Present the current best bounds on the effect of the disorder ***

Spin systems may alter their properties when placed in non-homogeneous environments. In this section, we consider this effect for the case of a *random environment* (termed the *disorder*), formed from independent, local, random variables, and our focus is on the existence or absence of long-range order. We emphasize that the disorder is *quenched*; in other words, to sample a configuration of the system, one first samples an instance of the disorder and then samples a configuration from the model's disorder-dependent Hamiltonian.

To illustrate the topic, we mainly focus on the random-field spin systems described by the following formal Hamiltonians and disorder choices:

1. Random-field Ising model: Configurations are described by $\sigma : \mathbb{Z}^d \to \{-1, 1\}$. The disorder consists of $(\eta_v^{\text{RF-Ising}})_{v\in\mathbb{Z}^d}$, independent standard Gaussian random variables (i.e., of mean 0 and variance 1). The disorder strength is denoted $\lambda > 0$. The formal Hamiltonian is
$$H^{\text{RF-Ising},\eta^{\text{RF-Ising}},\lambda}(\sigma) := -\sum_{u\sim v}\sigma_u\sigma_v - \lambda\sum_v \eta_v^{\text{RF-Ising}}\sigma_v. \tag{14}$$

2. Random-field Potts model: Let $q \geq 2$ integer denote the number of states. Configurations are described by $\sigma : \mathbb{Z}^d \to \{1, 2 \ldots, q\}$. The disorder consists of $(\eta_{v,k}^{\text{RF-Potts}})_{v\in\mathbb{Z}^d,k\in\{1,\ldots,q\}}$, independent standard Gaussian random variables. The disorder strength is denoted $\lambda > 0$. The formal Hamiltonian is
$$H^{\text{RF-Potts},\eta^{\text{RF-Potts}},\lambda}(\sigma) := -\sum_{u\sim v} 1_{\sigma_u=\sigma_v} - \lambda\sum_v\sum_{k=1}^q \eta_{v,k}^{\text{RF-Potts}}1_{\sigma_v=k}. \tag{15}$$
The case $q = 2$ is equivalent to the random-field Ising model (the Hamiltonians differ only by the addition of a disorder dependent term).

3. Random-field Spin $O(n)$ model: Let $n \geq 1$ integer denote the number of components. Configurations are described by $\sigma : \mathbb{Z}^d \to \mathbb{S}^{n-1}$. The disorder consists of $(\eta_v^{\text{RF}-O(n)})_{v\in\mathbb{Z}^d}$, independent standard Gaussian random vectors in $\mathbb{R}^n$ (i.e., of mean 0 and identity covariance matrix). The disorder strength is denoted $\lambda > 0$. The formal Hamiltonian is
$$H^{\text{RF}-O(n),\eta^{\text{RF}-O(n)},\lambda}(\sigma) := -\sum_{u\sim v}\sigma_u\cdot\sigma_v - \lambda\sum_v \eta_v^{\text{RF}-O(n)}\cdot\sigma_v. \tag{16}$$
Here, we endow $\mathbb{R}^n$ with the Euclidean inner product $x \cdot y := \sum_{i=1}^n x_i y_n$ and norm $\|x\|^2 := x \cdot x$, and denote by $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n \colon \|x\| = 1\}$ the $(n-1)$-dimensional Euclidean sphere. The case $n = 1$ is again equivalent to the random-field Ising model.

We've restricted to Gaussian disorder for simplicity, but note that other disorder choices (typically having a rotationally-symmetric distribution around 0), are also of interest and are discussed in the literature.

We will mostly be interested in the properties of the models at low temperatures. In fact, in the presence of disorder, it turns out that the relevant phenomena already arise at *zero temperature*, and, mostly for simplicity, we will focus solely on this case. The zero-temperature measure, or *finite-volume ground state*, $\mathbb{P}^{\#,0}_{\Lambda,\tau}$ is defined as the limit in distribution of $\mathbb{P}^{\#,T}_{\Lambda,\tau}$ as $T \downarrow 0$. It is supported on the *minimizers* of the Hamiltonian $H^{\#}_{\Lambda,\tau}$, which we term *finite-volume ground configurations*[4]. In fact, in our examples above it is easily seen that

[4]In the literature, the term finite-volume ground states is often also used for these minimizers

there is a unique minimizer almost surely, so that $\mathbb{P}^{\#,0}_{\Lambda,\tau}$ is a delta measure (but note that there exist random λ, Λ, τ for which there are multiple minimizers).

An important role is played by the Gibbs measures of the model: These are the measures which arise as limits in distribution of $\mathbb{P}^{\#,T}_{\Lambda_n,\tau_n}$ for some sequence of domains $\Lambda_n \subset \mathbb{Z}^d$ which inrease to $\mathbb{Z}^d$ and some sequence of configurations τ_n, and also the convex combinations of these limits. The set of Gibbs measures is naturally random, depending on the realization of the disorder. Gibbs states at zero temperature are called *ground states*. They are supported on *ground configurations*, configurations σ which locally minimize the formal Hamiltonian $H^{\#}$ in the sense that if σ' differs from σ in finitely many vertices than $H^{\#}(\sigma') - H^{\#}(\sigma) \geq 0$ (noting that this energy difference is well defined, at least in the above examples, as only finitely many terms differ in the sums defining the Hamiltonians).

The above spin systems may be considered as perturbations of the corresponding pure (i.e., non-disordered) spin systems obtained by setting $\lambda = 0$ in the formal Hamiltonians. For instance, the random-field Ising model may be thought of as a perturbation of the Ising model, defined by the formal Hamiltonian

$$H^{\text{Ising}}(\sigma) := -\sum_{u \sim v} \sigma_u \sigma_v. \tag{17}$$

Our focus will then naturally be on the way in which the added disorder alters the properties of the underlying spin system.

2.4 The Imry-Ma phenomenon: Absence and preservation of long-range order in the presence of a random field

How does the addition of the random field affect these phase transitions? The added disorder naturally competes with the ferromagnetic interaction of the pure Hamiltonian and, at least intuitively, should weaken the long-range order. One may consider several parameter regimes according to the temperature T and disorder strength λ.

For a sufficiently high threshold temperature $T_0^{\#}(d)$, it follows from Dobrushin's uniqueness criterion *** ref *** that the model is disordered for all temperatures $T > T_0^{\#}(d)$ and all disorder strengths $\lambda \geq 0$ *** in the sense of exponential decay? ***. Moreover, for the random-field Ising model, it has been shown that one may take $T_0^{\text{RF-Ising}}(d) = T_c^{\text{Ising}}(d)$ *** ref Ding–Sun–Song [12] ***. It is apparently open to obtain a similar result for the random-field Potts models with $q \geq 3$ and the random-field $O(n)$ models with $n \geq 2$. *** check that it is indeed still open ***

There are several results in the literature showing that there exists a threshold disorder strength $\lambda_0^{\#}(d)$ such that the models are also disordered when the disorder strength $\lambda > \lambda_0^{\#}(d)$ at all temperatures T, including zero temperature! *** reference such results. For the XY model, reference Feldman. Is the general $O(n)$ case also done? Is the XY case also done at positive temperatures? ***

Exercise: Prove the above assertion at zero temperature for the random-field Ising and Potts models. *** can use a percolation argument with the points of large disorder. Can make this a guided exercise and reference [3, Appendix A] ***

Given the above results, interest is naturally directed towards the regime of low temperature and weak disorder strength. This was famously addressed in the physics literature by the work of Imry–Ma, who argued that the magnetized phase will be lost, in the presence of arbitrarily weak disorder, in dimension $d = 2$ for the random-field Ising and Potts models (and more general systems), and in all dimensions $d \leq 4$ for the random-field $O(n)$ model with $n \geq 2$. This prediction was famously made rigorous by the work of Aizenman–Wehr, who greatly extended its scope. Imry–Ma further predicted that the magnetized phase will be retained by the disordered system in higher dimensions (dimensions $d \geq 3$ for the random-field Ising and Potts models and dimensions $d \geq 5$ for the random-field $O(n)$ models with $n \geq 2$). For the random-field Ising model, this claim was under significant debate in the physics literature, with Parisi–Surlas *** presenting arguments against it. The debate was famously resolved by the rigorous works of Imbrie *** (at zero temperature) and Bricmont–Kupiainen *** (at all temperatures) who showed that the Imry–Ma prediction is correct: the magnetized phase is retained already in three dimensions.

*** Open problem: Long-range order for the random-field spin $O(n)$ model in dimensions $d \geq 5$ (even at zero temperature and even for the random-field XY model). ***

*** Can add here the $d \geq 3$ work of Ding–Liu–Xia that the critical temperature can be arbitrarily close to the pure Ising model if the disorder strength is sufficiently small. There is a related work of Ding–Huang–Xia in $d = 2$ at the critical temperature to find the critical scaling of the disorder strength with the size of the box. ***

The next sections discuss the Imry–Ma prediction in more detail. We first present a recent short proof of the existence of the magnetized phase in dimensions $d \geq 3$ due to Ding–Zhuang [13]. Then, we discuss quantitative aspects of the absence of phase transition in lower dimensions, presenting the work of Dario–Harel–Peled [10] and highlighting the many remaining open questions.

2.4.1 Long-range order in the random-field Ising and Potts models

In this section we present the argument of Ding–Zhuang [13] for the existence of long-range order in the random-field Ising model in dimensions $d \geq 3$, at low temperature and weak disorder. The argument can be thought of as a version of the famous Peierls argument for showing long-range order, adapted to disordered spin systems. It extends a technique of Fisher–Fröhlich–Spencer [14] which was introduced in an earlier attempt to settle the problem (this latter work gave strong support to the long-range order prediction by showing that it would occur if there were "no domain walls within domain walls"; see also *** Chalker ***).

The argument also adapts to the random-field Potts model, and gave the first proof of existence of a magnetized phase there.

Theorem 2.1. *For every $d \geq 3$ there exists $T_0 > 0$ and $\lambda_0 > 0$ such that for all $0 \leq T < T_0$ and $0 \leq \lambda < \lambda_0$,*

$$\lim_{L \to \infty} \mathbb{E}\left[\langle \sigma_0 \rangle_{\Lambda_L,+}^{\textit{RF-Ising}, \eta^{\textit{RF-Ising}}, T}\right] > 0. \tag{18}$$

To present the argument in its simplest form, we discuss only the zero temperature case random-field Ising model, leaving the extension to the other cases as an exercise *** add the exercise ***.

Fix $d \geq 3$. Let λ_0 be chosen sufficiently small and positive for the following arguments and fix a disorder strength $0 \leq \lambda < \lambda_0$. Fix $L \geq 0$ integer. For brevity, in the proof, we remove λ and L from most of the notation and write η for $\eta^{\text{RF-Ising}}$. We let σ^η be the, almost-surely unique, finite-volume ground configuration of the Ising model in Λ_L with $+$-boundary values. Also denote the finite-volume ground energy by

$$\mathrm{GE}^\eta := H^{\text{RF-Ising},\eta}_{\Lambda_L,+}(\sigma^\eta). \tag{19}$$

We denote the edge boundary of a set $A \subset \mathbb{Z}^d$ by

$$\partial A := \{\{u,v\} \in E(\mathbb{Z}^d)\colon |\{u,v\} \cap A| = 1\}. \tag{20}$$

For an integer $\ell \geq 1$ we let

$$\begin{aligned}&\mathcal{C}_\ell := \{A \subset \mathbb{Z}^d\colon A \text{ finite and connected, } A^c \text{ connected}, 0 \in A, |\partial A| = \ell\},\\ &\mathcal{C} := \cup_{\ell=0}^\infty \mathcal{C}_\ell\end{aligned} \tag{21}$$

The first step of the proof is to establish the containment of events,

$$\{\sigma_0^\eta = -1\} \subset \left\{\text{there exists } A \in \mathcal{C} \text{ for which } \mathrm{GE}^\eta - \mathrm{GE}^{\eta^A} \geq 2|\partial A|\right\}. \tag{22}$$

To see this, observe that if $\sigma_0^\eta = -1$ then there exists a (random) set $A \in \mathcal{C}$, $A \subset \Lambda_L$, such that $\sigma^\eta \equiv -1$ on the interior vertex boundary of A and $\sigma^\eta \equiv 1$ on the exterior vertex boundary of A. Suppose A is such a set. Define a new configuration and random field by flipping both the configuration and the random field on A,

$$\begin{aligned}\sigma_v^{\eta,A} &:= \begin{cases}-\sigma_v^\eta & v \in A\\ \sigma_v^\eta & v \notin A\end{cases},\\ \eta_v^A &:= \begin{cases}-\eta_v & v \in A\\ \eta_v & v \notin A\end{cases}.\end{aligned} \tag{23}$$

The discrete ± 1 symmetry of the random-field Ising model then yields the energy gap,

$$H^{\text{RF-Ising},\eta}(\sigma^\eta) - H^{\text{RF-Ising},\eta^A}(\sigma^{\eta,A}) \geq 2|\partial A|. \tag{24}$$

As $H^{\text{RF-Ising},\eta^A}(\sigma^{\eta,A}) \geq \mathrm{GE}^{\eta^A}$, it follows that there is a gap between the ground energies of the model with the original field η and the model with the flipped field η^A,

$$\mathrm{GE}^\eta - \mathrm{GE}^{\eta^A} \geq 2|\partial A|, \tag{25}$$

establishing the containment (22).

The argument will be (eventually) concluded by proving that, for each ℓ,

$$\mathbb{P}\left(\exists A \in \mathcal{C}_\ell \text{ such that } |\,\mathrm{GE}^\eta - \mathrm{GE}^{\eta^A}\,| \geq 2|\partial A|\right) \leq C_d \exp\left(-c_d \frac{\ell^{\frac{d-2}{d-1}}}{\lambda^2}\right) \tag{26}$$

(with $C_d, c_d > 0$ depending only on d), as, by (22), this will imply that

$$\sup_L \mathbb{P}(\sigma_0^\eta = -1) \leq \sum_{\ell \geq 1} C_d \exp\left(-c_d \frac{\ell^{\frac{d-2}{d-1}}}{\lambda^2}\right) < 1 \tag{27}$$

for sufficiently small λ.

The proof of (26) makes use of the concentration properties of the distribution of the ground energy. The first and fundamental ingredient is the following consequence of the Gaussian isoperimetric inequality of Borell and Tsirelson–Ibragimov–Sudakov *** ref? ***.

Theorem 2.2 (Concentration of maxima of Gaussian processes)**.** *Let T be a compact set. Let $(X_t)_{t\in T}$ be a continuous Gaussian process (not necessarily centered). Denote $M_t := \max_{t\in T} X_t$. Then $\mathbb{E}(M_t) < \infty$ and for every $u > 0$,*

$$\mathbb{P}(|M_t - \mathbb{E}(M_t)| \geq u) \leq 2e^{-\frac{u^2}{2\sigma_T^2}} \tag{28}$$

with $\sigma_T^2 := \sup_{t\in T} \mathrm{Var}(X_t)$.

This result is applied conditionally. For each finite $A \subset \mathbb{Z}^d$, write η_{A^c} for the restriction of η to A^c. Observe that conditionally on η_{A^c}, GE^η is the minimum of a Gaussian process on the compact set $T = \{-1, 1\}^{\Lambda_L}$, whose maximal variance is $\lambda^2|A\cap\Lambda_L| \leq \lambda^2|A|$. Theorem 2.2 thus implies that, almost surely,

$$\mathbb{P}\left(\left|\,\mathrm{GE}^\eta - \mathbb{E}(\mathrm{GE}^\eta \mid \eta_{A^c})\right| \geq u \mid \eta_{A^c}\right) \leq 2e^{-\frac{u^2}{2\lambda^2|A|}}. \tag{29}$$

This will be applied through the following useful corollary.

Corollary 2.3. *There exist $C, c > 0$ such that for each $A \subset \mathbb{Z}^d$ and $u > 0$,*

$$\mathbb{P}\left(\left|\,\mathrm{GE}^\eta - \mathrm{GE}^{\eta^A}\,\right| \geq u\right) \leq Ce^{-c\frac{u^2}{\lambda^2|A|}}, \tag{30}$$

and also for each $A, A' \subset \mathbb{Z}^d$ and $u > 0$,

$$\mathbb{P}\left(\left|\,\mathrm{GE}^{\eta^{A'}} - \mathrm{GE}^{\eta^A}\,\right| \geq u\right) \leq Ce^{-c\frac{u^2}{\lambda^2|A\Delta A'|}}, \tag{31}$$

where $A\Delta A'$ is the symmetric difference of A and A'.

Proof. The essential point is that, almost surely, $\mathbb{E}(\mathrm{GE}^\eta \mid \eta_{A^c}) = \mathbb{E}(\mathrm{GE}^{\eta^A} \mid \eta_{A^c})$, which follows from the fact that η^A has the same distribution as η and $\eta^A = \eta$ on A^c. It thus follows from (29) that, almost surely,

$$\mathbb{P}\left(\left|\,\mathrm{GE}^\eta - \mathrm{GE}^{\eta^A}\,\right| \geq u \mid \eta_{A^c}\right) \leq Ce^{-c\frac{u^2}{\lambda^2|A|}}, \tag{32}$$

The inequality (30) follows by taking the expectation of (32). Inequality (31) follows from (30) by replacing η with $\eta^{A'}$ (which has the same distribution as η). $\square$

To understand (26) better, observe first that the same bound holds for a fixed deterministic finite set $A \subset \mathbb{Z}^d$ by (30) and the isoperimetric inequality

$$|A| \leq C_d|\partial A|^{d/(d-1)}. \tag{33}$$

Indeed,

$$\mathbb{P}(\mathrm{GE}^{\eta} - \mathrm{GE}^{\eta^A} \geq 2|\partial A|) \leq C \exp\left(-c\frac{|\partial A|^2}{\lambda^2 |A|}\right) \leq C \exp\left(-c_d \frac{|\partial A|^{\frac{d-2}{d-1}}}{\lambda^2}\right) \tag{34}$$

where we use the convention that the values of the positive C, c, C_d, c_d may change from expression to expression, with C, C_d only increasing and c, c_d only decreasing (but C, c remain absolute constants and C_d, c_d depend only on d).

However, the estimate (34) does not suffice to establish (26) via a union bound, since the number of subsets $A \in \mathcal{C}$ with $|\partial A| \leq \ell$ is at least $c_d \exp(C_d \ell)$ (this may be argued directly. One may also consult [19] or [6, Theorem 6 and Theorem 7], noting the equivalence in [7, Appendix A]). Instead, the estimate (26) is derived from the concentration bound (31) using a coarse-graining technique (or chaining argument) introduced by Fisher–Fröhlich–Spencer [14] in a closely-related context. We proceed to elaborate on this technique.

Given a set $A \subset \mathbb{Z}^d$ and integer $N \geq 1$, let A_N be the N-coarse-grained version of A defined as the union of all cubes $B \subset \mathbb{Z}^d$, of the form $v + \{0, 1, \ldots, N-1\}^d$ with $v \in N\mathbb{Z}^d$, which satisfy $|A \cap B| \geq \frac{1}{2}|B|$. We consider all possible coarse grainings of sets in $\mathcal{C}_\ell$,

$$\mathcal{C}_\ell^N := \{A_N \colon A \in \mathcal{C}_\ell\}. \tag{35}$$

The following important inputs are established in [14] *** for $d = 3$ and maybe special value of the parameter; see also [7] for extensions ***

Proposition 2.4. *For each integer $\ell, N \geq 1$,*

$$|\mathcal{C}_\ell^N| \leq C_d e^{C_d \frac{\ell}{N^{d-1}} \log(N+1)} \tag{36}$$

and, for each $A \in \mathcal{C}_\ell$,

$$|A_{2N} \Delta A_N| \leq C_d N \ell. \tag{37}$$

*** Very roughly, $|\partial A_N| \approx |\partial A|$ so that A_N may be regarded as a set with surface volume at most $|\partial A|/N^{d-1}$ after shrinking the lattice $\mathbb{Z}^d$ by a factor N. This is complicated, however, by the fact that A_N need not be connected or have connected complement ***

One may then prove (26) via the following chaining argument. Write the telescopic expansion

$$\mathrm{GE}^{\eta} - \mathrm{GE}^{\eta^A} = \sum_{k=0}^{K-1} \left(\mathrm{GE}^{\eta^{A_{2^{k+1}}}} - \mathrm{GE}^{\eta^{A_{2^k}}}\right) \tag{38}$$

where we note that $A_{2^0} = A_1 = A$ and where we choose K sufficiently large that $A_{2^K} = \emptyset$ (so that $\eta^{A_{2^K}} = \eta$). Specifically, choosing K so that 2^K has order $\ell^{\frac{1}{d-1}}$ suffices by the isoperimetric inequality (33) *** do we need to explain here that if A_{2^K} was non-empty then its boundary length would be much more than ℓ which would be a contradiction? ***. Then, for each choice of positive coefficients $(\alpha_k)_{k=0}^{K-1}$ summing to 1 we have, using Proposition 2.4,

$$\begin{aligned}
&\mathbb{P}\left(\exists A \in \mathcal{C}_\ell \text{ such that } |\,\mathrm{GE}^{\eta} - \mathrm{GE}^{\eta^A}\,| \geq 2|\partial A|\right) \\
&\leq \sum_{k=0}^{K-1} \mathbb{P}\left(\exists A \in \mathcal{C}_\ell \text{ such that } |\,\mathrm{GE}^{\eta^{A_{2^{k+1}}}} - \mathrm{GE}^{\eta^{A_{2^k}}}\,| \geq 2\alpha_k \ell\right) \\
&\leq \sum_{k=0}^{K-1} \sum_{\substack{B\in\mathcal{C}_\ell^{2^k}, B'\in\mathcal{C}_\ell^{2^{k+1}} \\ \exists A\in\mathcal{C}_\ell \text{ with } B=A_{2^k}, B'=A_{2^{k+1}}}} \mathbb{P}\left(|\,\mathrm{GE}^{\eta^{B'}} - \mathrm{GE}^{\eta^B}\,| \geq 2\alpha_k \ell\right) \\
&\leq \sum_{k=0}^{K-1} \sum_{\substack{B\in\mathcal{C}_\ell^{2^k}, B'\in\mathcal{C}_\ell^{2^{k+1}} \\ \exists A\in\mathcal{C}_\ell \text{ with } B=A_{2^k}, B'=A_{2^{k+1}}}} Ce^{-c\frac{\alpha_k^2\ell^2}{\lambda^2|B\Delta B'|}} \\
&\leq \sum_{k=0}^{K-1} C_d e^{C_d(k+1)\frac{\ell}{2^{k(d-1)}}} e^{-c_d\frac{\alpha_k^2\ell}{\lambda^2 2^k}} \qquad (39)
\end{aligned}$$

which one may check is less than the right-hand side of (26) when $0 \leq \lambda \leq \lambda_0$ with $\lambda_0 \leq c_d$ positive but sufficiently small, and letting $\alpha_k = \gamma 2^{-\frac{1}{4}\min\{k,K-k\}}$ with γ a normalizing constant ensuring that the α_k sum to 1.

*** Exercise: Extend argument to low, positive temperatures. Change ground energy to free energy.

Exercise: Extend argument to random-field Potts model. ***

2.4.2 Quantitative estimates on the absence of magnetization in low-dimensional systems

List of notation to introduce

- $\mathcal{S}$ $\mathcal{S}_{q,L}^{\tau}$
- For almost every η, there exists a unique maximiser in the definition of $F_{L,\tau}(\eta)$, we denote it by $\sigma_v^{\eta,\lambda,\tau}$
- Let us introduce the function energy for the ground state of the disordered Potts model with specified boundary condition: for a fixed $\tau \in \{1,\ldots,q\}^{\partial\Lambda_L}$,

$$F_{L,\tau}(\eta) := \sup_{\sigma\in\mathcal{S}_{q,L}^{\tau}} \left(\sum_{\substack{u\sim v \\ \{u,v\}\cap\Lambda_L\neq\emptyset}} 1_{\sigma_u=\sigma_v} + \sum_{x\in\Lambda_L}\sum_{k=1}^{q} \eta_{v,k} 1_{\sigma_v=k} \right)$$

*** Here we will review results from Dario–Harel–Peled [10]. ***

Theorem 2.5 (Quantitative Imry-Ma phenomenon for the random-field Potts model)**.** *Consider the random-field Potts model*

$$\mathbb{E}\left[\sup_{\tau\in\{1,\ldots,q\}^{\partial\Lambda_L}} \left| \frac{\left|\left\{v\in\Lambda_L \,:\, \sigma_v^{\eta,\lambda,\tau} = k\right\}\right|}{|\Lambda_L|} - \frac{1}{q} \right| \right] \leq \frac{C}{\sqrt[4]{\ln\ln L}}$$

Proof. We first collect a few (standard) properties of the function $F_{L,\tau}$:

- The function $\eta \mapsto F_{L,\tau}(\eta)$ is a supremum of a finite number of linear functions (in the variable η). This observation implies that this function is convex, differentiable almost everywhere and we have
$$\frac{\partial F_{L,\tau}(\eta)}{\partial \eta_{k,i}} = 1_{\sigma_v^{\eta,\lambda,\tau}=k}$$
- There exists a constant $C < \infty$ such that For any pair of boundary conditions $\tau, \tau' \in \mathcal{S}_{q,L}^{\tau}$ and any realization of the external field η,
$$|F_{L,\tau}(\eta) - F_{L,\tau'}(\eta)| \leq C\, |\partial \Lambda_L| \tag{40}$$
- If we write $\eta = (\eta$

$\square$

*** Point to exercise (maybe in appendix?) on the absence of a magnetized phase for the two-dimensional random-field Ising model at zero temperature ***

*** Open problem: Uniformity of distribution of random-field Potts spin at the origin in dimension $d = 2$. ***

*** Mention also quantum version [2] and its accompanying papers in the physics literature ***

3 First-passage percolation

***

(B_δ) in the perturbation of sets lemma is actually the good set. Consider calling it (G_δ) or something similar to imply this.

Copy introduction to perturbation of weights lemma from coalescence paper where there is more motivation and earlier uses.

Add relatively simple application of perturbation of weights to give that $\xi \geq \frac{1}{d+1}$.

Is there a nice application that we can include where the perturbation of weights lemma is used in the tail? or with more general p? Or can we at least cite papers where this is done?

***

First-passage percolation is a model for a random metric space, formed by a random perturbation of an underlying base space. Since its introduction by Hammersley–Welsh in 1965 [17], it has been studied extensively in the probability and statistical physics literature. We refer to [18] for general background and to [5] for more recent results.

We study first-passage percolation on the hypercubic lattice $(\mathbb{Z}^d, E(\mathbb{Z}^d))$, $d \geq 2$, in an independent and identically distributed (IID) random environment. The model is specified by a *weight distribution* ν, which is a probability measure on the non-negative reals. It is defined by assigning each edge $e \in E(\mathbb{Z}^d)$ a random passage time (also termed weight) τ_e

with distribution ν, independently between edges. Then, each finite path p in $\mathbb{Z}^d$ is assigned the random passage time

$$T(p) := \sum_{e \in p} \tau_e, \tag{41}$$

yielding a random metric T on $\mathbb{Z}^d$ by setting the *passage time between* $u, v \in \mathbb{Z}^d$ to

$$T(u,v) := \inf_p T(p), \tag{42}$$

where the infimum ranges over all finite paths connecting u and v. Any path achieving the infimum is termed a *geodesic* between u and v. A unique geodesic exists when G is atomless and will be denoted $\gamma(u,v)$. The focus of first-passage percolation is the study of the large-scale properties of the random metric T and its geodesics.

The passage time of the geodesic between given endpoints is naturally a function of the weights assigned to all edges. To what extent is this passage time *influenced* by the weight assigned to a specific edge? This notion is formalized here by the *probability* that the geodesic passes through that edge. It is clear that the influence of edges near the endpoints cannot be uniformly small, but it is not clear whether the influence diminishes uniformly for edges far from the endpoints. This issue was highlighted by Benjamini–Kalai–Schramm [8] in their seminal study of the variance of the passage time, where the following problem, later termed *the BKS midpoint problem*[5], was posed: Consider the geodesic between 0 and v. Does the probability that it passes at distance 1 from $v/2$ tend to zero as $\|v\| \to \infty$? The following more general version may also be expected:

$$\begin{gathered}\text{show that for any } \epsilon > 0 \text{ there is } r(\epsilon) > 0 \text{ such that for each } v \in \mathbb{Z}^d \setminus \{0\} \\ \text{and all } e \in E(\mathbb{Z}^d) \text{ with } d(e, \{0, v\}) > r(\epsilon) \text{ we have } \mathbb{P}(e \in \gamma(0,v)) < \epsilon\end{gathered} \tag{43}$$

with $d(e, \{0, v\})$ denoting the distance of e from the closer endpoint.

On the square lattice ($d = 2$), The BKS midpoint problem was resolved positively by Damron–Hanson [9] under the assumption that the limit shape boundary is differentiable and then resolved unconditionally by Ahlberg-Hoffman [1] (in the more general version (43)). Recently, assuming that the limit shape has more than 40 extreme points, the authors [11] provided a quantitative version of (43), showing that $\mathbb{P}(e \in \gamma(0,v))$ is smaller than a negative power of $d(e, \{0, v\})$. In all dimensions $d \geq 2$, an optimal, up to sub-power factors, quantitative version of (43) was obtained by Alexander [4] under assumptions on the model which are still unverified (the proof relies on assumptions on the limit shape and on the passage time fluctuations). Unconditionally, the BKS midpoint problem, and its generalization (43), remain open when $d \geq 3$.

Our first main result shows that, in all dimensions, there can be at most a constant number of exceptional edges in (43). Let us state this precisely. We work with the following class of weight distributions: assume that for some $b > a > 0$ and $\alpha > 0$,

$$\begin{gathered}\nu \text{ is supported on the interval } [a,b] \text{ and is absolutely continuous} \\ \text{with a density } \rho \text{ satisfying } \rho(x) \geq \alpha \text{ for almost all } x \in [a,b].\end{gathered} \tag{44}$$

[5]Though it is referred to as the BKS midpoint problem, the problem was discussed by earlier authors. For instance, it is closely related to the discussion in (9.22) of Kesten's lectures [18].

Theorem 3.1. *Let $d \geq 2$. Suppose that the weight distribution ν satisfies* (44)*. Then, for any $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ depending on d, ν and ε, such that for all $v \in \mathbb{Z}^d$,*

$$\left|\left\{e \in E(\mathbb{Z}^d) : \mathbb{P}(e \in \gamma(0, v)) \geq \epsilon\right\}\right| \leq C_\epsilon. \tag{45}$$

*** Shall we track the dependence of C_ε on ε when the density of ν is also bounded above? ***

Note that for small $\epsilon > 0$ the left-hand side of (45) also has a complementary lower bound $c_\epsilon > 0$, as there must be edges at a constant distance from the endpoints of the geodesic which have a constant influence.

*** Should add the basic delocalization bound $\xi \geq \frac{1}{d+1}$ and the quantitative influence bound (as in the introduction to the paper?) ***

3.1 Perturbing the weights

The following lemma is the main technical tool required for the proof of the main theorems. It quantifies the amount by which the joint distribution of a vector of independent random variables with a distribution satisfying (44) is altered by small additive perturbations. It is taken from [11, Lemma 2.12, Remark 2.15 and Remark 2.16]. *** A version of this lemma has been used also in [**?**, Lemma 2.1] ***. Let us note that a distribution ν satisfying (44) is the image of the standard Gaussian distribution under an increasing Lipschitz function and therefore [11, Remark 2.16] holds for such a distribution.

*** Put the introduction to the perturbation of weights from the coalescence paper ***

Lemma 3.2. *Let ν be a distribution satisfying* (44)*. Then, there exist $C_0 > 0$ and*

- *Measurable subsets $(G_\delta)_{\delta>0}$ of $[a, b]$ with $\lim_{\delta\downarrow 0} \nu(G_\delta) = 1$,*
- *For each $\sigma \in [0, 1]$, an increasing bijection $g^+_\sigma : [a, b] \to [a, b]$.*

such that the following holds:

1. *For any $\sigma \in [0, 1]$, $t \leq g^+_\sigma(t) \leq t + C_0\sigma$ for $t \in [a, b]$ and for all $\delta > 0$,*

$$g^+_\sigma(t) \geq t + \delta\sigma \quad \textit{for } w \in G_\delta. \tag{46}$$

2. *For any $p > 1$, an integer $n \geq 1$, a vector $s = (s_1, \ldots, s_n) \in [0, 1]^n$ and a measurable $A \subset \mathbb{R}^n$ we have*

$$\mathbb{P}\Big(\big(g^+_{s_1}(X_1), \ldots, g^+_{s_n}(X_n)\big) \in A\Big) \geq \exp\left(-\frac{p\|s\|_2^2}{2(p-1)}\right) \cdot \mathbb{P}\big((X_1, \ldots, X_n) \in A\big)^p, \tag{47}$$

where $X_1, X_2, \ldots, X_n$ are independent random variables with distribution ν.

*** Would be good to add a proof. Perhaps, to make the proof simpler, under the extra assumption that the density of ν is also bounded above.

Make the constant C_0 explicit in terms of the parameters a, b, α?

Why not allow negative σ and s? e.g., $\sigma \in [-1, 1]$. ***

*** Consider remarking that if the density is also bounded above by β then we get a quantitative bound on $\nu(G_\delta) \geq 1 - C\beta\delta$. ***

3.2 Number of edges with a constant influence

In this section we prove Theorem 3.1.

Fix $\varepsilon > 0$ and $v \in \mathbb{Z}^d$. For $x \in \mathbb{Z}^d$, we write $\gamma_x := \gamma(x, v + x)$ and $T_x := T(x, v + x)$ for the geodesic and passage time between x and $v + x$. Let

$$E := \{e \in E(\mathbb{Z}^d) \colon \mathbb{P}(e \in \gamma_0) \geq \varepsilon\}. \tag{48}$$

be the set of edges with influence at least ε on T_0. Our goal is to show that $|E|$ is bounded above uniformly in v (i.e., the bound is a function solely of the influence threshold ε, the dimension d and the weight distribution ν).

Assume $E \neq \emptyset$ as otherwise there is nothing to prove. For $r > 0$, let

$$B_r := \{x \in \mathbb{Z}^d \colon \|x\|_1 \leq r\} \tag{49}$$

where $\|x\|_1$ is the ℓ_1 norm of x (the graph distance between x and 0 in $\mathbb{Z}^d$).

Lemma 3.3. *For each $p > 1$, $r, t > 0$ it holds that for all $x \in B_r$,*

$$\mathbb{P}(|\gamma_x \cap E| \geq ...) \geq e^{-\frac{pt^2}{2(p-1)}} (\frac{\varepsilon}{2})^p \tag{50}$$

Proof. Define the event $A := \{|\gamma_0 \cap E| \geq \frac{1}{2}\varepsilon |E|\}$. Noting that $\mathbb{E}|\gamma_0 \cap E| \geq \varepsilon E$ by the definition of E, we deduce from Markov's inequality (applied to $|E \setminus \gamma_0|$) that

$$\mathbb{P}(A) \geq \frac{\frac{1}{2}\varepsilon}{1 - \varepsilon} \geq \frac{1}{2}\varepsilon. \tag{51}$$

Define $s : E(\mathbb{Z}^d) \to [0, 1]$ by $s_e := \frac{t}{\sqrt{|E|}} 1_{e \in E}$. We apply Lemma 3.2 to the weight vector $(\tau_e)_{e \in \mathbb{Z}^d}$ (formally, as the lemma is stated for a finite vector of random variables, we apply it to the finite sub-collection of the weight vector on which our event A depends on). Thus we obtain a new weight vector defined by $\tau_e^+ := g_{s_e}^+(\tau_e)$. Correspondingly, we let γ_x^+ and T_x^+ be the geodesic and passage time between x to $v + x$ evaluated with the weight vector (τ_e^+). We regard the event A as the set of (τ_e) for which it holds, and correspondingly define

$$A^+ := \{(\tau_e^+) \in A\} = \left\{ |\gamma_0^+ \cap E| \geq \frac{1}{2}\varepsilon |E| \right\}. \tag{52}$$

By Lemma 3.2 we then have

$$\mathbb{P}(A^+) \geq e^{-\frac{p\|s\|_2^2}{2(p-1)}} \mathbb{P}(A)^p = e^{-\frac{pt^2}{2(p-1)}} \mathbb{P}(A)^p \geq e^{-\frac{pt^2}{2(p-1)}} \left(\frac{\varepsilon}{2}\right)^p \tag{53}$$

using (51) in the last inequality. Next, we wish to compare τ_e^+ to τ_e via (46). To this end, let $\delta_\varepsilon > 0$ be sufficiently small so that for all integer $k \geq 1$, denoting by $\mathrm{Bin}(k, p)$ a random variable having the binomial distribution with k trials and success probability p,

$$\mathbb{P}\left(\mathrm{Bin}(k, p) \leq \frac{\varepsilon}{4} k\right) \leq \frac{1}{2} e^{-\frac{pt^2}{2(p-1)}} \left(\frac{\varepsilon}{2}\right)^p. \tag{54}$$

$\square$

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